The nondeterministic time hierarchy theorem

(Adapted from <http://blog.computationalcomplexity.org/2011/04/new-proof-of-nondeterministic-time.html>)

THEOREM. For any two time constructible functions t1, t2: N →N, where t1(n) is o(t2(n)/log t2(n)), NTIME(t1(n)) is properly contained in NTIME(t2(n)).

PROOF. The following NTM decides a language in NTIME(t2(n)) but not in NTIME(t1(n)).

N=”On input w= <M>01m0y, where y ∈ {0,1}\*, (reject if w is not in this form)

1. Let n be the length of w.
2. Compute t2(n) using time constructibility and store ⎡ t2(n)/log t2(n)⎤ in binary.
3. If |y|< t1(|w|-|y|), then

Simulate <M> on w0 for ⎡ t2(n)/log t2(n) ⎤ steps.

Simulate <M> on w1 for ⎡ t2(n)/log t2(n) ⎤ steps.

Accept if both simulations accept, reject otherwise.

1. If |y|= t1(|w|-|y|), then

Simulate <M> on <M>01m0 for ⎡ t2(n)/log t2(n) ⎤ steps on the computation path specified by y

Reject if this simulation accepts, accept otherwise.

1. Reject.”

L(N) is clearly in NTIME(t2(n)). (t1 is also computed using time constructibility in the program.)

Assume that L(N) is in in NTIME(t1(n)). Then there exists a NTM M which runs in that time and recognizes L(N). Does M accept <M>01m0 (for m sufficiently large so that the simulations mentioned in N’s code have time to complete)?

M accepts <M>01m0

if and only if

M accepts <M>01m0s for all 1-symbol strings s

if and only if

M accepts <M>01m0s for all 2-symbol strings s

…

if and only if

M accepts <M>01m0s for all t1(|<M>01m0|)-symbol strings s

if and only if

M rejects <M>01m0

Contradiction.