Boğaziçi University, Dept. of Computer Engineering

CMPE 58N, MONTE CARLO METHODS

Spring 2012, Midterm

Name: ________________________________

Student ID: ___________________________

Signature: _____________________________

• Please print your name and student ID number and write your signature to indicate that you accept the University honour code.

• During this examination, you may use any notes, books or laptops. You can even lookup resources on the internet; however communication with fellow students is not allowed.

• Read each question carefully and show all your work. Underline your final answer to each question.

• There are 6 questions. Point values are given in parentheses.

• You have 180 minutes to do all the problems.

<table>
<thead>
<tr>
<th>Q</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Score</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Max</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>120</td>
</tr>
</tbody>
</table>
1. **(Gamma Distribution)** The gamma density is given by

\[
\mathcal{G}(\lambda; a, b) = \exp((a - 1) \log \lambda - b \lambda - \log \Gamma(a) + a \log b)
\]

(a) Suppose we have

\[
p_1(x) = \mathcal{G}(x; a_1, b_1) \\
p_2(x) = \mathcal{G}(x; a_2, b_2)
\]

Express

\[
p_1(x)p_2(x)
\]

as a Gamma density times a constant.

(b) Suppose we have

\[
p(x_1) = \mathcal{G}(x_1; \alpha_1, \beta_1) \\
p(x_2) = \mathcal{G}(x_2; \alpha_2, \beta_2)
\]

Find the density of \(z\) where

\[
z = x_1x_2
\]

(20 points)
2. **Circles** Suppose we have only access to unit Gaussian $\mathcal{N}(x; 0, 1)$ random number generator and no uniformly random numbers. Describe methods for sampling

(a) uniformly on the unit circle, i.e., on $A = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$

(b) uniformly on a unit circular region, i.e., on $A = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\}$

(c) uniformly on a unit circular band, i.e., on $A_e = \{(x_1, x_2) : 1 - e \leq x_1^2 + x_2^2 \leq 1 + e\}$ where $0 < e < 1$

Try to be as efficient as possible. For each method sketch a proof why your method works.

(20 points)
3. **Simplex** Consider a set $A \subset \mathbb{R}^3$ such that $A = \{(x_1, x_2, x_3) : x_i \geq 0 \text{ and } \sum x_i = 1\}$. Suppose we wish to generate points uniformly random on $A$ and someone proposes the following method.

- Sample $u_i$ uniformly on $(0, 1]$ for $i = 1 \ldots 3$. Let $s = \sum u_i$. Set $x_i = u_i/s$ for $i = 1 \ldots 3$

Prove that this method works or disprove and show that it does not work.

*(20 points)*
4. **IS** Suppose we will use importance sampling with a Gaussian proposal of form $q = \mathcal{N}(x; 0, v)$ for estimating expectations under the density $p = \mathcal{U}(x; 0, 1)$. What is the best $v$ in terms of minimizing the variance of the importance weights?

*(20 points)*
5. **LLN and CLT** Suppose we are throwing a loaded coin with an unknown probability $\pi$ of heads. How many trials would be needed to figure out with 50 percent confidence that indeed the coin is biased, i.e. $\pi \neq 1/2$.

(20 points)
6. Suppose you are given a directed acyclic graph (DAG) \( G = (V, E) \) where \( V \) is the vertex set and \( E \) is the edge set. Describe a method to sample uniformly from all topological orderings of \( G \), i.e. permutations of \( V \) that conform to \( G \).

For example: \( A \leftarrow B \rightarrow C \) has two topological orders: \( \sigma_1 = (B, A, C) \) and \( \sigma_2 = (B, C, A) \). Both permutations conform to \( G \). Here your algorithm should output \( \sigma_1 \) with probability 0.5. On the other hand \( A \rightarrow B \rightarrow C \) has only a single topological order \( (A, B, C) \).

To find a topological order of \( G \), (1) remove a node \( v \) that doesn’t have any edges pointing to it, (2) remove all the edges from \( v \) to the other nodes from the graph, (3) goto one if the graph \( G \) has still vertices. The elimination sequence is a possible topological order.

\( (20\ points) \)