## Boğaziçi University, Dept. of Computer Engineering

# CMPE 482, NUMERIC LINEAR ALGEBRA AND ITS APPLICATIONS

### Spring 2014, Midterm 1

Name: ______________________________

Student ID: ______________________________

Signature: ______________________________

- Please print your name and student ID number and write your signature to indicate that you accept the University honour code.
- During this examination, you may not use any notes or books.
- Read each question carefully and **WRITE CLEARLY**. Unreadable answers will not get any credit.
- For each question you do not know the answer and leave blank, you can get %10 of the points, if you write only "I don't know the answer but I promise to think about this question and learn its solution".
- There are **5** questions. Point values are given in parentheses.
- You have **120 minutes** to do all the problems.

| Q | 1 | 2 | 3 | 4 | 5 | Total |
|---|---|---|---|---|---|---|
| Score | | | | | | |
| Max | 20 | 20 | 20 | 20 | 20 | 100 |

Matlab Functions

```
[Q R] = qr(A)          % A == Q*R;       QR Factorization
[Qh Rh] = qr(A,0)      % A == Qh*Rh;     Reduced QR Factorization
[U S V] = svd(A)       % A == U*S*V'     Singular Value Decomposition
[U S V] = svd(A, 0)    % A == U*S*V'     Reduced Singular Value Decomposition
x = R\b                % x == inv(R)*b   Solves a linear system where R is triangular
```

1. (a) Suppose you are given a reduced SVD of a matrix $A$. Describe a method for getting the Moore-Penrose inverse $A^+$. *(10 points)*

   (b) If $E = uv^*$ is an outer product, do we have $\|E\|_F = \|u\|_F \|v\|_F$, where $\|\cdot\|_F$ is the Frobenius norm? Prove or give a counter example. *(10 points)*

2. (Oblique Projector) Suppose you are given a $m \times n_1$ matrix $S_1$ and a $m \times n_2$ matrix $S_2$ such that $n_1 < m, n_2 < m$ but $n_1 + n_2 \geq m$ such that $S = [S_1 \;\; S_2]$ is of full column rank.

   (a) Derive the expression for a (possibly oblique) projector $P$ that will project $v \in \mathbb{R}^m$ onto the range space of $S_1$ along the range space of $S_2$.

   (b) Consider the example

$$S_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad\qquad S_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

   Find $P$ and for all $x$ on the 2-norm ball, carefully draw the image $Px$.

*(20 points)*

3. (a) Given a $N \times N$ projection matrix $P$, prove that $\|P\|_2 = 1$ if and only if $P$ is orthogonal. (Hint: For the if part, use the SVD. For the only if part, consider the space $S_1 = \{v : v = Px; x \in \mathbb{R}^N\}$ and $S_2 = \{w : w = (I-P)x; x \in \mathbb{R}^N\}$. Take $u$ orthogonal to $S_2$. Use contradiction. )

(a) ($P$ is an orthogonal projection $\Rightarrow \|P\|_2 = 1$) We let $\|\cdot\| \equiv \|\cdot\|_2$ If $P$ is an orthogonal projector, we know that $P = P^*$. Let the SVD be

$$\begin{aligned} P &= U\Sigma V^* \\ \|P\| &= \|\Sigma\| \end{aligned}$$

We have

$$\begin{aligned} P &= P^2 = P * P^* = U\Sigma V^* V\Sigma U^* = U\Sigma^2 U^\top \\ \|P\|_2 &= \|U\Sigma^2 U^\top\|_2 = \|\Sigma^2\|_2 = \|\Sigma\|_2^2 \end{aligned}$$

which implies $\|\Sigma\| = \|\Sigma\|^2$, hence $\|\Sigma\| = 1$

(b) ($\|P\|_2 = 1 \Rightarrow P$ is an orthogonal projection ) Now assume that $\|P\|_2 = 1$ but $P$ is not an orthogonal projection. Choose $u \perp S_2$. Construct

$$s = Pu \in S_1$$

By orthogonality of $u$ to $S_2$ we have

$$u^*(I-P)u = 0$$

$$u^*u = u^*Pu$$

$$\begin{aligned} |u^*u| &= |u^*Pu| \\ \|u^*\|^2 &\leq \|u^*\|\|Pu\| \\ \|u^*\| &\leq \|Pu\| \end{aligned}$$

But by definition of a norm and from the assumption $\|P\| = 1$ we have

$$\|Pu\| \leq \|P\|\|u\| \leq \|u\|$$

This implies

$$\|u\| \leq \|Pu\| \leq \|u\|$$

$\|u\| = \|Pu\|$. But
But as $S_1$ and $S_2$ are not orthogonal we have

$$\begin{aligned} 0 &< |u^*P^*(I-P)u| \\ &= |u^*P^*u - u^*P^*Pu| \\ &= |u^*P^*u - \|Pu\|^2| \\ &= |u^*u - \|Pu\|^2| \\ &= |\|u\|^2 - \|Pu\|^2| \\ &= |\|u\|^2 - \|u\|^2| \\ &= 0 \end{aligned}$$

Contradiction.

*(20 points)*

4. (Intersection of Subspaces) Consider two matrices $A \in \mathbb{R}^{m \times n_1}$ and $B \in \mathbb{R}^{m \times n_2}$ where $n_1 < m$ and $n_2 < m$, but $r = n_1 + n_2 - m > 0$. Let $S_A = \text{span}(A)$ and $S_B = \text{span}(B)$, with orthogonal compliments $S_A^\perp = \text{null}(A^*)$ and $S_B^\perp = \text{null}(B^*)$.

   (a) Derive a method that by using three $QR$ decompositions, finds a matrix $Q \in \mathbb{R}^{m \times r}$ where the columns of $Q$ are an orthonormal basis for $S_A \cap S_B$. (Hint: The intersection is orthogonal to the linear subspace spanned by the orthogonal compliments, $(S_A \cap S_B)^\perp = S_A^\perp + S_B^\perp$)

   (b) Fill in the rest of the following Matlab program to implement your approach.

```
m = 5;
n1 = 4;
n2 = 3;
A = randn(m, n1);
B = randn(m, n2);
r = n1 + n2 - m;

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Q = ------------

m = 5;
```

```
n1 = 4;
n2 = 3;
A = randn(m, n1);
B = randn(m, n2);
r = n1 + n2 - m;
[Qa Ra] = qr(A);
[Qb Rb] = qr(B);
C = [ Qa(:,(n1+1):end)  Qb(:,(n2+1):end)];
[Qc Rc] = qr(C);
Q = Qc(:, (m-r)+1:end)
```

*(20 points)*

5. (Basis Regression) Suppose we are given $N$ noisy measurements of an unknown function $y = f(x)$. That is, we have $y = [y_1, \ldots, y_N]^\top$ where $y_i = f(x_i) + \epsilon_i$, where $\epsilon_i$ are unknown. Moreover, suppose we know that the function $f$ has the following algebraic form

$$f(x) \quad = \quad \theta_1 \cos(x) + \theta_2 \sin(x) + \theta_3 \exp(-x) + 1$$

Write a Matlab program that estimates the $\theta$ parameters and finds the projector $P$ that projects $y$ onto $f(x)$.

(a) State the solutions in terms of factors of an SVD.

(b) Fill in the rest of the following Matlab program to implement your approach.

```
% Assume y and x are two vectors of size N x 1
N = length(y);

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*(20 points)*