

Signal Processing First



Lecture 7

Fourier Series & Spectrum

READING ASSIGNMENTS



- This Lecture:
 - **Fourier Series in Ch 3, Sects 3-4, 3-5 & 3-6**
 - Replaces pp. 62-66 in Ch 3 in DSP First
 - Notation: \mathbf{a}_k for Fourier Series
- Other Reading:
 - Next Lecture: Sampling

LECTURE OBJECTIVES

- **ANALYSIS** via Fourier Series

- For **PERIODIC** signals: $x(t+T_0) = x(t)$

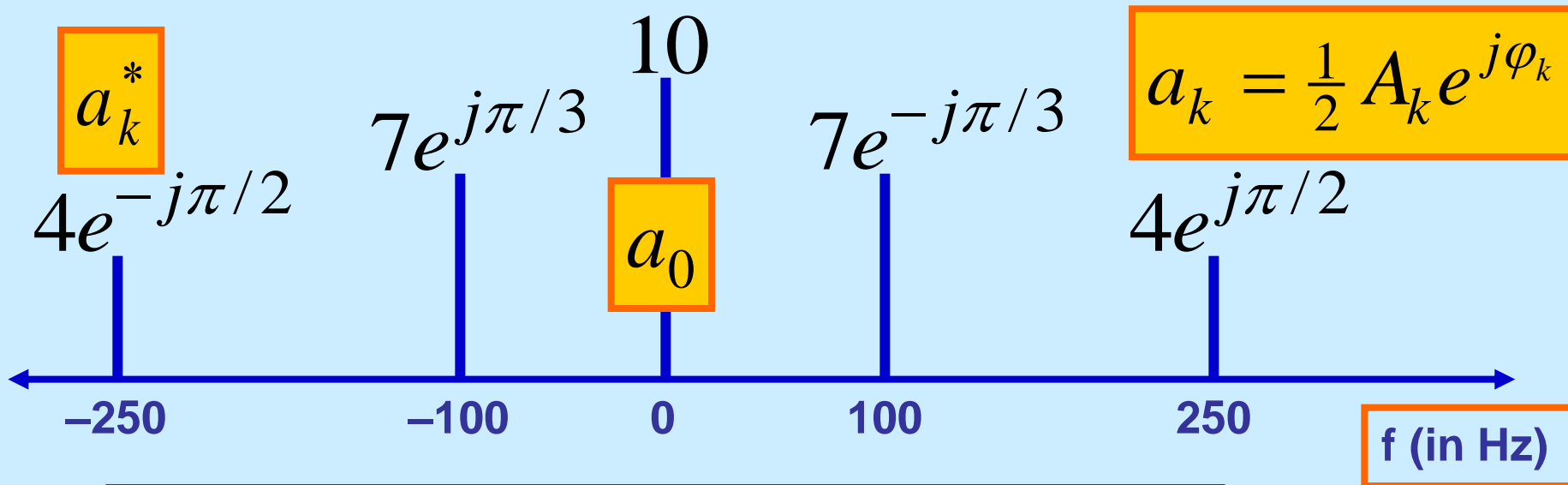
$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi k / T_0)t} dt$$

- **SPECTRUM** from Fourier Series

- a_k is Complex Amplitude for k-th Harmonic

SPECTRUM DIAGRAM

- Recall Complex Amplitude vs. Freq



$$x(t) = a_0 + \sum_{k=1}^N \left\{ a_k e^{j2\pi f_k t} + a_k^* e^{-j2\pi f_k t} \right\}$$

Harmonic Signal

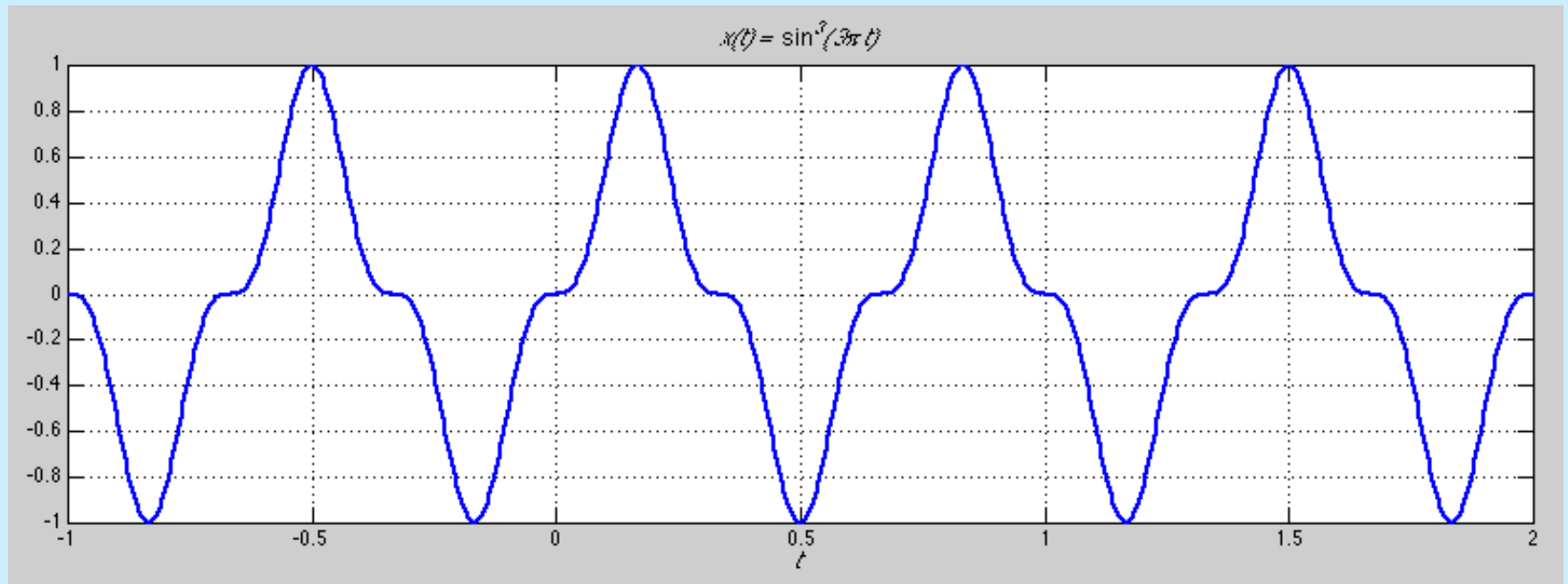
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$2\pi(f_0) = \omega_0 = \frac{2\pi}{T_0} \quad \text{or} \quad T_0 = \frac{1}{f_0}$$

