Lecture Slides for

INTRODUCTION TO

Machine Learning

2nd Edition

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CHAPTER 8:

Nonparametric Methods

Nonparametric Estimation

- Parametric (single global model), semiparametric (small number of local models)
- Nonparametric: Similar inputs have similar outputs
- Functions (pdf, discriminant, regression) change smoothly
- Keep the training data;“let the data speak for itself”
- Given x, find a small number of closest training instances and interpolate from these
- Aka lazy/memory-based/case-based/instance-based learning

Density Estimation

- Given the training set $X=\{x^t\}_t$ drawn iid from $p(x)$
- Divide data into bins of size h
- Histogram:

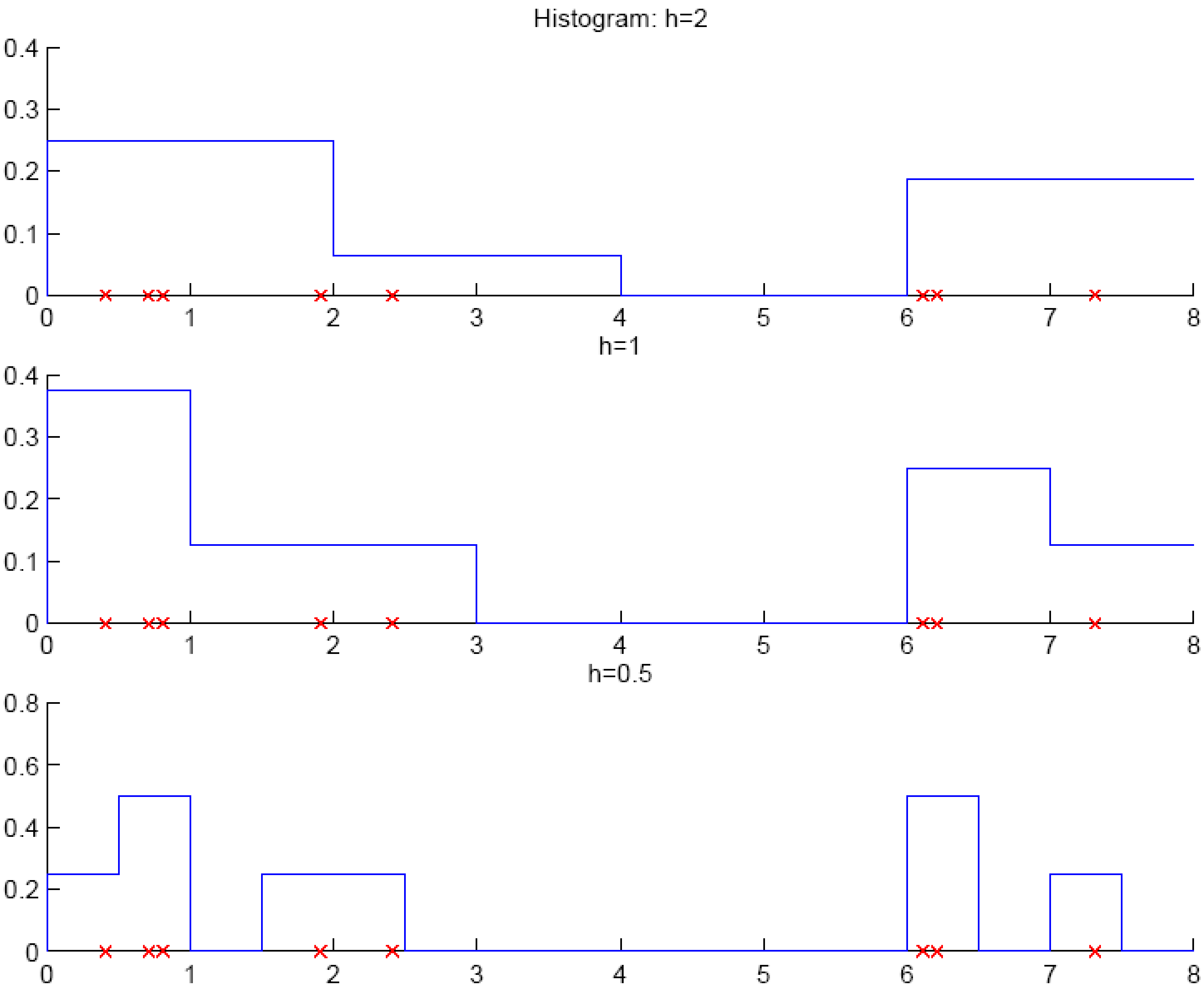
$$\hat{p}(x) = \frac{\#\{x^t \text{ in the same bin as } x\}}{Nh}$$

