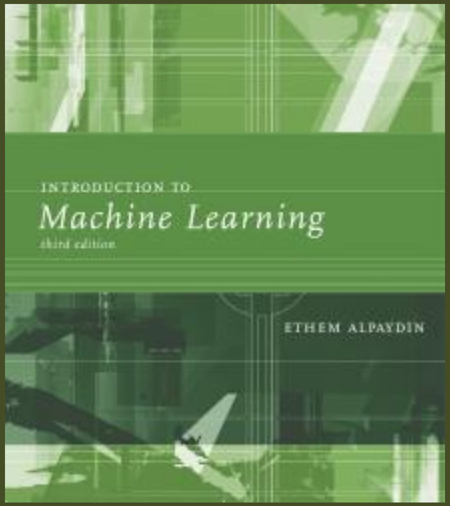


Lecture Slides for

INTRODUCTION TO MACHINE LEARNING

3RD EDITION

ETHEM ALPAYDIN

alpaydin@boun.edu.tr
http://www.cmpe.boun.edu.tr/~ethem/i2ml3e

CHAPTER 3:

BAYESIAN DECISION THEORY

Probability and Inference

- Result of tossing a coin is ∈ {Heads,Tails}
- Random var $X \in \{1,0\}$

  Bernoulli: $P\{X=1\} = p_o^X (1 - p_o)^{(1-X)}$
- Sample: $\mathbf{X} = \{x^t\}_{t=1}^N$

  Estimation: $p_o = \#\{Heads\}/\#\{Tosses\} = \sum_t x^t / N$
- Prediction of next toss:

  Heads if $p_o > ½$, Tails otherwise

Classification

- Credit scoring: Inputs are income and savings.
  Output is low-risk vs high-risk
- Input: $\mathbf{x} = [x_1, x_2]^T$,Output: C Î {0,1}
- Prediction:

$$\text{choose} \begin{cases} C = 1 \text{ if } P(C = 1 | x_1, x_2) > 0.5 \\ C = 0 \text{ otherwise} \end{cases}$$

or

$$\text{choose} \begin{cases} C = 1 \text{ if } P(C = 1 | x_1, x_2) > P(C = 0 | x_1, x_2) \\ C = 0 \text{ otherwise} \end{cases}$$

Bayes' Rule

prior *likelihood*

posterior

$$P(C \mid \mathbf{x}) = \frac{P(C)\,p(\mathbf{x} \mid C)}{p(\mathbf{x})}$$

evidence

$$P(C=0) + P(C=1) = 1$$

$$p(\mathbf{x}) = p(\mathbf{x} \mid C=1)P(C=1) + p(\mathbf{x} \mid C=0)P(C=0)$$

$$p(C=0 \mid \mathbf{x}) + P(C=1 \mid \mathbf{x}) = 1$$

Bayes' Rule: K>2 Classes

$$P(C_i \mid \mathbf{x}) = \frac{p(\mathbf{x} \mid C_i)P(C_i)}{p(\mathbf{x})}$$

$$= \frac{p(\mathbf{x} \mid C_i)P(C_i)}{\sum_{k=1}^{K} p(\mathbf{x} \mid C_k)P(C_k)}$$

$P(C_i) \geq 0$ and $\sum_{i=1}^{K} P(C_i) = 1$

choose C_i if $P(C_i \mid \mathbf{x}) = \max_k P(C_k \mid \mathbf{x})$

Losses and Risks

- Actions: α_i
- Loss of α_i when the state is C_k : λ_{ik}
- Expected risk (Duda and Hart, 1973)

$$R(\alpha_i \mid \mathbf{x}) = \sum_{k=1}^{K} \lambda_{ik} P(C_k \mid \mathbf{x})$$

$$\text{choose } \alpha_i \text{ if } R(\alpha_i \mid \mathbf{x}) = \min_k R(\alpha_k \mid \mathbf{x})$$

Losses and Risks: 0/1 Loss

$$\lambda_{ik} = \begin{cases} 0 \text{ if } i = k \\ 1 \text{ if } i \neq k \end{cases}$$

$$\begin{aligned} R(\alpha_i \mid \mathbf{x}) &= \sum_{k=1}^{K} \lambda_{ik} P(C_k \mid \mathbf{x}) \\ &= \sum_{k \neq i} P(C_k \mid \mathbf{x}) \\ &= 1 - P(C_i \mid \mathbf{x}) \end{aligned}$$

For minimum risk, choose the most probable class

Losses and Risks: Reject

$$\lambda_{ik} = \begin{cases} 0 & \text{if } i = k \\ \lambda & \text{if } i = K+1, \quad 0 < \lambda < 1 \\ 1 & \text{otherwise} \end{cases}$$

$$R(\alpha_{K+1} \mid \mathbf{x}) = \sum_{k=1}^{K} \lambda P(C_k \mid \mathbf{x}) = \lambda$$

$$R(\alpha_i \mid \mathbf{x}) = \sum_{k \neq i} P(C_k \mid \mathbf{x}) = 1 - P(C_i \mid \mathbf{x})$$

choose C_i if $P(C_i \mid \mathbf{x}) > P(C_k \mid \mathbf{x}) \;\; \forall k \neq i$ and $P(C_i \mid \mathbf{x}) > 1 - \lambda$