Example

$$x(t) = \sin^3(3\pi t)$$



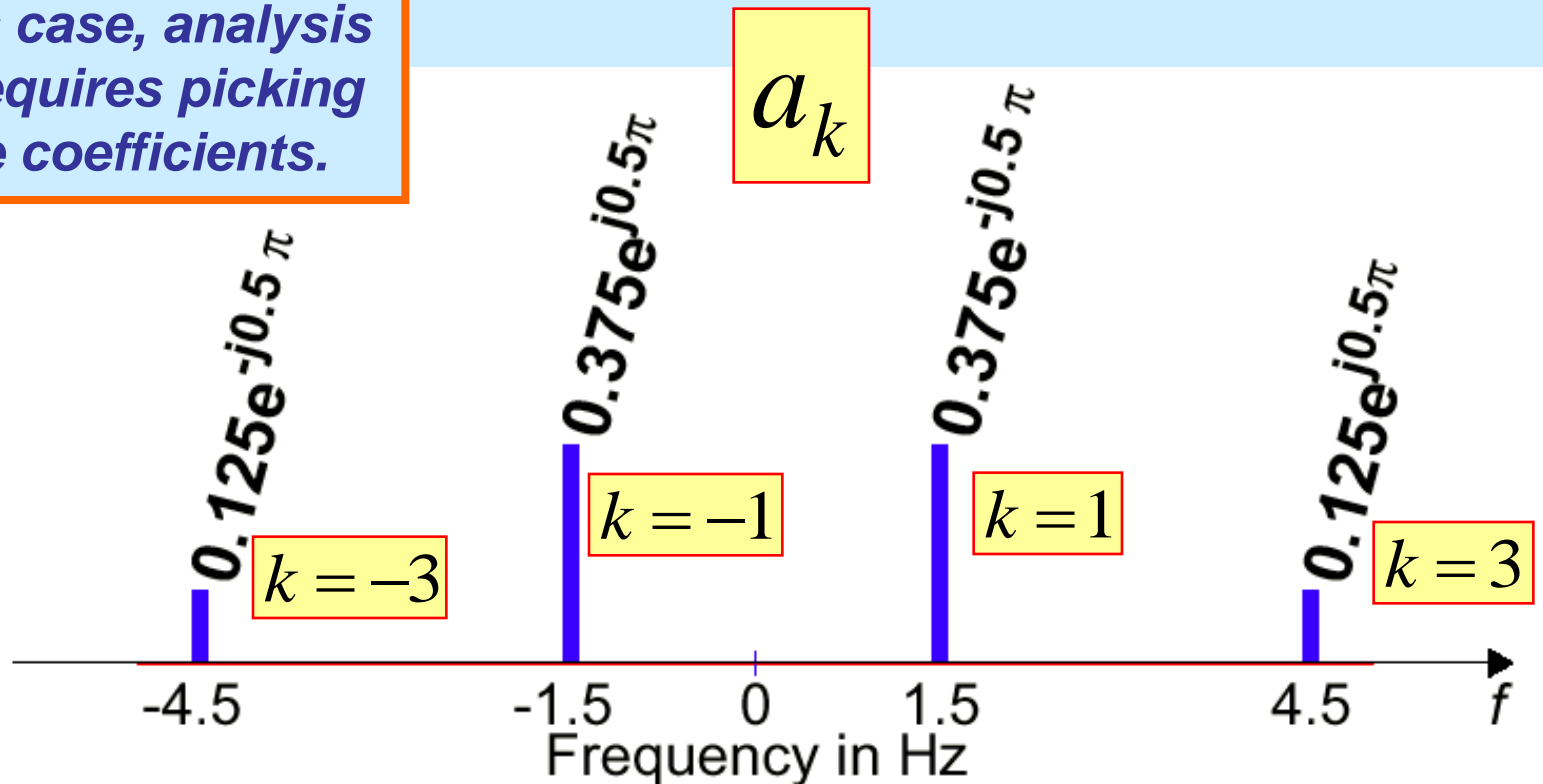
$$x(t) = \left(\frac{j}{8}\right)e^{j9\pi t} + \left(\frac{-3j}{8}\right)e^{j3\pi t} + \left(\frac{3j}{8}\right)e^{-j3\pi t} + \left(\frac{-j}{8}\right)e^{-j9\pi t}$$

Example

$$x(t) = \sin^3(3\pi t)$$

$$x(t) = \left(\frac{j}{8}\right)e^{j9\pi t} + \left(\frac{-3j}{8}\right)e^{j3\pi t} + \left(\frac{3j}{8}\right)e^{-j3\pi t} + \left(\frac{-j}{8}\right)e^{-j9\pi t}$$

In this case, analysis just requires picking off the coefficients.



