Signal Processing First

Lecture 4
Spectrum Representation

© 2003, JH McClellan & RW Schafer

READING ASSIGNMENTS

- This Lecture:
  - Chapter 3, Section 3-1

- Other Reading:
  - Appendix A: Complex Numbers

  - Next Lecture: Ch 3, Sects 3-2, 3-3, 3-7 & 3-8

© 2003, JH McClellan & RW Schafer

LECTURE OBJECTIVES

- Sinusoids with DIFFERENT Frequencies
  - SYNTHESIZE by Adding Sinusoids

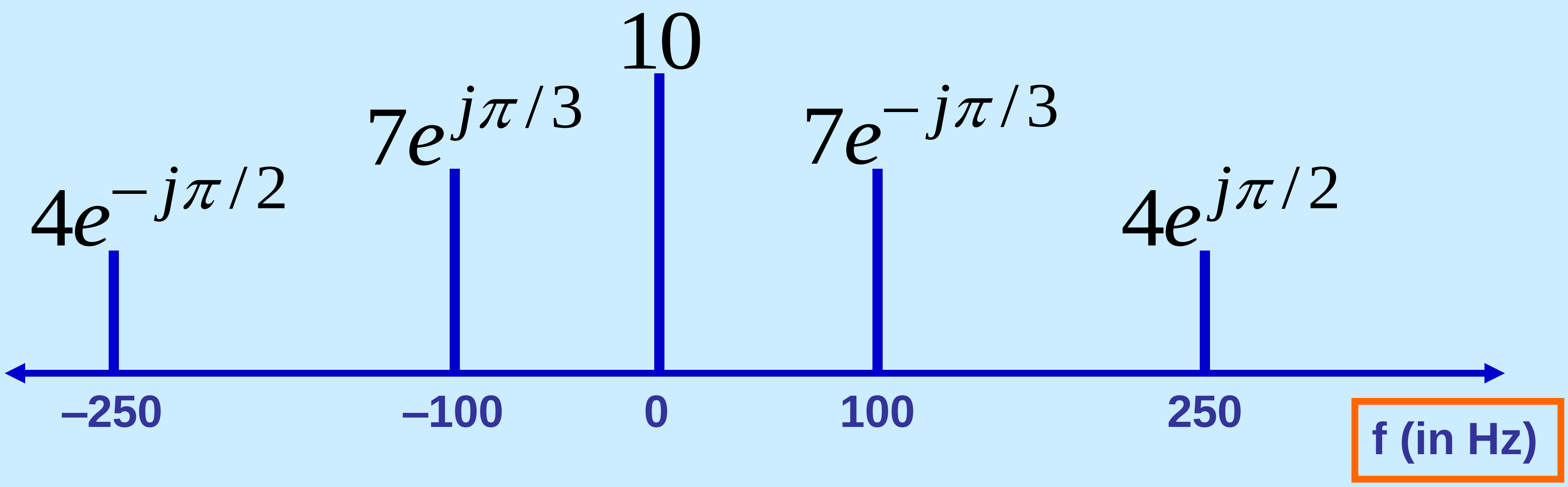
$$x(t) = \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

