A New Algorithm for Optimum Multiuser Detection in Synchronous CDMA Systems

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Abstract: Optimum multiuser detection for Code Division Multiple Access (CDMA) systems requires the solution of an NP-hard combinatorial optimization problem. It is well known that the computational complexity of the optimum multiuser detector is exponential with the number of active users in the system. In order to reduce the complexity of the optimum multiuser detection, we propose a Reduced Complexity Maximum Likelihood (RCML) algorithm that includes a set of novel certain boundary rules and characteristics. We investigate the performance and complexity tradeoffs for the RCML algorithm by conducting a set of simulations; Maximum Likelihood (ML) detection as a reference for performance comparisons, and relaxation based Semidefinite Programming (SDPB) algorithm as a reference for complexity comparisons. We show that the RCML algorithm is a promising algorithm for its computational savings over relaxation based algorithms in lightly-to-moderately loaded CDMA systems, and for its optimality in highly loaded CDMA systems.

Keywords: Multiuser detection, CDMA, Branch and bound algorithms, Complexity and performance

1. Introduction

The optimum multiuser detection problem for Code Division Multiple Access (CDMA) systems is an NP-hard problem, i.e., its computational complexity increases exponentially with the number of users [1]. In general, the exact solution of the NP-hard combinatorial optimization problem for CDMA systems can be achieved by an exhaustive search over all feasible solutions. It has been shown that the optimum multiuser detection is possible within polynomial time complexity for certain special conditions [1–4]. For example, if a set of signature sequences where all cross correlations are negative is used, the optimum multiuser detection problem is solvable with a third-order polynomial complexity [2].

The exponential complexity of optimum multiuser detection for CDMA systems has driven researchers to focus on designing suboptimum schemes with an acceptable tradeoff between complexity and performance [5–15]. Suboptimum multiuser detectors can mainly be classified into three categories. The first two categories are linear multiuser detectors and nonlinear or feedback interference suppression detectors [16–19]. The third category, which is the related interest in this research, includes suboptimum algorithms that reduce the complexity of the optimum detector by devising a criterion that allows them to examine a subset of all the possible bit combinations examined by the optimum detector. This category can be further classified into tree-search type algorithms, local search algorithms and global search algorithms. For example, the detectors proposed in [20–23] are based on the breadth-first M- and T-algorithms. In such detectors, either the outputs of the matched filters are used [21, 23] or a whitening filter output is used to improve the performance of the algorithm [20, 22, 23]. Another tree-search type algorithm is sequential decoding for an asynchronous channel [24]. The complexity of this detector is a random variable and it depends on the noise level and user characteristics. For the global minimum search algorithms, the performance and complexity of the algorithms are determined by the number of discrete local minima, which might be exponential in some cases [8, 25–30].

In combinatorial optimization, the optimum multiuser detection problem for systems employing binary signaling is known as the unconstrained bivalent quadratic programming problem and it belongs to a class with many different applications [28, 29, 31]. Most of the approaches are based on the breadth-first tree search algorithms [14, 31–35]. Unlike the tree-search type algorithms proposed in the theory of communications field, which are suboptimum, the search algorithms in combinatorial optimization are exhaustive and strive for the optimum solution [1]. The pruning and bounding in such algorithms are based on some characteristics and boundary conditions derived from the optimum solution. However, these algorithms overlook additional characteristics of CDMA systems for which we may improve the speed of convergence.