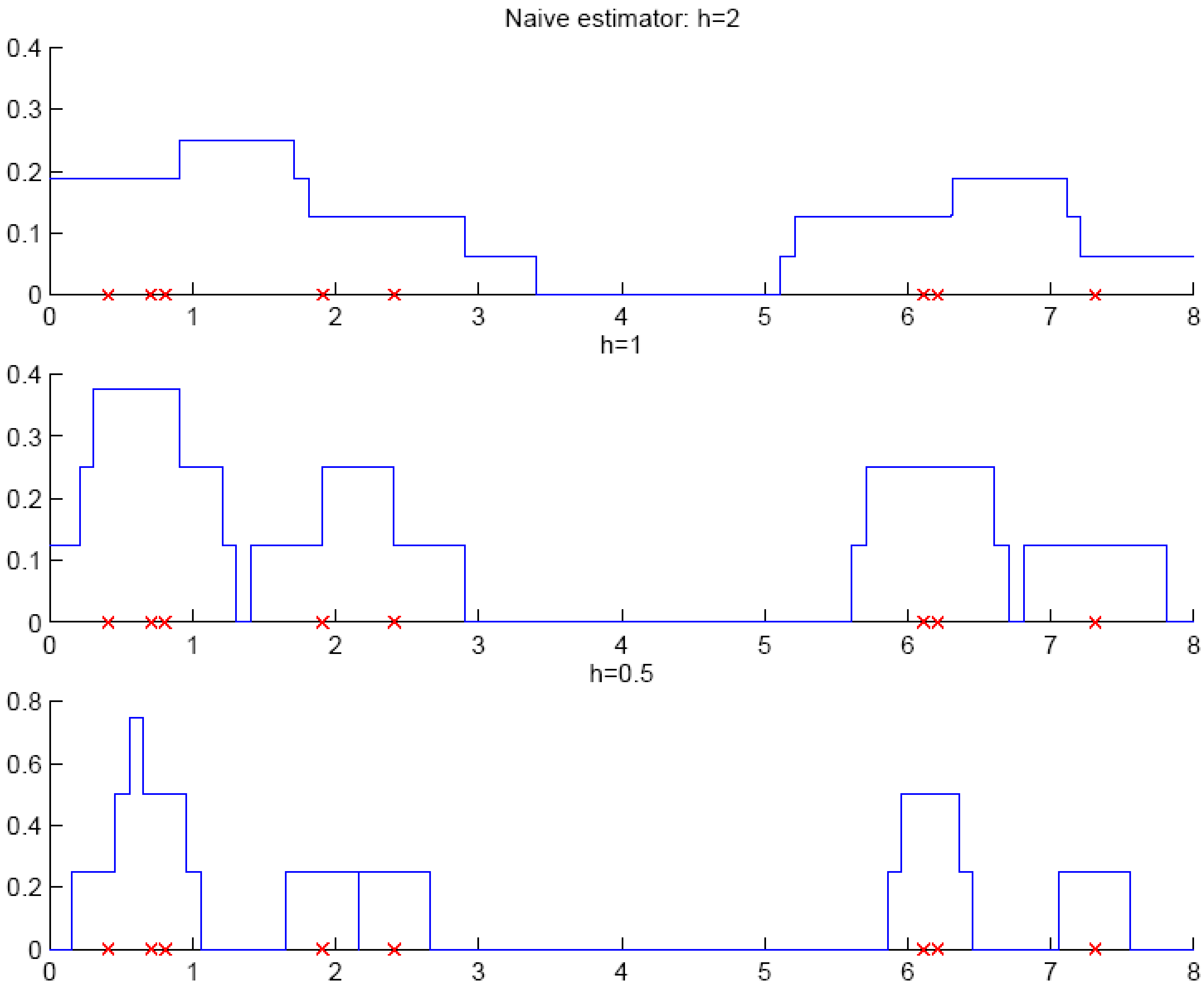
- Naive estimator:

$$\hat{p}(x) = \frac{\#\{x - h < x^t \le x + h\}}{2Nh}$$

or

$$\hat{p}(x) = \frac{1}{Nh}\sum_{t=1}^{N} w\left(\frac{x - x^t}{h}\right) \quad w(u) = \begin{cases} 1/2 & \text{if } |u| < 1 \\ 0 & \text{otherwise} \end{cases}$$


Histogram: h=2
0.4
0.3
0.2
0.1
0
0 1 2 3 4 5 6 7 8
h=1
0.4
0.3
0.2
0.1
0
0 1 2 3 4 5 6 7 8
h=0.5
0.8
0.6
0.4
0.2
0
0 1 2 3 4 5 6 7 8


