Boğaziçi University, Dept. of Computer Engineering

CMPE 58N, MONTE CARLO METHODS

Spring 2014, Midterm

Name: _____

Student ID: _____

Signature: _____

- Please print your name and student ID number and write your signature to indicate that you accept the University honour code.
- During this examination, you may NOT use any books or laptops. You can use an A4 cheat sheet.
- Read each question carefully and show all your work. Underline your final answer to each question.
- There are $\mathbf{5}$ questions. Point values are given in parentheses.
- You have $\mathbf{180\ minutes}$ to do all the problems.

Q	1	2	3	4	5	Total
Score						
Max	20	20	20	20	20	100

1. (a) *(1 pts)* What is the law of large numbers ?

   (b) *(1 pts)* What is the CLT ?

   (c) *(1 pts)* What does the CLT says about Monte Carlo integration?

   (d) *(1 pts)* Suppose you have generated independent samples from the joint density $p(x, y)$. How would you estimate the expected value $\langle x \rangle$?

   (e) *(1 pts)* Box Muller method for generating Gaussian random variables is an instance of which sampling method ?

   (f) *(1 pts)* What is 'the weight function'?

   (g) *(1 pts)* What is 'the variance of importance weights'?

   (h) *(1 pts)* Is reducing the variance of importance weights a good objective in importance sampling ? Why?

(i) *(1 pts)* What is the 'optimum proposal' in importance sampling?

(j) *(1 pts)* How do you 'postprocess' samples generated by importance sampling $x^{(i)}$ to make them unweighted in the sense that you can use the estimator $\sum_i f(x^{(i)})/N$ to estimate $\langle f \rangle$?

(20 points)

2. Suppose $x_i \sim \mathcal{E}(x_i; \lambda_i) = \lambda_i \exp(-\lambda_i x_i)$ for $i = 1, 2$. Find the densities of

   (a) $s = x_1 + x_2$

   (b) $y = x_1/(x_1 + x_2)$

(20 points)

3. **(Rejection and Importance Sampling)** Consider a symmetric triangular distribution p with support $[-1, 1]$ and a symmetric triangular distribution on $[-\tau, \tau]$ as the proposal q,

   (a) What is the highest efficiency as a function of τ for rejection sampling from p? Draw a figure. (Efficiency is the ratio of the number of samples generated by the numbers that are proposed)

   (b) What is the importance weight expression as a function of τ?

   (c) What is the variance expression for the importance weights as a function of τ ?

(20 points)

4. **Markov Chains**

(a) *(2 pts)* What is meant by 'the stationary (invariant) distribution of a Markov chain'?

(b) *(2 pts)* Under what conditions is the stationary distribution of a MC unique ?

(c) *(2 pts)* What is meant when one says 'the Markov chain converged to the stationary distribution' ?

(d) *(2 pts)* Show that when two Markov transition matrices T_1 and T_2 have the same unique invariant distribution, their mixture $T = \lambda T_1 + (1-\lambda)T_2$ has the same invariant distribution.

(e) *(2 pts)* What is detailed balance?

(f) *(2 pts)* Give an example of a Markov chain on two states, that doesn't satisfy detailed balance.

(g) *(2 pts)* Suppose a chain doesn't satisfy detailed balance. Can it have a unique stationary distribution ?

(h) *(2 pts)* What is a periodic Markov chain?

(i) *(2 pts)* What is the total variation distance of two densities ?

(j) *(2 pts)* How can you estimate numerically the rate of convergence of a Markov chain on a small finite set ?

(20 points)

5. Suppose we have the following procedure, expressed in a Matlab like language. rand() returns a uniform random number on $[0, 1]$

```
x(1) = 1
lp = log([3 1])
for t=2:T,
    xp = (rand()<0.5) + 1;
    la = lp(xp) - lp(x(t-1));
    if log(rand()) < la,
        x(t) = xp;
    else
        x(t) = x(t-1);
    end
end

E = sum(x == 1)/T;
```

   (a) Find the Markov transition matrix $T = p(x_t|x_{t-1})$ that this procedure corresponds to. What is its stationary distribution?

   (b) What is the expected value of E? What is its variance?

(20 points)