This paper proposes a Reduced Complexity Maximum Likelihood (RCML) algorithm that includes a set of new certain boundary rules and characteristics. These are used to reduce the complexity while providing the optimal solution regardless of the constraints for the cross-correlation and received signal levels in CDMA systems. This paper is an extended version of the works given in [36, 37] and introduces the related simulation results for bit-synchronous CDMA systems. The rest of the paper is organized as follows: Section 2 presents the system model and the optimum multiuser detector for bit-synchronous CDMA systems. Section 3 presents the RCML algorithm for optimum detection and its possible extensions for devising a suboptimum detector. Section 4 presents the simulations and discussions. Section 5 concludes the work.
2. Synchronous CDMA system

For a synchronous CDMA system operating in Additive White Gaussian Noise (AWGN) channel, the equivalent low-pass received waveform can be expressed as [1]:

\[ y(t) = \sum_{k=1}^{K} \sqrt{E_k} s_k(t) b_k + n(t) \quad 0 \leq t \leq T \] (1)

where \( K \) is the number of users, \( E_k, s_k(t) \) and \( b_k \in \{-1, 1\} \) represent energy per bit, unit-energy signature waveform and bit value of the \( k \)-th user, respectively; \( T \) is the bit interval and \( n(t) \) is the noise. The receiver consists of a bank of filters matched to the signature waveforms assigned to the users and a multiuser detector. The output of the filter matched to the signature waveform of user \( k \) and sampled at \( T \) is achieved by the following equation.

\[ y_k = \int_{0}^{T} y(t) s_k(t) dt = \sqrt{E_k} b_k + \sum_{i \neq k}^{K} \sqrt{E_i} \rho_{ik} b_i + n_k \] (2)

where,

\[ n_k = \int_{0}^{T} s_k(t) n(t) dt, \quad \text{and} \quad \rho_{ik} = \int_{0}^{T} s_i(t) s_k(t) dt \]

where \( \rho_{ik} \) denotes the cross correlation of the signature waveforms of \( i \) and \( k \) and \( n_k \) denotes the noise at the output of the \( k \)-th matched filter. The matched filter outputs are sufficient statistics for optimal multiuser detection and can be expressed in vector form as follows:

\[ y = [y_1, y_2, \ldots, y_K]^T = R\mathbf{b} + \mathbf{n} \] (3)

where \( R \) is the normalized cross correlation matrix of the signature waveforms, \( R_{ij} = \rho_{ij} \), and,

\[ E = \text{diag} \left[ \sqrt{E_1}, \sqrt{E_2}, \ldots, \sqrt{E_K} \right]_{KxK}, \]

\( \mathbf{n} \) is the noise vector with autocorrelation matrix \( \mathbb{E}[\mathbf{n}\mathbf{n}^T] = \frac{N_0}{2} R \), and \( N_0 \) is the one-sided noise power spectral density of a zero-mean AWGN source.

The Maximum Likelihood (ML) receiver selects the bits \( \mathbf{b} \) that maximize the following metric:

\[ \Omega = 2\mathbf{b}^T E y - \mathbf{b}^T E^T R \mathbf{b} \] (4)

and thus we have Equation (5) as follows:

\[ \hat{\mathbf{b}} = \arg \max_{\mathbf{b} \in \{-1, 1\}^K} 2\mathbf{b}^T E y - \mathbf{b}^T E^T R \mathbf{b} \] (5)

The optimum receiver for the synchronous case consists of \( K \) single-user matched filters followed by a detector that computes the metrics for the \( 2^K \) possible transmitted information bits represented by the vector \( \mathbf{b} \) and selects the vector \( \hat{\mathbf{b}} \) that gives the largest metric value.

Consequently, dynamic programming can be used to find the longest path in a layered directed graph that maximizes the ML metric [1]. Based on the ML metric \( \Omega \), Equation (5) can be reduced to:

\[ \hat{\mathbf{b}} = \arg \max_{\mathbf{b} \in \{-1, 1\}^K} \left( 2 \sum_{i=1}^{K} \sqrt{E_i} y_i b_i - \sum_{i=1}^{K} \sum_{j=i+1}^{K} \sqrt{E_i} \rho_{ij} b_i b_j \right) \] (6)

\[ \hat{\mathbf{b}} = \arg \max_{\mathbf{b} \in \{-1, 1\}^K} \left( K \sum_{i=1}^{K} \sqrt{E_i} y_i b_i - \sum_{i=1}^{K-1} \sum_{j=i+1}^{K} \sqrt{E_i} \rho_{ij} b_i b_j \right) \] (7)