Naive estimator: h=2
h=1
h=0.5
0.4
0.3
0.2
0.1
0
0.8
0.6
0 1 2 3 4 5 6 7 8

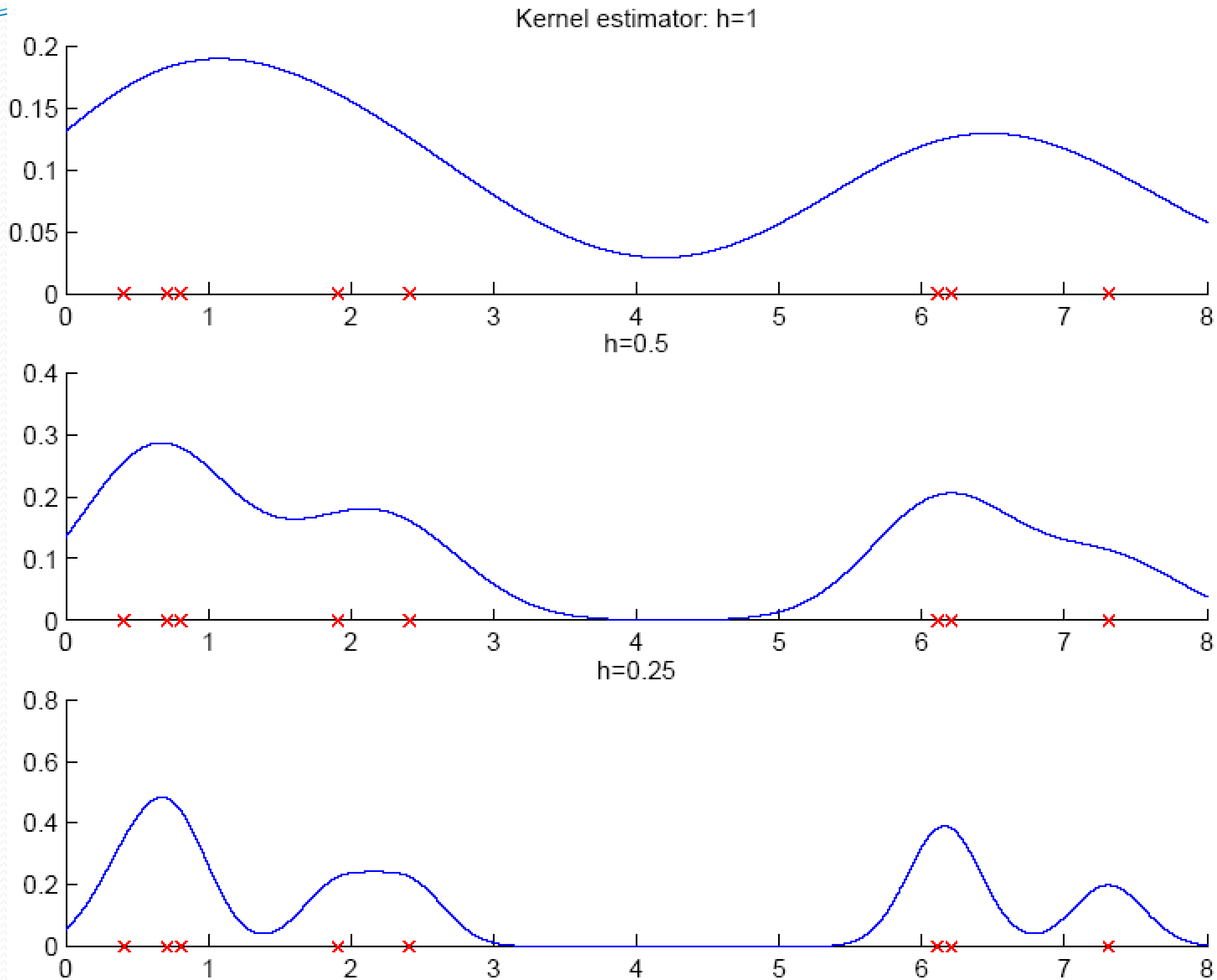
Kernel Estimator

- Kernel function, e.g., Gaussian kernel:

$$K(u)=\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{u^2}{2}\right]$$

- Kernel estimator (Parzen windows)

$$\hat{p}(x)=\frac{1}{Nh}\sum_{t=1}^{N}K\left(\frac{x-x^t}{h}\right)$$

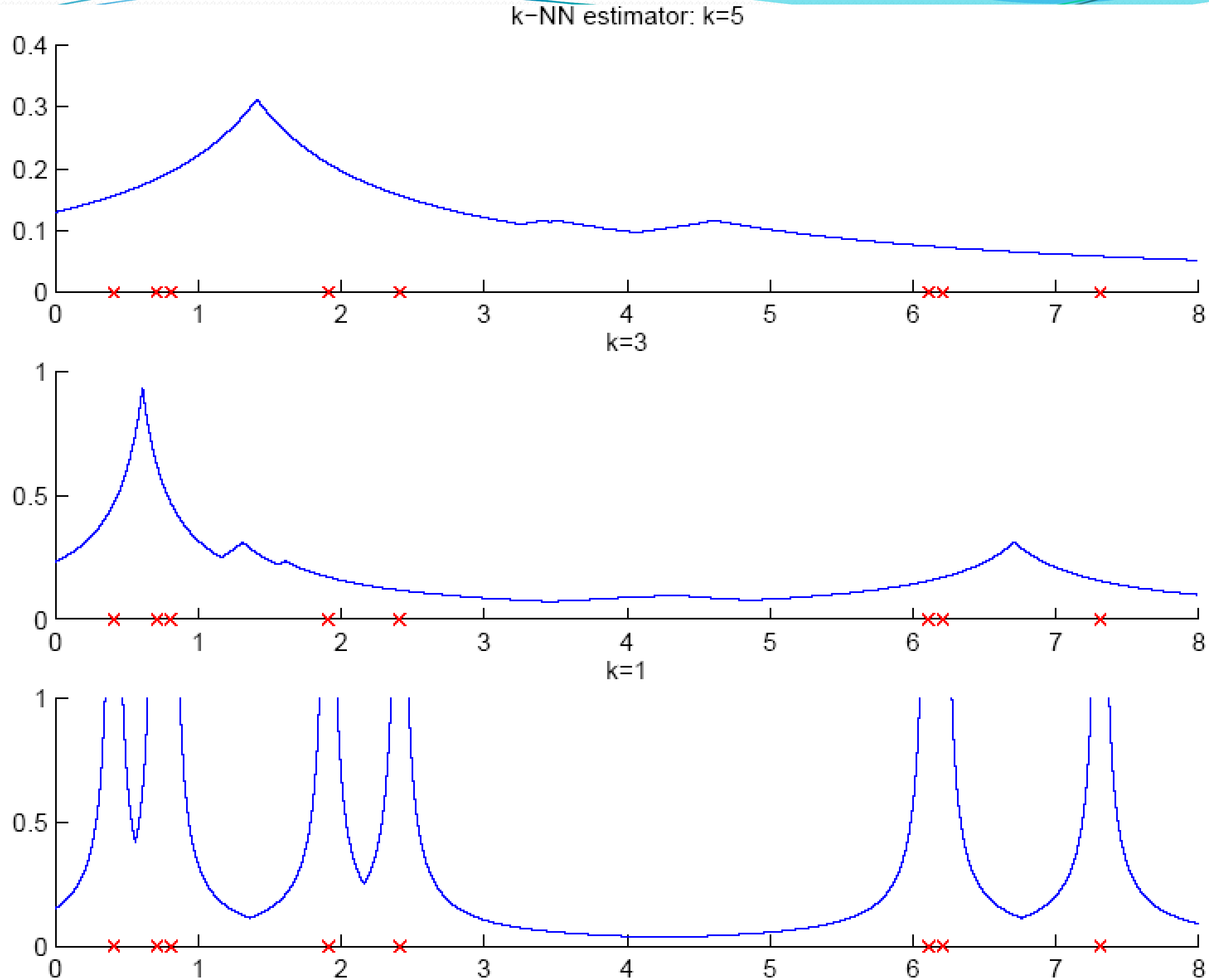

Kernel estimator: h=1
0.2
0.15
0.1
0.05
0
0
1
2
3
4
5
6
7
8
h=0.5
0.4
0.3
0.2
0.1
0
h=0.25
0.8
0.6
0.4
0.2
0

k-Nearest Neighbor Estimator

- Instead of fixing bin width *h* and counting the number of instances, fix the instances (neighbors) *k* and check bin width

$$\hat{p}(x) = \frac{k}{2Nd_k(x)}$$

$d_k(x)$, distance to *k*th closest instance to *x*


k−NN estimator: k=5
0.4
0.3
0.2
0.1
0
0 1 2 3 4 5 6 7 8
k=3
1
0.5
0
0 1 2 3 4 5 6 7 8
k=1
1
0.5
0
0 1 2 3 4 5 6 7 8

