Boğaziçi University, Dept. of Computer Engineering

CMPE 58N, MONTE CARLO METHODS

Spring 2012, Midterm

Name: ______________________________

Student ID: ______________________________

Signature: ______________________________

- Please print your name and student ID number and write your signature to indicate that you accept the University honour code.
- During this examination, you may use any notes, books or laptops. You can even lookup resources on the internet; however communication with fellow students is not allowed.
- Read each question carefully and show all your work. Underline your final answer to each question.
- There are **6** questions. Point values are given in parentheses.
- You have **180 minutes** to do all the problems.

Q	1	2	3	4	5	6	Total
Score							
Max	20	20	20	20	20	20	120

1. **(Gamma Distribution)** The gamma density is given by

$$\mathcal{G}(\lambda; a, b) = \exp((a-1)\log\lambda - b\lambda - \log\Gamma(a) + a\log b)$$

   (a) Suppose we have

$$\begin{aligned} p_1(x) &= \mathcal{G}(x; a_1, b_1) \\ p_2(x) &= \mathcal{G}(x; a_2, b_2) \end{aligned}$$

   Express

$$p_1(x)p_2(x)$$

   as a Gamma density times a constant.

   (b) Suppose we have

$$\begin{aligned} p(x_1) &= \mathcal{G}(x_1; \alpha_1, \beta_1) \\ p(x_2) &= \mathcal{G}(x_2; \alpha_2, \beta_2) \end{aligned}$$

   Find the density of z where

$$z = x_1 x_2$$

(20 points)

2. **Circles** Suppose we have only access to unit Gaussian $\mathcal{N}(x; 0, 1)$ random number generator and no uniformly random numbers. Describe methods for sampling

   (a) uniformly on the unit circle, i.e., on $A = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$

   (b) uniformly on a unit circular region, i.e., on $A = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\}$

   (c) uniformly on a unit circular band, i.e., on $A_e = \{(x_1, x_2) : 1 - e \leq x_1^2 + x_2^2 \leq 1 + e\}$ where $0 < e < 1$

   Try to be as efficient as possible. For each method sketch a proof why your method works.

   (20 points)

3. **Simplex** Consider a set $A \subset R^3$ such that $A = \{(x_1, x_2, x_3) : x_i \geq 0 \text{and} \sum_i x_i = 1\}$. Suppose we wish to generate points uniformly random on A and someone proposes the following method.

   - Sample u_i uniformly on $(0, 1]$ for $i = 1 \ldots 3$. Let $s = \sum_i u_i$. Set $x_i = u_i/s$ for $i = 1 \ldots 3$

   Prove that this method works or disprove and show that it does not work.

   (20 points)

4. **IS** Suppose we will use importance sampling with a Gaussian proposal of form $q = \mathcal{N}(x; 0, v)$ for estimating expectations under the density $p = \mathcal{U}(x; 0, 1)$. What is the best v in terms of minimizing the variance of the importance weights?

(20 points)

5. **LLN and CLT** Suppose we are throwing a loaded coin with an unknown probability π of heads. How many trials would be needed to figure out with 50 percent confidence that indeed the coin is biased, i.e. $\pi \neq 1/2$.

(20 points)

6. Suppose you are given a directed acyclic graph (DAG) $G = (V, E)$ where V is the vertex set and E is the edge set. Describe a method to sample uniformly from all topological orderings of G, i.e. permutations of V that conform to G.

   For example: $A \leftarrow B \rightarrow C$ has two topological orders: $\sigma_1 = (B, A, C)$ and $\sigma_2 = (B, C, A)$. Both permutations conform to G. Here your algorithm should output σ_i with probability 0.5. On the other hand $A \rightarrow B \rightarrow C$ has only a single topological order (A, B, C).

   To find a topological order of G, (1) remove a node v that doesn't have any edges pointing to it, (2) remove all the edges from v to the other nodes from the graph, (3) goto one if the graph G has still vertices. The elimination sequence is a possible topological order.

   (20 points)