Name:
Burak Tepedelen

CMPE 598

Lecture 10

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SHOR'S ALGORITHM

Given a positive integer, find its factors. There exists a fast classical algorithm for detecting whether the number is prime. If so, problem solved. There exists a fast classical algorithm for detecting whether the number is a power (ie. of the form $a^b forb > 1$)
$x = a^b$
$\log x = b * \log a$ so b cannot be bigger than $\log x$
Binary search: : If you can find a factor, you can find all other factors as well using the same method repeatedly.
Assume you are given a number pq where p and q are primes.
We want to factor a large integer N.

- FACTORING is reduced to finding a non-trivial square root of 1 modulo $N \Rightarrow y^2 \equiv (mod 15) y \in (1, 2, 3, \ldots, N-1)$
  if N=15
  trivial square roots:
  $1^2 = 1(mod 15)$ $(-1)^2 = 1(mod 15)$
  $14^2 = 1(mod 15)$
  example non-trivial root:
  $4^2 = 1(mod 15)$
  if y is a non-trivial square root of 1 mod N, then N divides $y^2 - 1 \equiv (y+1)(y-1)$, but N does not divide neither (y-1) not (y+1) so this

means that $\gcd(y-1, N) > 1$ because if y-1 and N were relatively prime,then since N divides $(y-1)(y+1)$, it would have to divide $(y+1)$.

- Finding such a root is reduced to computing the order of a random integer modulo N.
  Pick a random number $X(modN)$, order of $X(modN)$ is the smallest number r such that $x^r \equiv 1(modN)$
  Note: Until this step everything can be solved by classical machines in a fast manner

- The order of an integer is precisely the period of a particular periodic superposition

2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8
2	4	8	1	2	4	8	1

  Classical FFT's input is an M-dimensional complex valued vector alpha (Where M is a power of 2) and output is a an M-dimensional complex valued vector beta.

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{M-1} \end{bmatrix} = \frac{1}{\sqrt{M}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & w & w^2 & \dots & w^{m-1} \\ 1 & w^2 & w^4 & \dots & w^{2(m-1)} \\ 1 & w^3 & w^6 & \dots & w^{3(m-1)} \\ 1 & w^4 & w^8 & \dots & w^{4(m-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{m-1} & w^{2(m-1)} & \dots & w^{(m-1)(m-1)} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{M-1} \end{bmatrix}$$

  Where W is a complex Mth root of unity.
  FFT runs in $O(M \log M)$ steps.

- And, periods of superposition can be found by the quantum FFT.
  Since whole β cannot be accessed because of the quantum reality this is called "Quantum Fourier Sampling".
  Input: A superposition of $M = \log M$ qubits $|\alpha >= \sum_{i=0}^{M-1} \alpha_i |j>$.
  Method: We'll use $O(m^2)$ quantum operations to obtain superposition $|\beta >= \sum_{i=0}^{M-1} \beta_i |j>$.
  Output: A random m-bit number j (ie. $0 \leq j \leq M-1$), from the probability distribution $P_i[j] = |B_j|^2$.

Suppose the input to quantum Fourier sampling is periodic with the period k, for some k that divides M. Then the output will be a multiple of M/k , and it is equally likely to be any of the k multiples of M/k.

$|\alpha >= \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{M-1} \end{bmatrix}$ is such that $\alpha_j = \alpha_i$ whenever $i = j(modk)$ where k is a particular integer that divides M. So there areM/k repetitions of same sequence $(\alpha_0, \alpha_1, \ldots, \alpha_{M-1})$ of length k. AND SUPPOSE EXACTLY ONE OF THE K NUMBERS IS NONZERO say α_j.

Suppose the vector $(\alpha_0, \alpha_1, \ldots, \alpha_{M-1})^T$ is periodic with period k with no offset. Thus $|\alpha >= \sum_{j=0}^{M/k-1} \sqrt{\frac{k}{M}} |jk >$(non-zero terms are $\alpha_0, \alpha_1, \ldots$)

Claim: $|\beta >= \frac{1}{\sqrt{k}} \sum_{j=0}^{k-1} |\frac{jM}{k}$

In the input vector, the coefficient of α_l is $\frac{k}{M}$ if k divides l,and zero otherwise. The j_{th} coefficient of $|\beta >$ is $|\beta >= \frac{1}{\sqrt{M}} \sum_{l=0}^{M-1} w^{jl} \alpha_l = \frac{\sqrt{k}}{M} \sum_{i=0}^{Wk-1} w^{jik}$.

So this sum is the geometric series $1 + w^{jk} + w^{2jk} + \ldots$ containing $\frac{M}{k}$ terms and with ratio w^{jk}. There are two cases. If the ratio is exactly 1, which happens if $jk = 0(modM)$, then the sum of the series is just the number of terms.

If the ratio isn't 1, apply the usual formula for geometric series to find that sum.

So $\beta_j = \frac{1}{\sqrt{k}}$ is M divides jk, and is zero otherwise.