STRATEGY: $x(t) \rightarrow a_k$

■ ANALYSIS

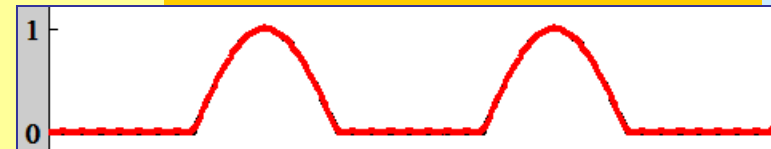
- Get representation from the signal
- Works for PERIODIC Signals
- Fourier Series
 - Answer is: an INTEGRAL over one period

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j\omega_0 k t} dt$$

FS: Rectified Sine Wave $\{a_k\}$

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt \quad (k \neq \pm 1)$$

Half-Wave Rectified Sine



$$a_k = \frac{1}{T_0} \int_0^{T_0/2} \sin\left(\frac{2\pi}{T_0}t\right) e^{-j(2\pi/T_0)kt} dt$$

$$= \frac{1}{T_0} \int_0^{T_0/2} \frac{e^{j(2\pi/T_0)t} - e^{-j(2\pi/T_0)t}}{2j} e^{-j(2\pi/T_0)kt} dt$$

$$= \frac{1}{j2T_0} \int_0^{T_0/2} e^{-j(2\pi/T_0)(k-1)t} dt - \frac{1}{j2T_0} \int_0^{T_0/2} e^{-j(2\pi/T_0)(k+1)t} dt$$

$$= \frac{e^{-j(2\pi/T_0)(k-1)t}}{j2T_0(-j(2\pi/T_0)(k-1))} \Bigg|_0^{T_0/2} - \frac{e^{-j(2\pi/T_0)(k+1)t}}{j2T_0(-j(2\pi/T_0)(k+1))} \Bigg|_0^{T_0/2}$$

FS: Rectified Sine Wave

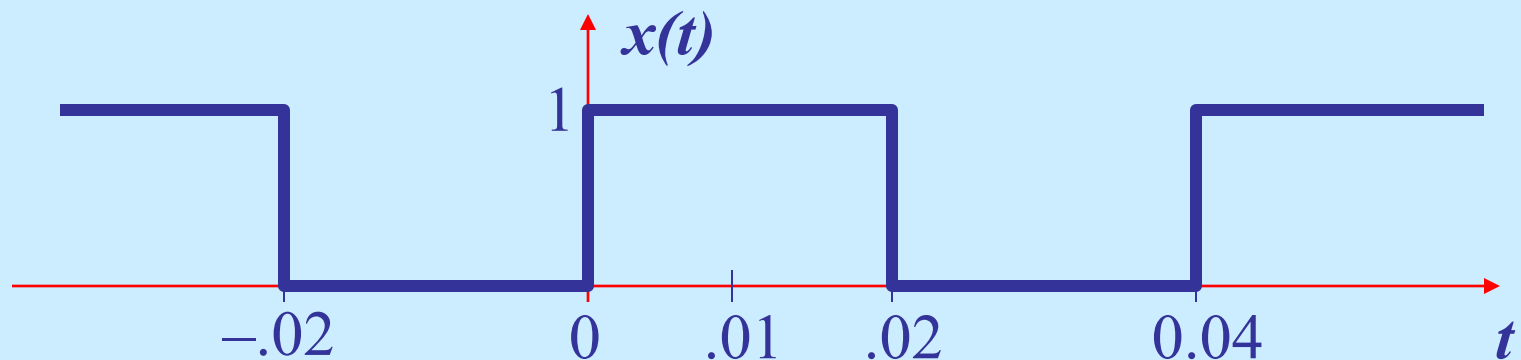
$\{a_k\}$

$$\begin{aligned}
 a_k &= \frac{e^{-j(2\pi/T_0)(k-1)t} \Big|_0^{T_0/2}}{j2T_0(-j(2\pi/T_0)(k-1))} - \frac{e^{-j(2\pi/T_0)(k+1)t} \Big|_0^{T_0/2}}{j2T_0(-j(2\pi/T_0)(k+1))} \\
 &= \frac{1}{4\pi(k-1)} \left(e^{-j(2\pi/T_0)(k-1)T_0/2} - 1 \right) - \frac{1}{4\pi(k+1)} \left(e^{-j(2\pi/T_0)(k+1)T_0/2} - 1 \right) \\
 &= \frac{1}{4\pi(k-1)} \left(e^{-j\pi(k-1)} - 1 \right) - \frac{1}{4\pi(k+1)} \left(e^{-j\pi(k+1)} - 1 \right) \\
 &= \left(\frac{k+1-(k-1)}{4\pi(k^2-1)} \right) \left(-(-1)^k - 1 \right) = \begin{cases} 0 & k \text{ odd} \\ \pm \frac{1}{j4} & k = \pm 1 \\ \frac{-1}{\pi(k^2-1)} & k \text{ even} \end{cases}
 \end{aligned}$$

SQUARE WAVE EXAMPLE

$$x(t) = \begin{cases} 1 & 0 \leq t < \frac{1}{2} T_0 \\ 0 & \frac{1}{2} T_0 \leq t < T_0 \end{cases}$$

for $T_0 = 0.04$ sec.