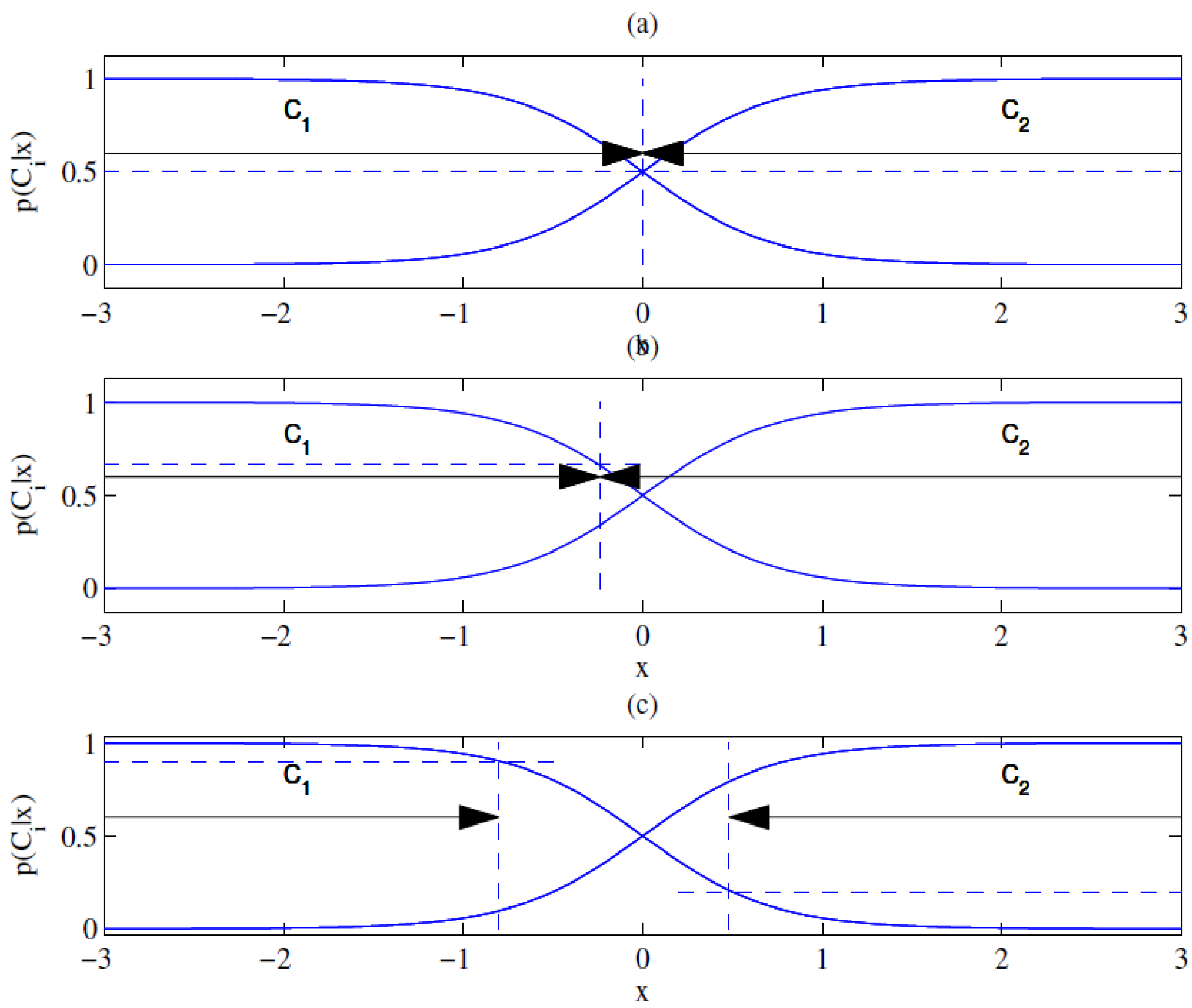
reject otherwise

Different Losses and Reject

Equal losses

Unequal losses

With reject

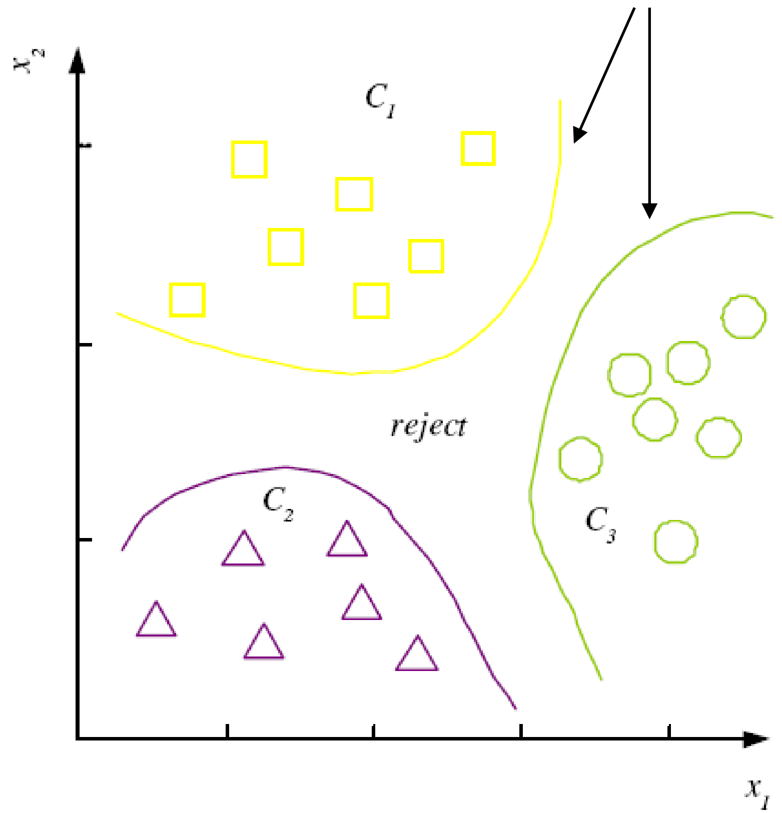


Discriminant Functions

$$\text{choose } C_i \text{ if } g_i(\mathbf{x}) = \max_k g_k(\mathbf{x})$$

$$g_i(\mathbf{x}) = \begin{cases} -R(\alpha_i \mid \mathbf{x}) \\ P(C_i \mid \mathbf{x}) \\ p(\mathbf{x} \mid C_i)P(C_i) \end{cases}$$

$$g_i(\mathbf{x}),\ i = 1, \ldots, K$$



K decision regions $\mathcal{R}_1, \ldots, \mathcal{R}_K$

$$\mathcal{R}_i = \{\mathbf{x} \mid g_i(\mathbf{x}) = \max_k g_k(\mathbf{x})\}$$

K=2 Classes

- Dichotomizer (K=2) vs Polychotomizer (K>2)
- $g(\boldsymbol{x}) = g_1(\boldsymbol{x}) - g_2(\boldsymbol{x})$

$$\text{choose} \begin{cases} C_1 \text{ if } g(\mathbf{x}) > 0 \\ C_2 \text{ otherwise} \end{cases}$$

- *Log odds:* $\log \frac{P(C_1 | \mathbf{x})}{P(C_2 | \mathbf{x})}$

Utility Theory

- Prob of state k given exidence $\mathbf{x}$: $P(S_k | \mathbf{x})$
- Utility of α_i when state is k: U_{ik}
- Expected utility:

$$EU(\alpha_i | \mathbf{x}) = \sum_k U_{ik} P(S_k | \mathbf{x})$$

Choose α_i if $EU(\alpha_i | \mathbf{x}) = \max_j EU(\alpha_j | \mathbf{x})$

Association Rules

- Association rule: $X \rightarrow Y$
- *People who buy/click/visit/enjoy X are also likely to buy/click/visit/enjoy Y.*