Multivariate Data

- Kernel density estimator

$$\hat{p}(\mathbf{x}) = \frac{1}{Nh^d} \sum_{t=1}^{N} K\left(\frac{\mathbf{x} - \mathbf{x}^t}{h}\right)$$

Multivariate Gaussian kernel

spheric

$$K(\mathbf{u}) = \left(\frac{1}{\sqrt{2\pi}}\right)^d \exp\left[-\frac{\|\mathbf{u}\|^2}{2}\right]$$

ellipsoid

$$K(\mathbf{u}) = \frac{1}{(2\pi)^{d/2}|\mathbf{S}|^{1/2}} \exp\left[-\frac{1}{2}\mathbf{u}^T\mathbf{S}^{-1}\mathbf{u}\right]$$

Nonparametric Classification

- Estimate $p(\boldsymbol{x}|C_i)$ and use Bayes' rule
- Kernel estimator

$$\hat{p}(\mathbf{x}|C_i) = \frac{1}{N_i h^d} \sum_{t=1}^{N} K\left(\frac{\mathbf{x}-\mathbf{x}^t}{h}\right) r_i^t \quad \hat{P}(C_i) = \frac{N_i}{N}$$

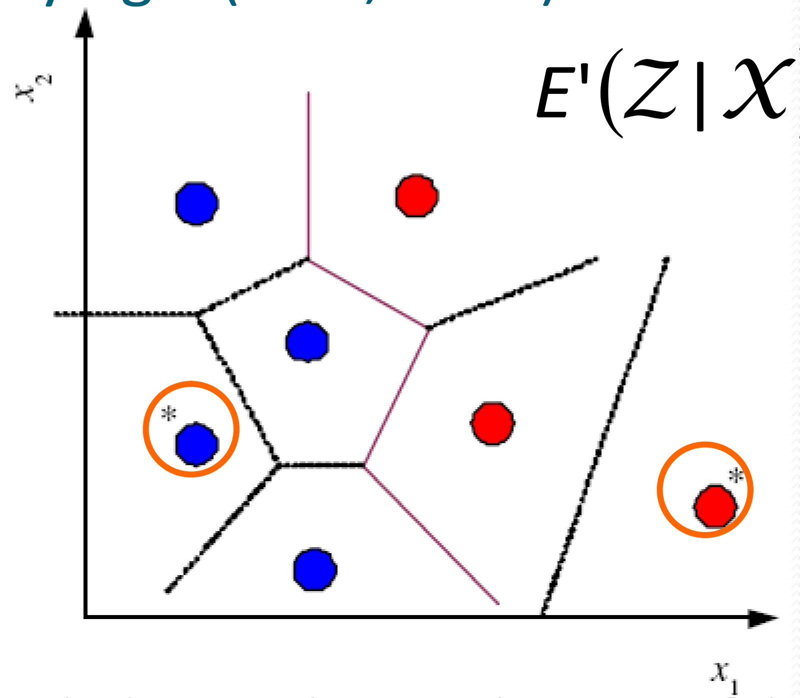
$$g_i(\mathbf{x}) = \hat{p}(\mathbf{x}|C_i)\hat{P}(C_i) = \frac{1}{Nh^d} \sum_{t=1}^{N} K\left(\frac{\mathbf{x}-\mathbf{x}^t}{h}\right) r_i^t$$

- k-NN estimator

$$\hat{p}(\mathbf{x}|C_i) = \frac{k_i}{N_i V^k(\mathbf{x})} \quad \hat{P}(C_i|\mathbf{x}) = \frac{\hat{p}(\mathbf{x}|C_i)\hat{P}(C_i)}{\hat{p}(\mathbf{x})} = \frac{k_i}{k}$$

Condensed Nearest Neighbor

- Time/space complexity of *k*-NN is O (*N*)
- Find a subset Z of X that is small and is accurate in classifying X (Hart, 1968)



$$E'(Z|\mathcal{X}) = E(\mathcal{X}|Z) + \lambda|Z|$$

Condensed Nearest Neighbor

- Incremental algorithm: Add instance if needed

$\mathcal{Z} \leftarrow \emptyset$

Repeat

For all $\boldsymbol{x} \in \mathcal{X}$ (in random order)

Find $\boldsymbol{x}' \in \mathcal{Z}$ s.t. $\|\boldsymbol{x} - \boldsymbol{x}'\| = \min_{\boldsymbol{x}^j \in \mathcal{Z}} \|\boldsymbol{x} - \boldsymbol{x}^j\|$

$\boxed{\text{If class}(\boldsymbol{x}) \neq \text{class}(\boldsymbol{x}') \text{ add } \boldsymbol{x} \text{ to } \mathcal{Z}}$