Fourier Coefficients a_k

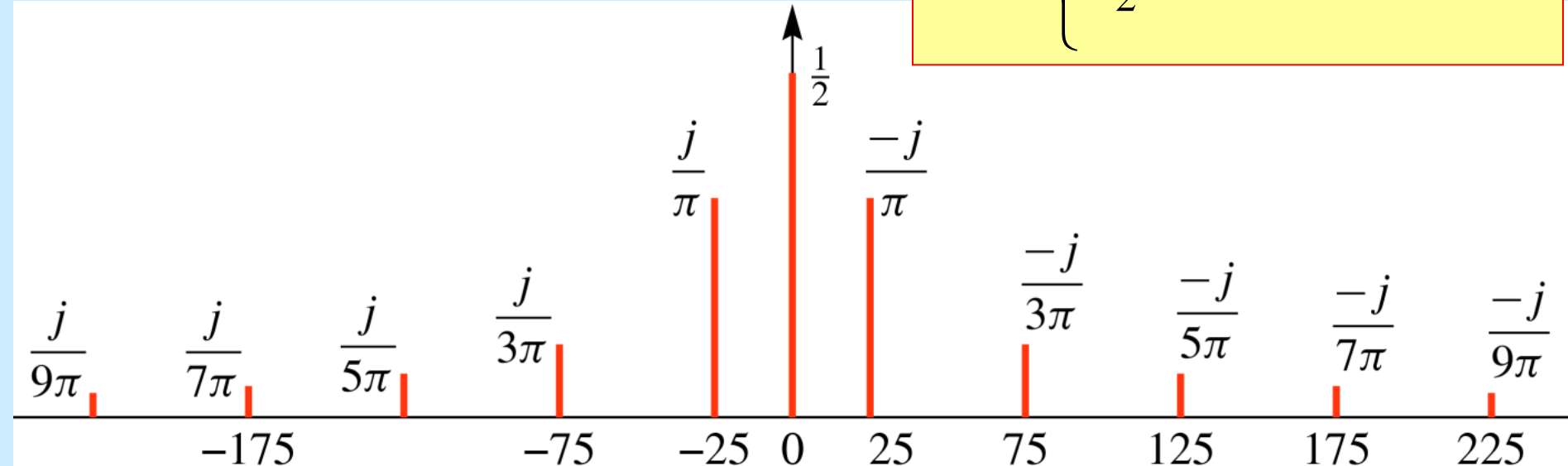
- a_k is a function of k
 - Complex Amplitude for k -th Harmonic
 - This one doesn't depend on the period, T_0

$$a_k = \frac{1 - (-1)^k}{j2\pi k} = \begin{cases} \frac{1}{j\pi k} & k = \pm 1, \pm 3, \dots \\ 0 & k = \pm 2, \pm 4, \dots \\ \frac{1}{2} & k = 0 \end{cases}$$

Spectrum from Fourier Series

$$\omega_0 = 2\pi / (0.04) = 2\pi(25)$$

$$a_k = \begin{cases} \frac{-j}{\pi k} & k = \pm 1, \pm 3, \dots \\ 0 & k = \pm 2, \pm 4, \dots \\ \frac{1}{2} & k = 0 \end{cases}$$



Fourier Series Synthesis

- HOW do you APPROXIMATE $x(t)$?