- A rule implies association, not necessarily causation.

Association measures

- Support $(X \rightarrow Y)$:

$$P(X,Y) = \frac{\#\{\text{customers who bought } X \text{ and } Y\}}{\#\{\text{customers}\}}$$

- Confidence $(X \rightarrow Y)$:

$$P(Y \mid X) = \frac{P(X,Y)}{P(X)}$$

$$= \frac{\#\{\text{customers who bought } X \text{ and } Y\}}{\#\{\text{customers who bought } X\}}$$

- Lift $(X \rightarrow Y)$:

$$= \frac{P(X,Y)}{P(X)P(Y)} = \frac{P(Y \mid X)}{P(Y)}$$

Example

Transaction	Items in basket
1	milk, bananas, chocolate
2	milk, chocolate
3	milk, bananas
4	chocolate
5	chocolate
6	milk, chocolate

SOLUTION:

$$\begin{aligned}
\text{milk} \rightarrow \text{bananas} &: \quad \text{Support} = 2/6, \text{Confidence} = 2/4 \\
\text{bananas} \rightarrow \text{milk} &: \quad \text{Support} = 2/6, \text{Confidence} = 2/2 \\
\text{milk} \rightarrow \text{chocolate} &: \quad \text{Support} = 3/6, \text{Confidence} = 3/4 \\
\text{chocolate} \rightarrow \text{milk} &: \quad \text{Support} = 3/6, \text{Confidence} = 3/5
\end{aligned}$$

Apriori algorithm (Agrawal et al., 1996)

- For (X,Y,Z), a 3-item set, to be frequent (have enough support), (X,Y), (X,Z), and (Y,Z) should be frequent.
- If (X,Y) is not frequent, none of its supersets can be frequent.
- Once we find the frequent *k*-item sets, we convert them to rules: X, Y → Z, ...

  and X → Y, Z, ...