1 Background

Here, we focus on the fundamental essentials of the theory as a first-order approximation to reality.

Suppose we have an asset (e.g. stock) with a price process \( S_t \). We have also a riskless asset (the bank account or the government bond, denoted by \( B \)).

An option is a financial instrument giving one the right but not the obligation to make a specified transaction at (or by) a specified date at a specified price.

- Call options give one the right to buy. Put options give one the right to sell.
  - European options give one the right to buy/sell on the specified date, the expiry date, when the option expires or matures.
  - American options give one the right to buy/sell at any time prior to or at expiry. Asian options depend on the average price over a period.
  - Lookback options depend on the maximum or minimum price over a period
  - Barrier options depend on some price level being attained

The price at which the transaction to buy/sell the underlying, on/by the expiry date (if exercised), is called the exercise price or the strike price \( K \).

The payoff is the value at expiry. If the asset is of price \( S_T \) and the strike price is \( K \), a European call option is worth \( C_T = (S_T - K)^+ \). For example, consider a call option with strike price \( K = 80 \) at time \( T \). If the stock at time \( T \) is 100 USD and we have a right to buy for 80, the option is worth 20 USD. Similarly,

1.1 The Bond, the riskless bank account

Fixed compound interest rate for borrowing and lending \( r \). If you have \( R_0 \) USD now, you will have at \( t \)

\[
R_t = R_0 \exp(rt)
\]

1.2 The PutCall Parity

This relation is independent of the model that is assumed for the stock-price behaviour. It is a model-independent result based on the no-arbitrage assumption.

Take a portfolio

\[
\Pi_t = S_t + P_t + C_t
\]

This portfolio is created as follows: at time \( t \)

- Buy 1 unit of stock at price \( S_t \)
- Buy a put option at strike price \( K \), with expiry time at \( T \)
• Write a call option at strike price $K$, with expiry time at $T$
We can have to outcomes:
• $S_T \geq K$.
  – The put option at strike price $K$ has no value, (as it gives only the right to sell at a cheaper price)
  – The call option we have written gives the buyer to buy at $K$, so we loose $S_T - K$.
  – The stock we have bought is now worth $S_T$

The conditional net price of the portfolio is
$$\Pi_T = S_T - (S_T - K) = K$$

• $S_T < K$.
  – The put option at strike price $K$ has value $K - S_T$, as we have the right to sell at a higher price
  – The call option we have written gives the buyer to buy at a higher price, so we loose nothing.
  – The stock we have bought is now worth $S_T$

The conditional net price of the portfolio is
$$\Pi_T = S_T + K - S_T = K$$

So whatever the outcome, the price of this portfolio at time $T$ will be $K$; it acts like government bond. So its price at time $t$ must be
$$\Pi_t = \exp(-r(T - t))K$$
otherwise it gives an arbitrage opportunity, a possibility to make money at no risk.

Strategies for exploiting the arbitrage opportunity:
• Portfolio is being sold cheaper $\Pi_t < \exp(-r(T - t))K$
  – At $t$: Lend money $\exp(-r(T - t))K$ from the bank, Buy the portfolio
  – At $T$: Pay back $\exp(-r(T - t))K \exp(r(T - t)) = K$ to the bank. Enjoy your $\exp(-r(T - t))K - \Pi_t$.

• Portfolio is being sold overpriced $\Pi_t > \exp(-r(T - t))K$
  – At $t$: Buy the negative of the portfolio:

2 Option Pricing via Monte Carlo

The model for a risk neutral evolution is
$$dS_t = rS_t dt + \sigma S_t dW_t$$
Here, $S_t$ is the price at time $t$. We also assume that there is the riskless bond with compound interest rate $r$. $W_t$ is a realisation from the Brownian process and $\sigma^2$ is the (constant) volatility.

The solution of this SDE is
$$\log S_t = \log S_0 + \left( r - \frac{1}{2} \sigma^2 \right) t + \sigma W_t$$

So at strike time, the price will be
$$S_T = S_0 \exp(\left( r - \frac{1}{2} \sigma^2 \right) T + \sigma W_T)$$
$$W_T \sim N(0; T)$$
The payoff for call options is

\[ C_T = (S_T - K)^+ \]

The payoff for put options is

\[ P_T = (K - S_T)^+ \]

The price of the call option at \( t = 0 \) will be

\[ e^{-rT} \langle C_T \rangle \]