\[ \hat{\mathbf{b}} = \arg \max_{\mathbf{b} \in \{-1, 1\}^K} \left( K \sum_{i=1}^{K} A_i b_i - \sum_{i=1}^{K-1} \sum_{j=i+1}^{K} B_j b_j \right) \] (8)

where \( A_i = \sqrt{E_i} y_i \) and \( B_j = -\sqrt{E_i} \rho_{ij} \forall i \in \{1, 2, \ldots, K\} \) and \( j > i \).

3. Reduced Complexity Maximum Likelihood (RCML) algorithm

In combinatorial optimization, Equation (5) (or Equation (7)) is known as unconstrained bivalent quadratic programming problem and it belongs to a class with many different applications [1]. The characteristics of the matrix \( E^T R E \), such as density (the ratio of the number of nonzero entries to the total number of entries) and sparsity, and the dimension of the problem mainly determine the complexity of the problem and the accuracy of the sub-optimal detectors [5]. When the matrix has low density or the diagonal of the matrix is dominant, indicating low cross-correlation, then finding the optimum solution becomes much easier. In this case, most of the proposed algorithms in the literature provide a significant improvement in performance compared to the conventional detector. Furthermore, it has been shown that careful design of the signature waveforms and thus the cross correlation matrix \( R \), results in significant reduction in the complexity of the optimum detection [1–3]. However, for a large number of users, higher than the processing gain, the design of such signature waveforms becomes extremely harder as indicated by the Welch bound [38].

In what follows we present the new algorithm for the bit-synchronous CDMA systems. Without loss of generality, we consider perfect power control. We want to maximize the following function:

\[ \Omega = \arg \max_{\mathbf{b} \in \{-1, 1\}^K} \left( \sum_{i=1}^{K} y_i b_i - \sum_{i=1}^{K} \sum_{j=1}^{K} \rho_{ij} b_i b_j \right) \] (9)
for a $K$-user system.

The above function can be expressed in terms of boundaries that define the decision regions for each possible vector: $b_i$; is the optimum solution if and only if,

$$
\Omega_m \geq \Omega_m' \quad \forall m = 1, 2, 3 \ldots K, m \neq n
$$

(10)

Table 1 provides an example for the above equations for $K = 3$ users for which we have the following ML metric:

$$
\Omega = y_1b_1 + y_2b_2 + y_3b_3 - \rho_{12}b_1b_2 - \rho_{13}b_1b_3 - \rho_{23}b_2b_3
$$

We assume that $b_m = [\ldots]$, an example vector of $b$, and generalize a bound in the following equation:

$$
y_i b_i \geq \sum_{j=1}^{K} \rho_{ij} b_i b_j
$$

(11)

For vectors that differ in two bits values a bound will be:

$$
y_i b_i + y_j b_j \geq \sum_{k=1}^{K} \rho_{ik} b_i b_k + \sum_{k=1}^{K} \rho_{jk} b_j b_k
$$

(12)

Similarly, comparing vectors differing in three and more bits will result in:

$$
y_i b_i + y_j b_j + y_k b_k \geq \sum_{m=1}^{K} \rho_{im} b_i b_m + \sum_{m=1}^{K} \rho_{jm} b_j b_m + \sum_{m=1}^{K} \rho_{km} b_k b_m
$$

(13)

and, finally we have the case where all the bits differ:

$$
\sum_{i=1}^{K} y_i b_i \geq 0
$$

(14)

In what follows, some of these boundaries are used to determine the branching and pruning criteria to be utilized in RCML algorithm.

$$
\text{If } y_i \geq \sum_{j=1}^{K} \rho_{ij} \text{, then } b_i = \text{sgn}(y_i)
$$

(15)

**Proof** (by contradiction): Assume Equation (15) does hold, and $b_i = -\text{sgn}(y_i)$. In this case,

If $y_i > 0$, then $b_i = -\text{sgn}(y_i) = -1$, thus, we have $-y_i > -\sum_{j=1}^{K} \rho_{ij}$, which contradicts with the assumption.