Until $\mathcal{Z}$ does not change

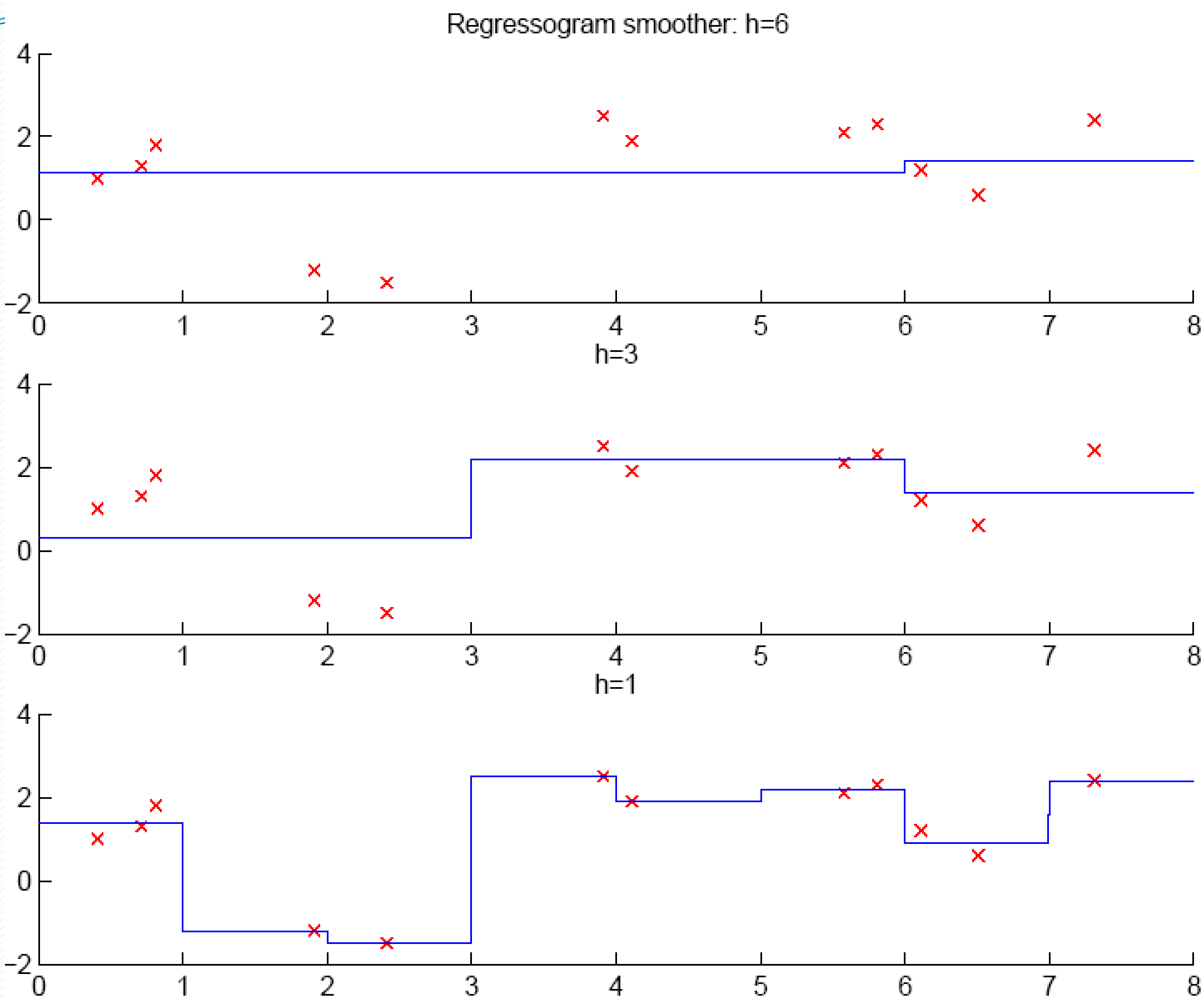
Nonparametric Regression

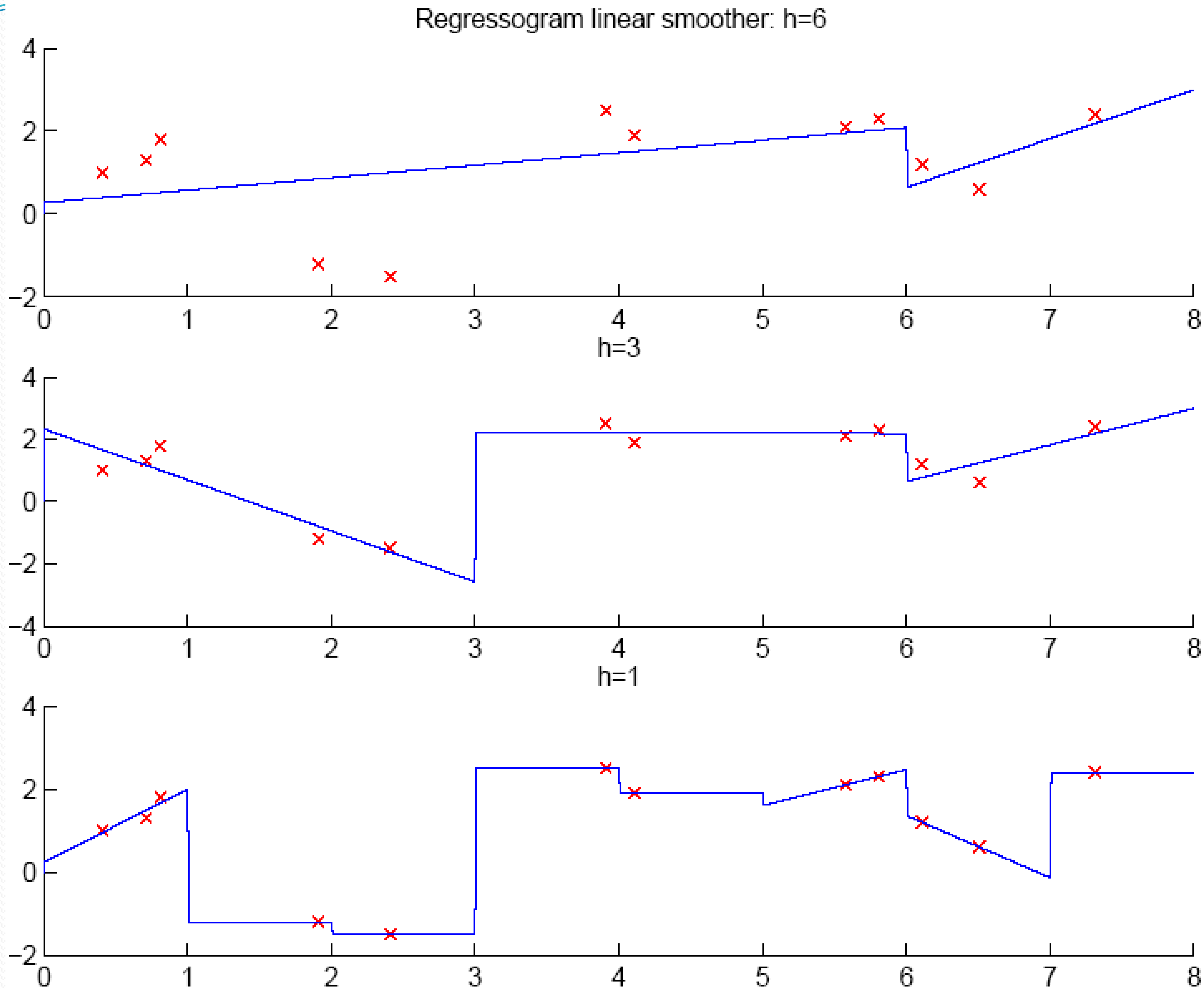
- Aka smoothing models
- Regressogram

$$\hat{g}(x) = \frac{\sum_{t=1}^{N} b(x, x^t) r^t}{\sum_{t=1}^{N} b(x, x^t)}$$

where

$$b(x, x^t) = \begin{cases} 1 & \text{if } x^t \text{ is in the same bin with } x \\ 0 & \text{otherwise} \end{cases}$$


Regressogram smoother: h=6
h=3
h=1


Regressogram linear smoother: h=6
h=3
h=1
4
2
0
−2
−4
0
1
2
3
4
5
6
7
8

Running Mean/Kernel Smoother

- Running mean smoother

$$\hat{g}(x) = \frac{\sum_{t=1}^{N} w\left(\frac{x - x^t}{h}\right) r^t}{\sum_{t=1}^{N} w\left(\frac{x - x^t}{h}\right)}$$

where

$$w(u) = \begin{cases} 1 & \text{if } |u| < 1 \\ 0 & \text{otherwise} \end{cases}$$