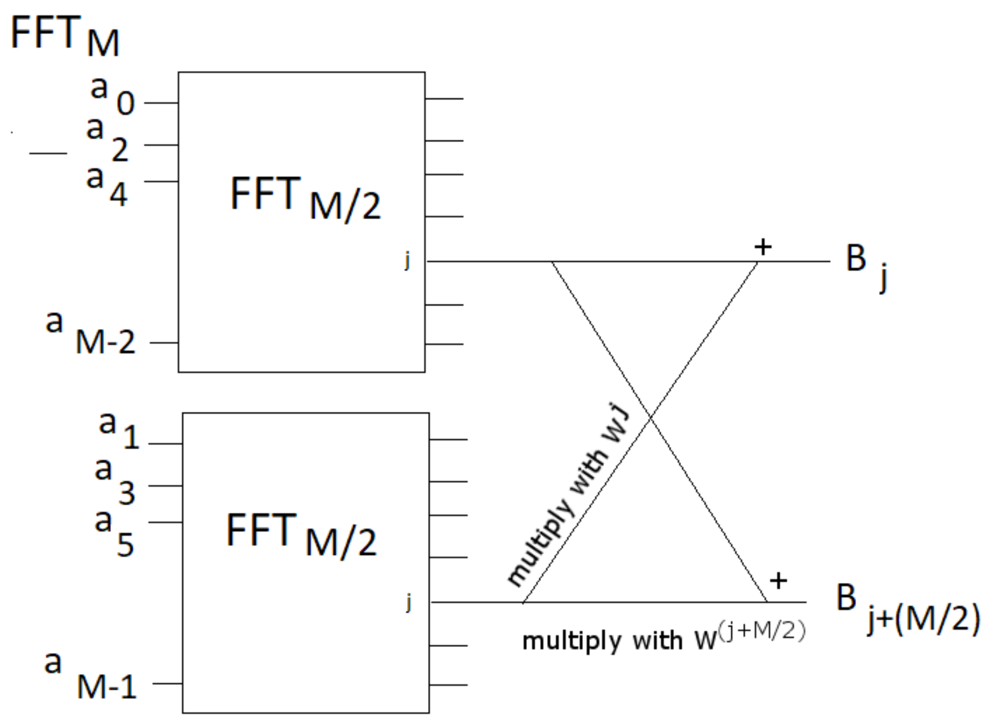
Also work (with little mod function) for the case where the offset isn't zero.

Suppose s independent samples are drawn uniformly from $0, \frac{M}{k}, \frac{2M}{k}, \frac{3M}{k}, \ldots, \frac{(k-1)M}{k}$. Then with probability at least $1 - \frac{k}{2^s}$, the greatest common divisor of these samples is M/k.

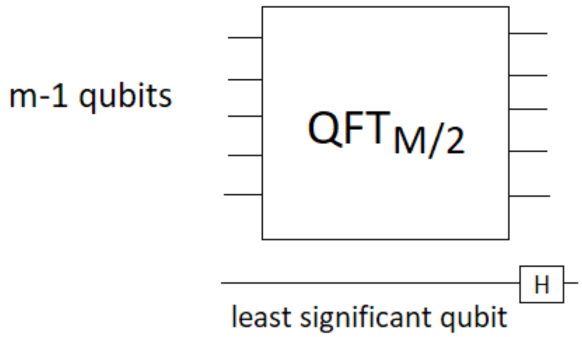
Proof: The only way this can fail is if all the samples are multiples of j. M/k, for some $j > 1$. So fix any integer $j \geq 2$.

The chance that particular sample is a multiple of $\frac{jM}{k}$ is at most $\frac{1}{j} \leq \frac{1}{2}$,

so the chance that all samples are multiples of $\frac{jM}{k}$ is at most $\frac{1}{2^s}$. The probability that this bad thing will happen for some $j \leq k$ is at most $k\frac{1}{2^s}$, since there are k candidates for number j.
How does the classical FFT work?
Divide and Conquer
From input $(\alpha_0, \alpha_1, \ldots, \alpha_{M-1})^T$ to output $(\beta_0, \beta_1, \ldots, \beta_{M-1})^T$.



In the quantum version, the input is now encoded in the 2^M amplitudes of $m = logM$ qubits. So the decomposition of the inputs to evens and odds is determined by the least significant qubit.
We will design a quantum circuit (subroutine) $QFT_M.QFT_{M/2}$ will applied to the remaining $m - 1$ qubits.
Sidenote:
For the 6 qubit system:



$$I(32) \bigotimes H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & \dots & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \dots & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$011000 \rightarrow even, 011001 \rightarrow odd$

$$\text{A} \begin{bmatrix} \alpha_y 0 \\ \alpha_y 1 \end{bmatrix} = \begin{bmatrix} \frac{\alpha_y 0 + \alpha_y 1}{\sqrt{2}} \\ \frac{\alpha_y 0 - \alpha_y 1}{\sqrt{2}} \end{bmatrix}$$

For each j, an operation is done in the classical FFT on the $(\frac{m}{2} + j)th$ wire. If j is represented by the $m-1$ bits $j_1, j_2, j_3, j_4, \dots j_{m-1}$ then $w^j = \prod_{l=1}^{m-1} w^{2jl}$.

$$\text{Gate with 2 qubits} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & W^2 \end{bmatrix}$$