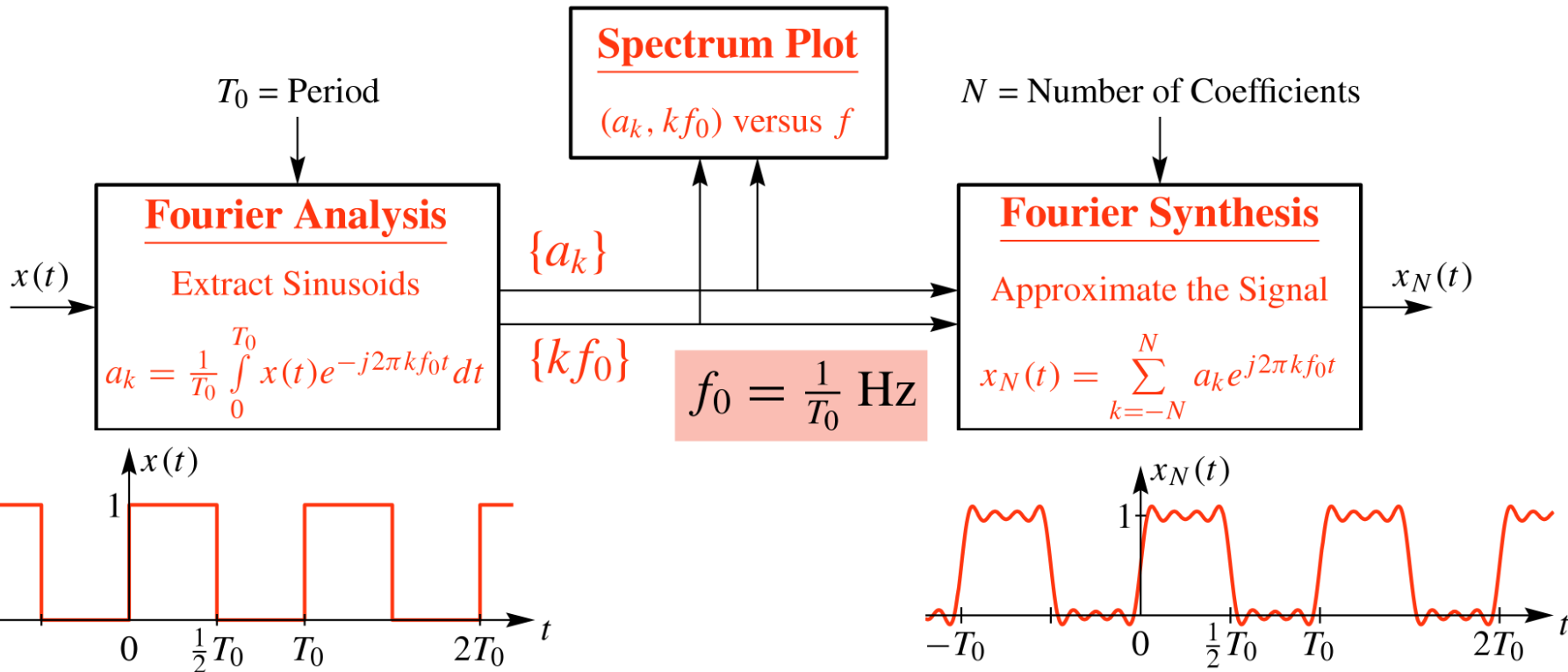
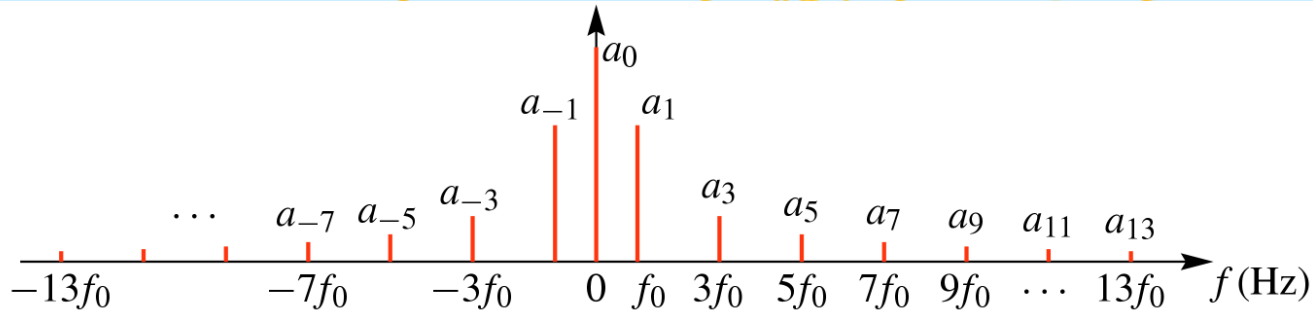
$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt$$