If $y_i < 0$, then $b_i = -\text{sgn}(y_i) = 1$, thus, we have $-y_i > \sum_{j=1}^{K} \rho_{ij}$, which contradicts with the assumption.

It is worth noting that the right hand-side of Equation (11) is upper-bounded by

$$
\sum_{j=1}^{K} \rho_{ij} b_i b_j \leq \sum_{j=1}^{K} |\rho_{ij}|
$$

(16)

**Rule 2.** Consider Equation (14).

If $|y_i| \geq \sum_{j=1}^{K} |y_j|$, then $b_i = \text{sgn}(y_i)$

(17)

else if $\sum_{i=1}^{M} |y_i| \geq \sum_{j=1}^{K} |y_j|$, prune $b_i = -\text{sgn}(y_i)$

(18)

**Proof of Equation (17)** (by contradiction): We assume Equation (5) is proven [1].

Then, Equation (17) is upper-bounded by $y_i b_i \geq -\sum_{j=1}^{K} y_j b_j = \sum_{j=1}^{K} |y_j|$. In this case,

If $y_i > 0$ then, for $b_i = -\text{sgn}(y_i) = 1$, we have $-y_i > -\sum_{j=1}^{K} |y_j|$, which is not true.

If $y_i < 0$ then, for $b_i = -\text{sgn}(y_i) = -1$, we have $-y_i > \sum_{j=1}^{K} |y_j|$, which is not true.

Hence, Equation (17) holds only for $b_i = \text{sgn}(y_i)$.

**Proof of Equation (18).** We assume Equation (10) is proven [1]. Assume $M = 2$ in Equation (18). Then we have $2^M = 4$ combinations of $b$ vectors:

$$
b_{m1} = [\ldots]_{1 \times K}, \quad b_{m2} = [\ldots]_{1 \times K},
$$

$$
b_{m3} = [\ldots]_{1 \times K}, \quad b_{m4} = [\ldots]_{1 \times K},
$$
According to the Equation (14) \( b_i = -\text{sgn}(y_i) \), \( i = 1, 2 \), \( b_m = [-\ldots -1]_K \). Can not be the optimal solution, and it can be pruned. For general \( M \), \( b_i = -\text{sgn}(y_i) \), \( i = 1, 2, \ldots, M \) can be pruned, thus reducing the complexity by \( 2^{K-M} \). Hence, Rule 2 holds.

**Rule 3.** If user \( i \) is optimally detected, then the problem can be reduced to \( K-1 \) users by subtracting the interference of user \( i \). Then Rule 1 and Rule 2 can be applied to the \( K-1 \) system. In the RCML algorithm, this rule is applied iteratively.

In optimum detection all the branches are examined and the branch resulting in the maximum metric value gives the optimum solution. In this paper, we refer to this algorithm the Maximum Likelihood (ML) detection using an exhaustive search, or shortly, the ML algorithm. In what follows we show how the rules derived in this paper can be employed to prune the search tree and thus reduce the complexity of optimal detection. This is why we call this algorithm the Reduced Complexity Maximum Likelihood (RCML) Algorithm.