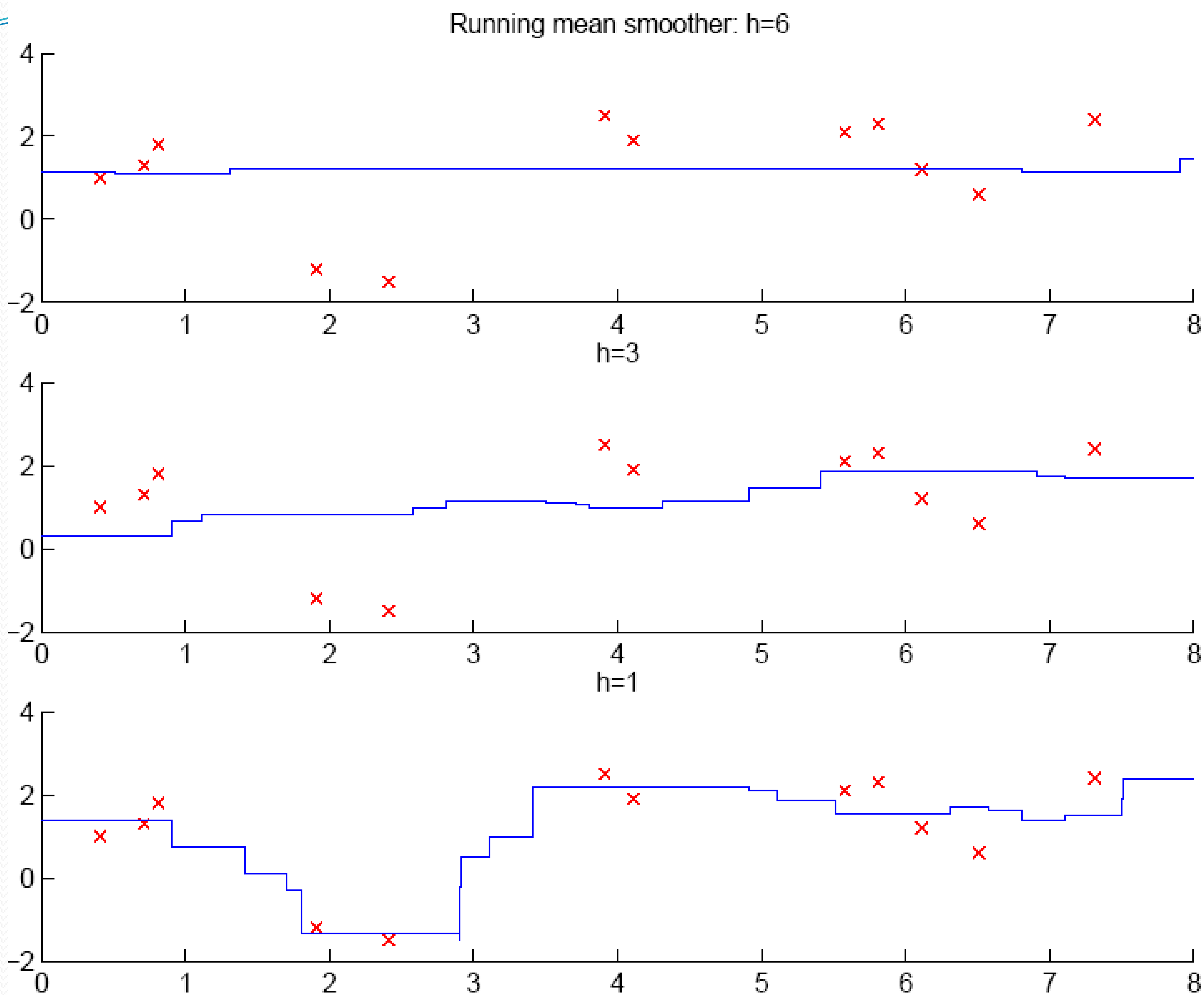
- Running line smoother

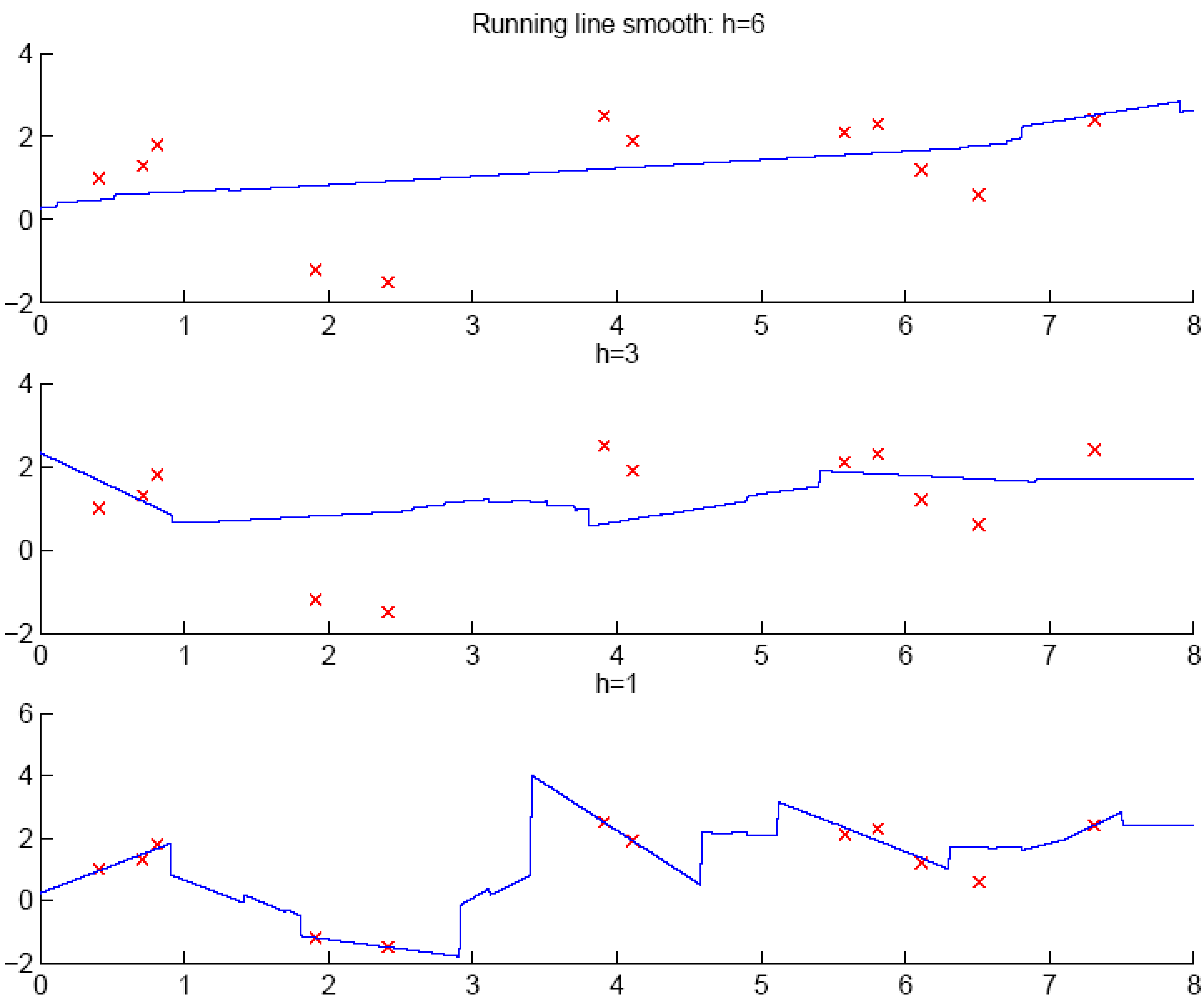
- Kernel smoother

$$\hat{g}(x) = \frac{\sum_{t=1}^{N} K\left(\frac{x - x^t}{h}\right) r^t}{\sum_{t=1}^{N} K\left(\frac{x - x^t}{h}\right)}$$