- Use FINITE number of coefficients

$$x(t) = \sum_{k=-N}^N a_k e^{j2\pi k f_0 t}$$

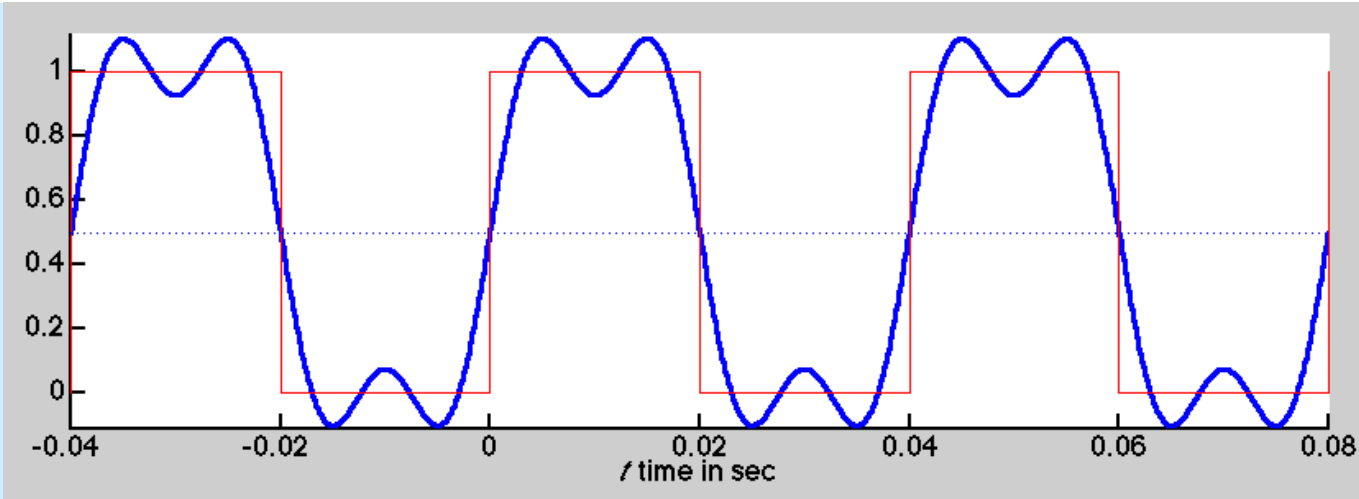
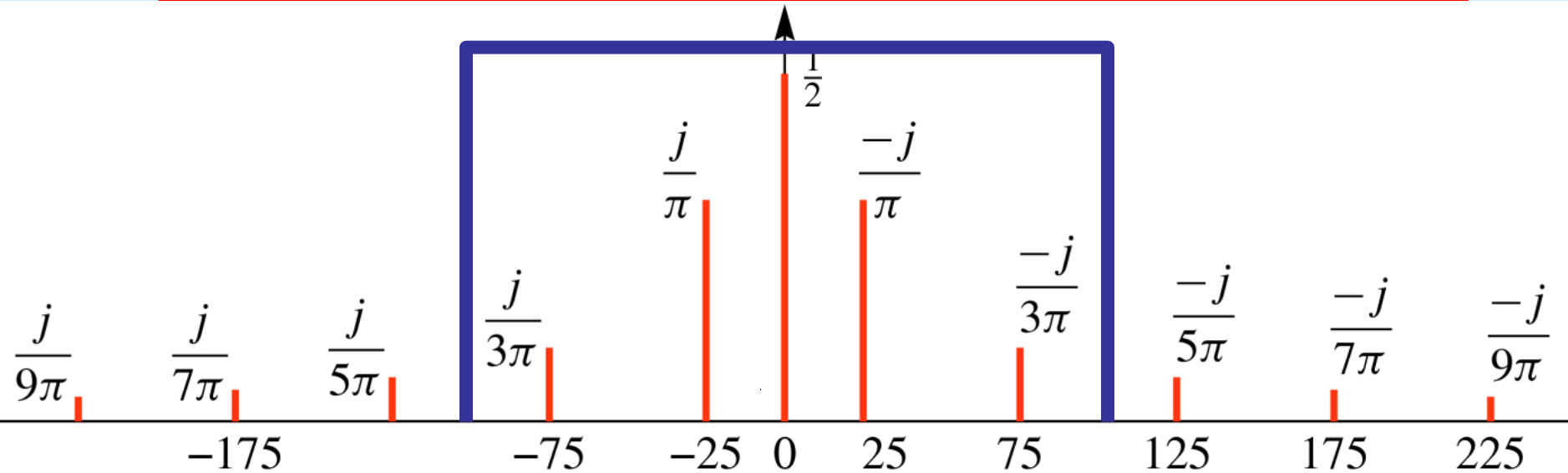
$$a_{-k} = a_k^* \quad \text{when } x(t) \text{ is real}$$

Fourier Series Synthesis



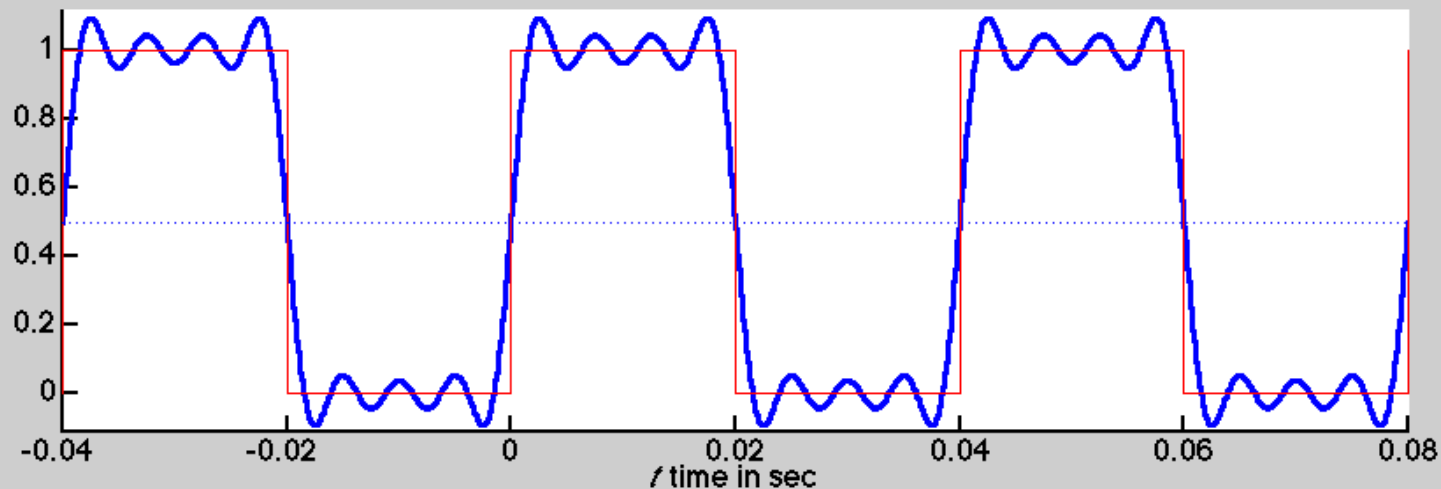
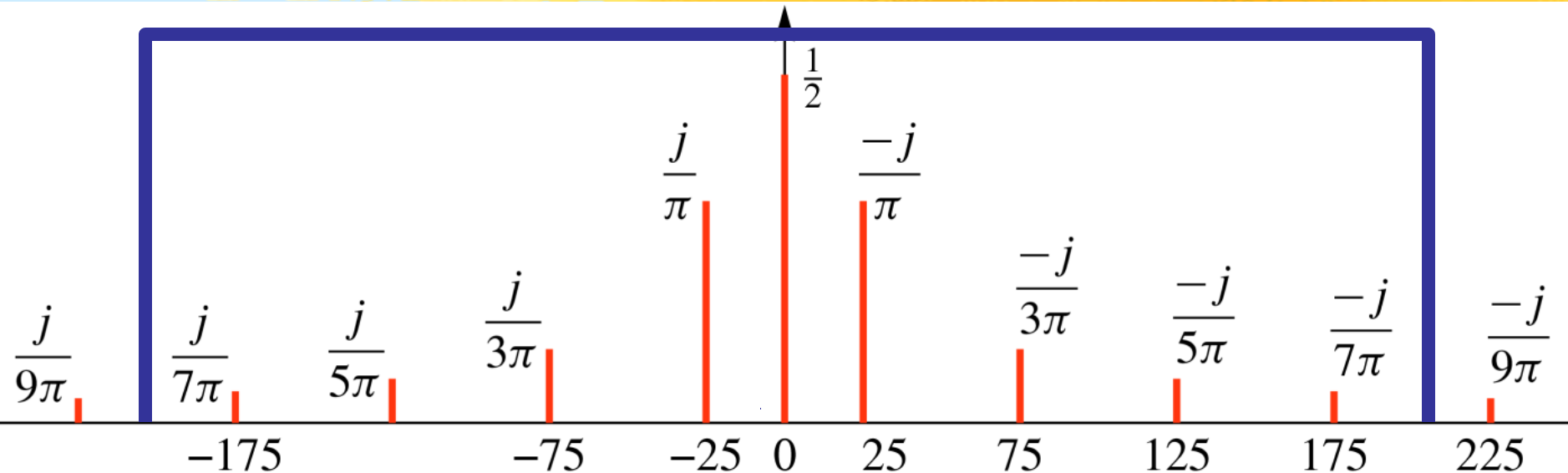
Synthesis: 1st & 3rd Harmonics