- SPECTRUM Representation
  - Graphical Form shows **DIFFERENT** Freqs

© 2003, JH McClellan & RW Schafer

FREQUENCY DIAGRAM

- Plot Complex Amplitude vs. Freq



© 2003, JH McClellan & RW Schafer

Another FREQ. Diagram

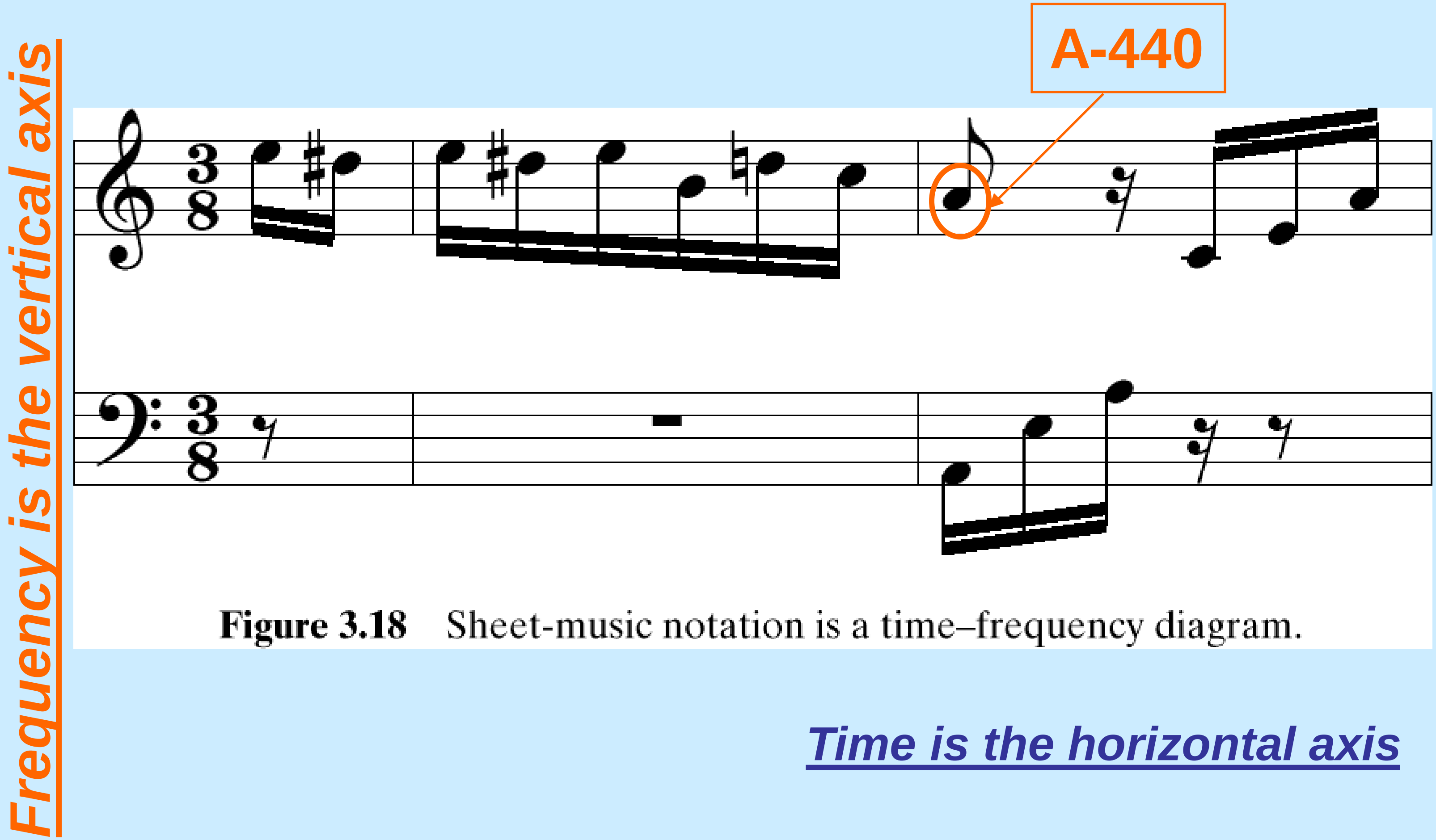


Figure 3.18 Sheet-music notation is a time–frequency diagram.

