CMPE 548 Monte Carlo methods
Assignment 2

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Abstract

In this assignment, you are expected submit one report (hardcopy) includes all results wanted in the questions.

Exercise 1. (**Bias of Perfect Monte Carlo**)

Definitions: Suppose, we can sample from the target density $\pi(x)$ directly. We sample N iid random variables (samples) $X^{(i)} \sim \pi(x)$ where $i \in \{1, \dots, N\}$. The 'perfect Monte Carlo estimate' of the density π is given by,

$$\pi_{\text{MC}}^N(x) = \frac{1}{N}\sum_{i=1}^{N} \delta_{X^{(i)}}(x)$$

where,

$$\delta_y(x) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

and it is also denoted as $\delta(x - y)$ and called as Dirac delta function[1]. By using the samples $\{X^{(i)}\}_{i=1}^N$ at hand, we would like to compute an expectation of the form,

$$\bar{\varphi} = \mathbb{E}_\pi[\varphi(X)] = \int \varphi(x)\pi(x)\mathrm{d}x$$

for some bounded test function $\varphi(x)$ (For the mean $\varphi(x) = x$). To that end, we'll use samples drawn from π and we denote the empirical perfect MC estimate of this expectation via $\tilde{\varphi}_{\text{MC}}$. This is defined as,

$$\tilde{\varphi}_{\text{MC}} = \mathbb{E}_{\pi^N}\left[\varphi(X)\right]$$

Question: Show that $\tilde{\varphi}_{\text{MC}}$ is unbiased, i.e., show,

$$\mathbb{E}\left[\tilde{\varphi}_{\text{MC}}\right] = \bar{\varphi}$$

[1] The following property of the Dirac function will be useful in this assignment,

$$f(y) = \int f(x)\delta_y(x)dx$$

Exercise 2. (**Bias of normalised importance sampling**)

Definitions: In the course, the IS technique is explained for estimating the expectations of normalised and unnormalised distributions. Here, we assume, we know the target density pointwise, i.e. we do not have to deal with normalisation. Your job is to show that the IS estimate of the mean in this setting is unbiased.

Recall that, given iid random variables (samples) $\{X^{(i)}\}_{i=1}^N$ from a proposal density $q(x)$, you have to calculate weights. Then the IS estimate of the expectation of some test function $\varphi(x)$ wrt π is given by,

$$\tilde{\varphi}_{\text{IS}} = \frac{1}{N} \sum_{i=1}^{N} W(X^{(i)}) \varphi(X^{(i)})$$

Question: Show that $\tilde{\varphi}_{\text{IS}}$ is unbiased, i.e.,

$$\mathbb{E}_q[\tilde{\varphi}_{\text{IS}}] = \bar{\varphi}$$

Exercise 3. (Importance sampling) Consider the target distribution

$$p(x) = \mathcal{N}(x; m, S)$$

when using the proposal

$$q(x) = \mathcal{N}(x; \mu, \Sigma)$$

- Derive the weight function for the target distribution
- Derive the analytic expression for the variance of importance weights