$$y(t) = \frac{1}{2} + \frac{2}{\pi} \cos(2\pi(25)t - \frac{\pi}{2}) + \frac{2}{3\pi} \cos(2\pi(75)t - \frac{\pi}{2})$$



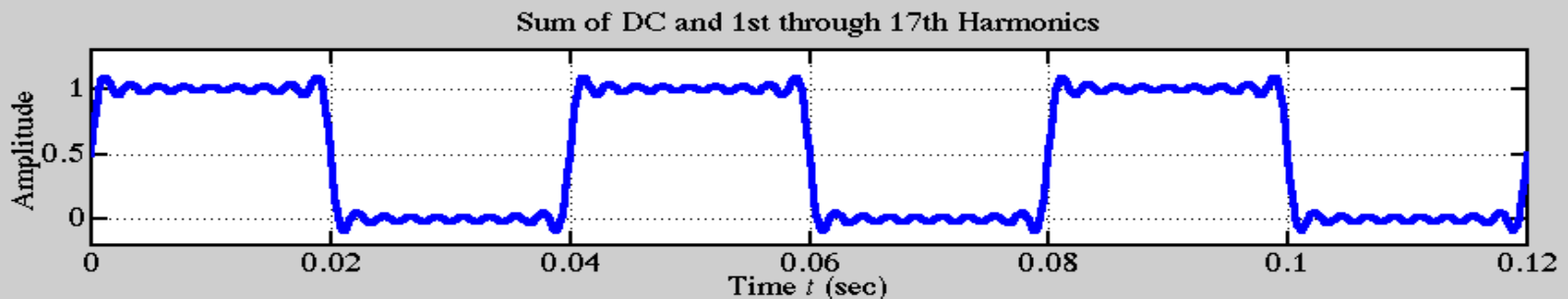
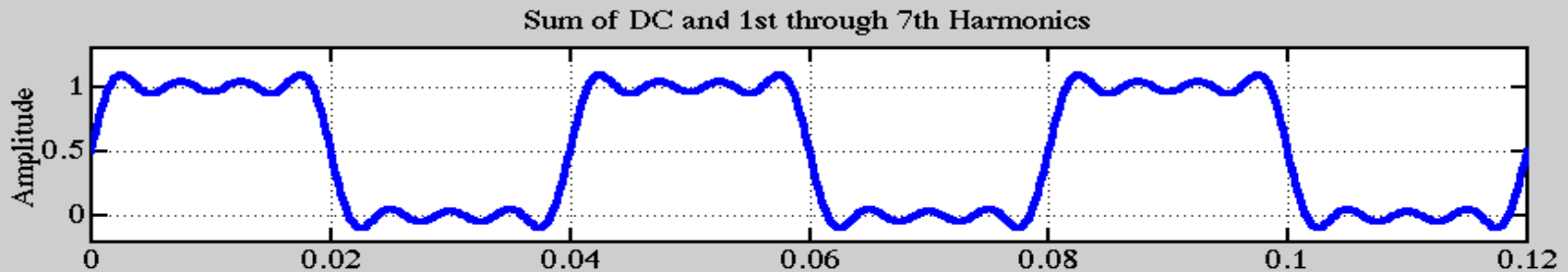
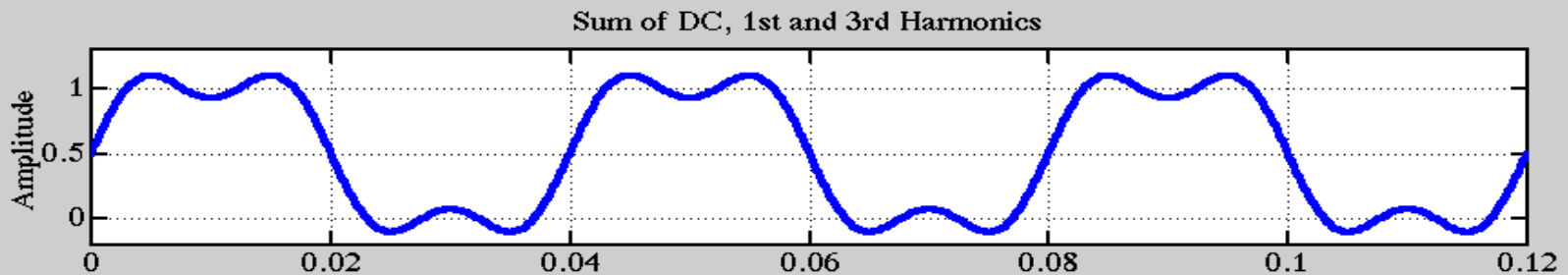
Synthesis: up to 7th Harmonic