Time is the horizontal axis

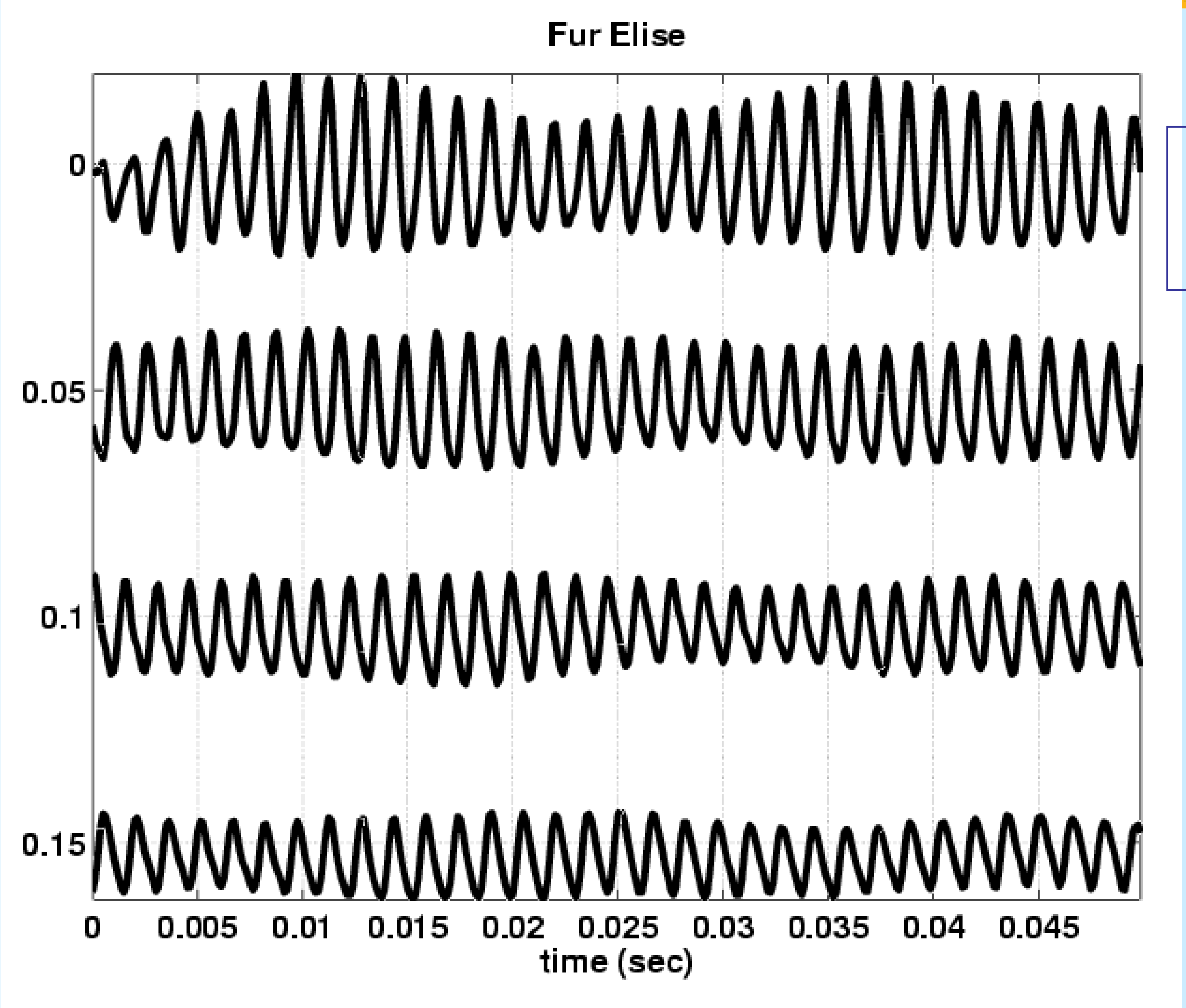
© 2003, JH McClellan & RW Schafer

MOTIVATION

- Synthesize Complicated Signals
  - Musical Notes
    - Piano uses 3 strings for many notes
    - Chords: play several notes simultaneously
  - Human Speech
    - Vowels have dominant frequencies
    - Application: computer generated speech
  - Can all signals be generated this way?
    - Sum of sinusoids?