**RCML Algorithm**

1. Start with sorting the observations \( y_i \), \( i = 1, \ldots, K \) in descending order. Without loss of generality, we assume that \( |y_1| \geq |y_2| \geq \ldots \geq |y_K| \).
2. Set \( i = 1, j = 1 \), and \( b_0^{(0)} = [000 \ldots 0] \).
3. Apply Rule 1 and Rule 2 for user \( i \).
   - If one of the conditions in these rules apply, then \( b_i = \text{sgn}(y_i) \)
   - else \( j = j + 2 \), and
     \[
     b_{i+1}^{(i+1)} = \begin{bmatrix}
     b_1^{(i+1)} \\
     b_2^{(i+1)}
     \end{bmatrix} = \begin{bmatrix}
     +1 & 0 & \ldots & 0 \\
     -1 & 0 & \ldots & 0
     \end{bmatrix}
     \]
4. \( i = i + 1 \).
5. \( k = 1, p = 1 \).
6. Subtract interference of users detected in \( b_k^{(i)} \).
7. Apply Rule 1 and Rule 2 for user \( i \) for the \((K-i+1)\)-user system.
   - If one of the conditions in these rules apply then
     \( b_i = \text{sgn}(y_i) \), and
     \( b_{k+1}^{(i)} = \text{sgn}(y_i) \)
   - else \( p = \max(p, j) + 1 \)
      \[
      b_{i+1}^{(i+1)} = \begin{bmatrix}
      b_1^{(i)} \\
      \vdots \\
      b_{p-1}^{(i)} \\
      b_k^{(i)}
      \end{bmatrix}
      \]
    \( b_{k+1}^{(i+1)} = 1 \)
    \( b_{p,i}^{(i+1)} = -1 \)
8. If \( k < j \) then \( k = k + 1 \) go to Step 6.
9. If \( i < k \) then \( j = p \) and go to Step 4.
10. The optimum solution is the bit vector \( b_p^{(0)} \) that results in the maximum value of Equation (9).

We should note that the RCML algorithm provides the optimal solution and thus the performance measure will be the same as the performance achieved by the ML algorithm. There may be cases that the above rules may not be applicable, yet the upper and the lower bound complexities are as follows: The complexity of the RCML algorithm is lower bounded by \( K \log (K) \). Assume that the entire received signal follows the Rule 1 or Rule 2 and Equation (17) holds for user 1. Then by subtracting the interference by using Rule 3, we have \( K-1 \) users where we can again apply Rule 1 or Rule 2. It is worth noting that having the above rules in an orderly fashion would not be realistic in a practical multuser system. On the other hand, the complexity of the algorithm is upper bounded by \( 2^{K-M} \). The upper bound will be reached when the rules cannot be applied throughout the search tree. The RCML algorithm explores all the possible solutions which are stored in the matrix \( b_{K}^{(p)} \) in every stage of the algorithm. As a suboptimal approach the number of bit vectors stored in every stage can be limited to certain size, \( S \), at the expense of some degradation in performance. In this paper, we consider the optimal solution yet we provide brief explanation of Semidefinite Programming algorithms that will be used as reference for comparison purposes.

### 3.1 Background in semidefinite programming

In [4], the authors introduced a semidefinite relaxation of the optimal detection problem as a complexity-limiting alternative, and showed that optimum detection can be achieved with polynomial complexity in the number of users. Cutting planes were added to strengthen this relaxation, and an interior-point method with polynomial complexity was used to solve the semidefinite programs arising from the relaxation. We consider this approach as the basic semidefinite programming (SDPB) which is presented in detail in [4]. The authors show that there is no appreciable performance difference between the semidefinite relaxation and optimal detection achieved by the ML algorithm. For a lightly loaded case, the basic semidefinite programming is sufficient to approximate the optimum detection. When the load increases in CDMA systems, additional cutting planes are required.

### 4. Simulations

In order to show the performance of the proposed RCML algorithm, we conducted an extensive set of simulations that are used to evaluate and compare its performance and complexity tradeoffs. In all of the following simulations, we consider a bit-synchronous CDMA system with binary random signature waveform of length 31. The obtained
results can be viewed as an upper bound for a system with a good set of signature waveforms. In all the simulations, we assumed perfect power control, i.e. all users have the same energies $E_1 = E_2 = \ldots = E_K$. The computational complexity is measured as the average CPU time (seconds) to achieve the optimum performance that is achieved by the ML algorithm. The CPU time is obtained in the MATLAB environment on a 600 Mhz Pentium III personal computer with 512MB of RAM. This environment is intentionally chosen to compare our results with the results given in [4].

