Boğaziçi University, Dept. of Computer Engineering

CMPE 482, NUMERIC LINEAR ALGEBRA AND ITS APPLICATIONS

Spring 2015, Midterm

Name: _____

Student ID: _____

Signature: _____

- Please print your name and student ID number and write your signature to indicate that you accept the University honour code.
- During this examination, you may not use any notes or books.
- Read each question carefully and **WRITE CLEARLY**. Unreadable answers will not get any credit.
- For each question you do not know the answer and leave blank, you can get %10 of the points, if you write only "I don't know the answer but I promise to think about this question and learn its solution".
- There are **5** questions. Point values are given in parentheses.
- You have **120 minutes** to do all the problems.

Q	1	2	3	4	5	Total
Score						
Max	20	20	20	20	20	100

Matlab Functions

```
[Q R] = qr(A)           % A == Q*R;       QR Factorization
[Qh Rh] = qr(A,0)       % A == Qh*Rh;     Reduced QR Factorization
[U S V] = svd(A)        % A == U*S*V'     Singular Value Decomposition
[U S V] = svd(A, 0)     % A == U*S*V'     Reduced Singular Value Decomposition
x = R\b                 % x == inv(R)*b   Solves a linear system where R is triangular
```

1. Suppose on a machine $\mathcal{M}$, each floating point number is represented using 4 bits only, of which 2 bits are the exponent with base 10.

   [Hint: Don't forget sign bits]

   (a) On the real line, mark and label all numbers that can be represented **exactly** by this floating point system.

   (b) What is the machine-epsilon for $\mathcal{M}$? Give the exact number?

(20 points)

2. (Stability) Suppose we have two algorithms, $\tilde{f}_1$ and $\tilde{f}_2$ that give solutions to the same problem f, given input y.

   (a) Suppose $\tilde{f}_1$ is known to be backward stable and $\tilde{f}_2$ is known to be stable. Which algorithm would you prefer, given their computational requirements are the same? Give a short reason. *(10 points)*

   (b) Suppose we come up with an algorithm $\tilde{f}_3$ that has very similar computational cost, but that satisfies

   $$\frac{\|\tilde{f}_3(y) - f(\tilde{y})\|}{\|f(\tilde{y})\|} = O(\varepsilon_{\text{machine}}^{1/3})$$

   for $\tilde{y}$ such that

   $$\frac{\|\tilde{y} - y\|}{\|y\|} = O(\varepsilon_{\text{machine}})$$

   Would you prefer $\tilde{f}_3$ over $\tilde{f}_1$ or $\tilde{f}_2$? Why or why not? Give the number of correct bits you expect to get in each algorithms if you are using single precision IEEE floating point numbers with a 24 bit mantissa. *(10 points)*

3. Suppose you are given the SVD of a matrix A. Describe how you

   (a) estimate the rank of A. *(5 points)*

   (b) Get an orthonormal basis for the range A. *(5 points)*

   (c) Get an orthonormal basis for the null space of A. *(5 points)*

   (d) An orthogonal projector onto the range of A. *(5 points)*

4. (Autoregression) Suppose we are given N noisy measurements. That is, we have $y = [y_1, \ldots, y_N]^\top$. We assume for $i = 3 \ldots N$ that $y_i = a_1 y_{i-1} + a_2 y_{i-2} + \epsilon_i$, where the errors ϵ_i are unknown. Write a Matlab/Octave program that estimates parameters $a = (a_1, a_2)$ that minimizes the norm of the error.

   (a) State the solution in terms of factors of an QR factorization.

   (b) Fill in the rest of the following Matlab program to implement your approach.

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% Assume y is a vector of size N x 1
N = length(y);

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a =
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```

(20 points)

5. (Small perturbations don't reduce rank) Suppose $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k$ are the nonzero singular values of a rank k matrix. Show that, if $\|E\|_2 < \sigma_k$, then we have $\text{rank}(A + E) \geq \text{rank}(A)$. *(20 points)*