© 2003, JH McClellan & RW Schafer

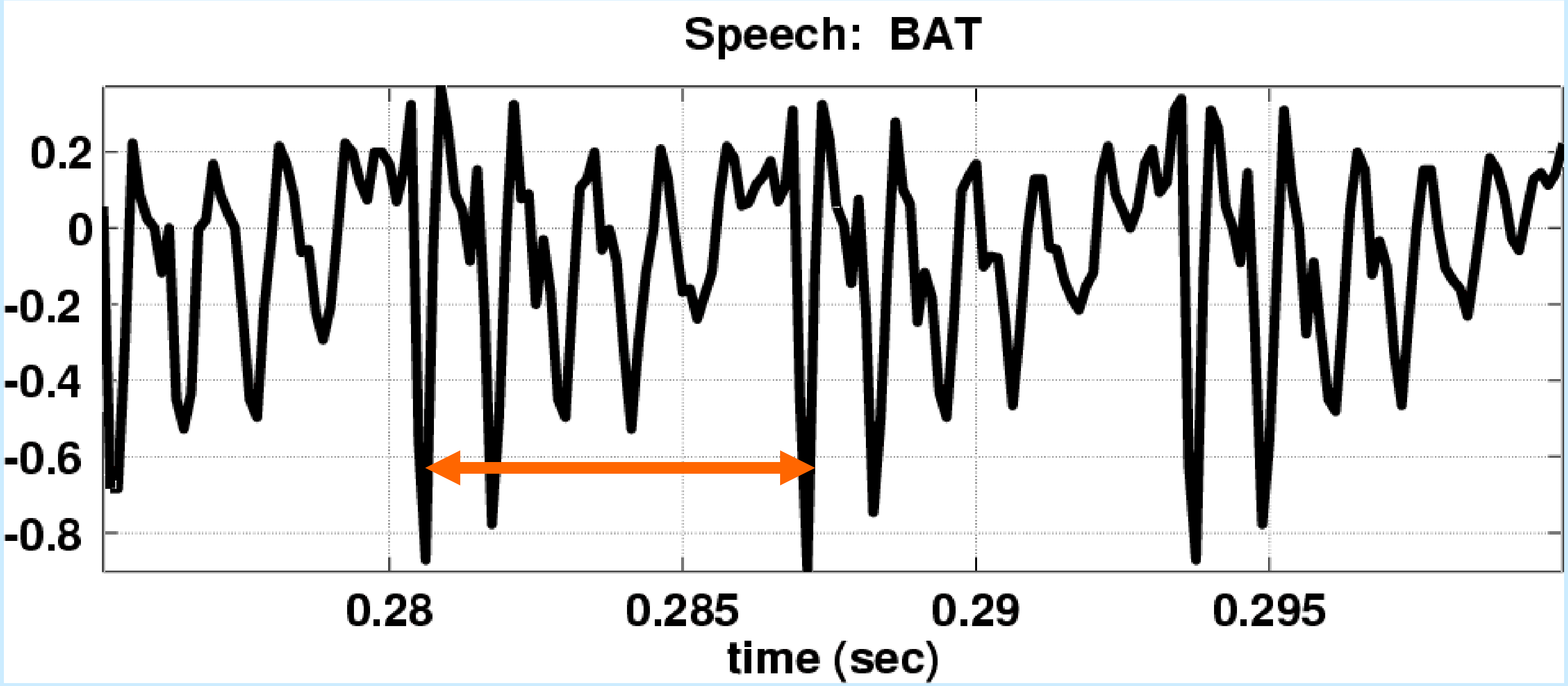
Fur Elise WAVEFORM



Beat Notes

Speech Signal: BAT

- Nearly **Periodic** in Vowel Region
  - Period is (Approximately) T = 0.0065 sec



© 2003, JH McClellan & RW Schafer

Euler's Formula Reversed

- Solve for cosine (or sine)

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$e^{-j\omega t} = \cos(-\omega t) + j\sin(-\omega t)$$

$$e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t)$$

$$e^{j\omega t} + e^{-j\omega t} = 2\cos(\omega t)$$

$$\boxed{\cos(\omega t) = \tfrac{1}{2}(e^{j\omega t} + e^{-j\omega t})}$$

© 2003, JH McClellan & RW Schafer

INVERSE Euler's Formula

- Solve for cosine (or sine)

$$\cos(\omega t) = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})$$

$$\sin(\omega t) = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$

© 2003, JH McClellan & RW Schafer

SPECTRUM Interpretation

- Cosine = sum of 2 complex exponentials:

$$A\cos(7t) = \frac{A}{2}e^{j7t} + \frac{A}{2}e^{-j7t}$$

One has a positive frequency
The other has negative freq.
Amplitude of each is half as big