$$y(t) = \frac{1}{2} + \frac{2}{\pi} \cos(50\pi t - \frac{\pi}{2}) + \frac{2}{3\pi} \sin(150\pi t) + \frac{2}{5\pi} \sin(250\pi t) + \frac{2}{7\pi} \sin(350\pi t)$$



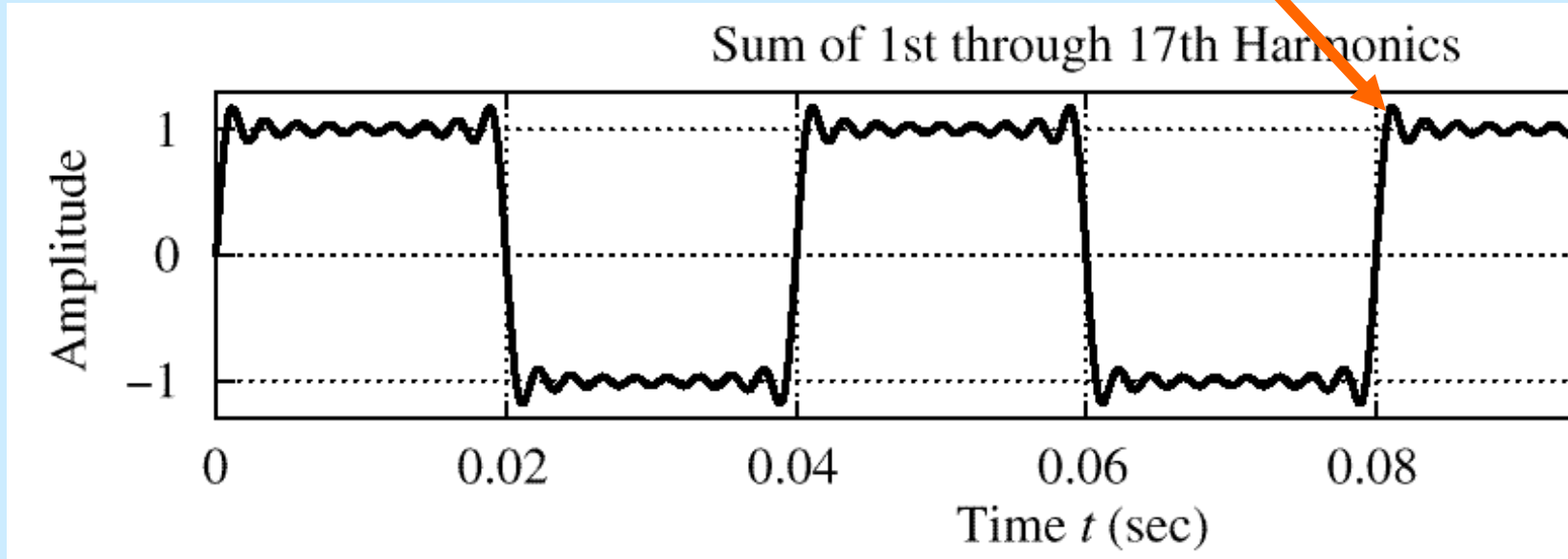
Fourier Synthesis

$$x_N(t) = \frac{1}{2} + \frac{2}{\pi} \sin(\omega_0 t) + \frac{2}{3\pi} \sin(3\omega_0 t) + \dots$$



Gibbs' Phenomenon

- Convergence at **DISCONTINUITY** of $x(t)$
 - There is always an **overshoot**
 - **9%** for the Square Wave case



Fourier Series Demos



- Fourier Series Java Applet

- Greg Slabaugh
 - Interactive

- http://users.ece.gatech.edu/mcclella/2025/Fsdemo_Slabaugh/fourier.html

- MATLAB GUI: fseriesdemo

- <http://users.ece.gatech.edu/mcclella/matlabGUIs/index.html>

fseriesdemo GUI