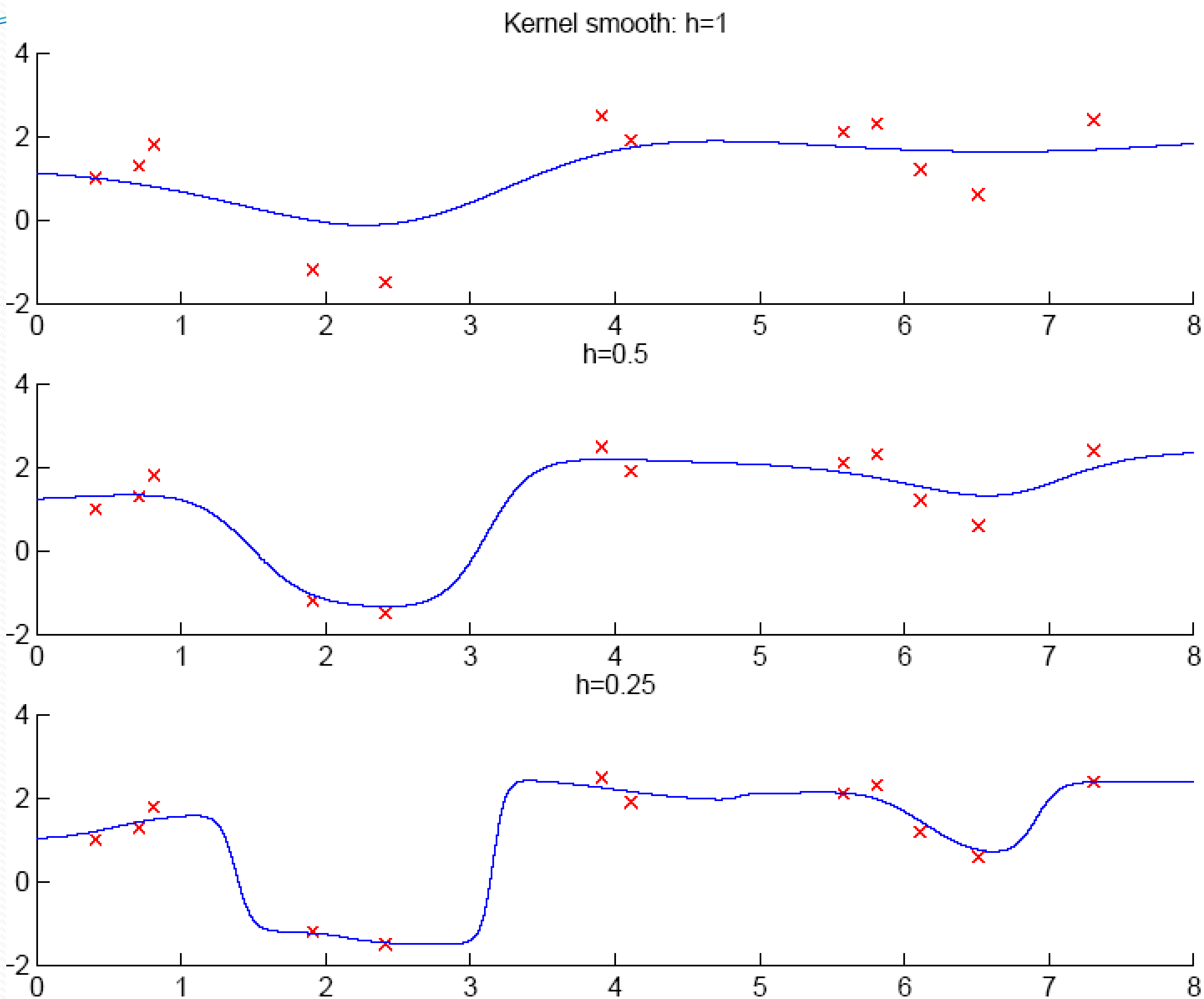
where *K*( ) is Gaussian

- Additive models (Hastie and Tibshirani, 1990)


Running mean smoother: h=6
h=3
h=1
4
2
0
−2
0
1
2
3
4
5
6
7
8

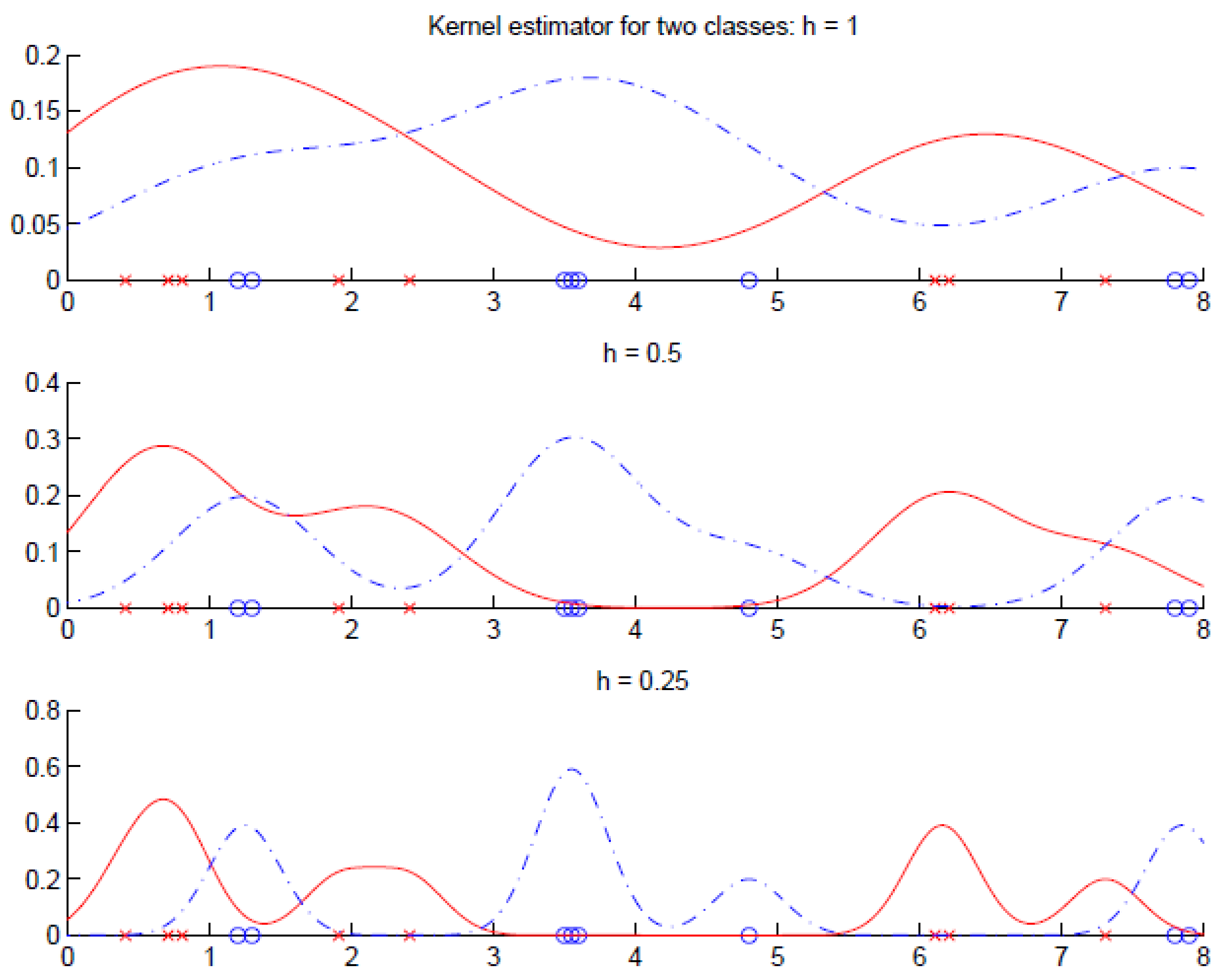


Running line smooth: h=6
h=3
h=1


Kernel smooth: h=1
h=0.5
h=0.25

How to Choose k or h?

- When k or h is small, single instances matter; bias is small, variance is large (undersmoothing): High complexity
- As k or h increases, we average over more instances and variance decreases but bias increases (oversmoothing): Low complexity
- Cross-validation is used to finetune k or h.


Kernel estimator for two classes: h = 1
0.2
0.15
0.1
0.05
0
0
1
2
3
4
5
6
7
8
h = 0.5
0.4
0.3
0.2
0.1
0
h = 0.25
0.8
0.6
0.4
0.2
0