In simulations, we evaluate the average computational times for the ML, SDPB and RCML algorithms. Figure 1 depicts the computational time (complexity) for these algorithms. As expected, the computational complexity for the ML algorithm increases exponentially with the number of users. The computational complexities for the RCML and SDPB algorithms are significantly lower than that of the ML algorithm. The complexity performances for the ML and SDPB algorithms, which are approximated using the curve-fitting technique [4], are as follows:

$$ C_{ML} \approx 40 \times 10^{-3} 2^K $$  \hspace{1cm} (19)

$$ C_{SDPB} \approx 1.4 \times 10^{-3} K^3 $$  \hspace{1cm} (20)

For lightly loaded CDMA systems ($K < 8$), the computational complexity of the ML algorithm is significantly smaller than that of the SDPB algorithm while it is somewhat closer to that of RCML algorithm. We observe that at $K = 13$ (not $K = 11$ as reported in [4]), both the ML and SDPB algorithms have the same computational complexity. For moderately loaded CDMA systems ($9 < K < 16$), the RCML algorithm has smaller complexity than that of both the ML and SDPB algorithms. If we were to decide on the performance and complexity tradeoffs for lightly loaded CDMA systems, one would achieve the optimal solution by using proper m-sequences with its third-order polynomial complexity [2]. In general, it is the highly loaded CDMA systems ($K > 16$) at which we appreciate the importance of branch and bound algorithms; both the SDPB and RCML algorithms require significantly smaller computational time than that of the ML algorithm.

Figures 2 and 3 depict the average bit error rate (BER) performances of the algorithms for moderately loaded and highly loaded CDMA systems, respectively. For smaller number of users ($K < 12$), the RCML algorithm significantly outperforms the SDPB algorithm in terms of computational complexity yet they both reach optimum detection. On the other hand, for highly loaded CDMA systems, the SDPB algorithm outperforms the RCML algorithm in terms of computational complexity at the expense of some performance degradation, for example, at least 1 dB performance degradation for

![Fig. 1. Average CPU time (sec) versus number of active users.](image1)

![Fig. 2. Average BER versus $E_b/N_0$ (dB) for lightly loaded CDMA system.](image2)

![Fig. 3. Average BER versus $E_b/N_0$ (dB) for highly loaded CDMA system.](image3)
K = 24 users. The RCML algorithm approaches the optimal solution at the expense of higher computational complexity than that of the SDPB algorithm yet significantly smaller computational complexity than the ML algorithm.

5. Conclusions

In this paper, we have proposed a new branch and bound based Reduced Complexity Maximum Likelihood (RCML) algorithm for bit-synchronous CDMA systems. We provide certain rules for the optimum multiuser detection with lowered complexity. In order to show the performance and complexity tradeoffs of the RCML algorithm, we conducted a series of simulations including Maximum Likelihood (ML) and Basic Semidefinite Programming (SDPB) algorithms. We show that the computational complexity for both the RCML and SDPB algorithms are significantly lower than that of the ML algorithm for moderately-to-highly loaded CDMA systems. We also show that the RCML algorithm outperforms the SDPB algorithm in terms of computational complexity for lightly loaded CDMA systems yet both reach the optimum multiuser detection. Therefore, the RCML algorithm is a promising algorithm for its computational savings over relaxation based algorithms in lightly-to-moderately loaded CDMA systems, and for its optimality in highly loaded CDMA systems. Furthermore, we have pointed out a way to further reduce the complexity of the RCML algorithm at the expense of degradation in performance.

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References


[34] Pardalos, P. M.; Rodgers, G. P.: Computational aspects of a branch and bound algorithm for quadratic zero-one programming. Computing 45 (1990), 131–144.


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