© 2003, JH McClellan & RW Schafer

NEGATIVE FREQUENCY

- Is negative frequency real?
- Doppler Radar provides an example
  - Police radar measures speed by using the Doppler shift principle
  - Let's assume 400Hz ←→60 mph
  - +400Hz means towards the radar
  - -400Hz means away (opposite direction)
  - Think of a train whistle

© 2003, JH McClellan & RW Schafer

SPECTRUM of SINE

- Sine = sum of 2 complex exponentials:

$$A\sin(7t) = \frac{A}{2j}e^{j7t} - \frac{A}{2j}e^{-j7t}$$

$$= \frac{1}{2}Ae^{-j0.5\pi}e^{j7t} + \frac{1}{2}Ae^{j0.5\pi}e^{-j7t}$$

$$\frac{-1}{j} = j = e^{j0.5\pi}$$

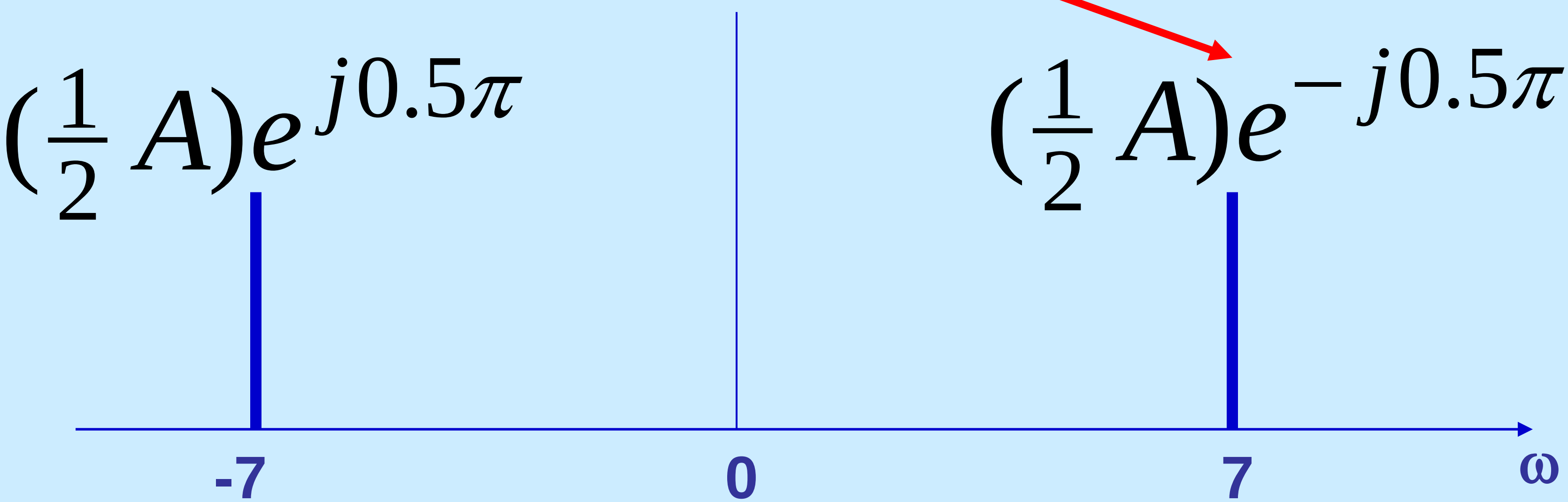
  - Positive freq. has phase = -0.5π
  - Negative freq. has phase = $+0.5\pi$

© 2003, JH McClellan & RW Schafer

GRAPHICAL SPECTRUM

EXAMPLE of SINE

$$A\sin(7t) = \tfrac{1}{2}Ae^{-j0.5\pi}e^{j7t} + \tfrac{1}{2}Ae^{j0.5\pi}e^{-j7t}$$

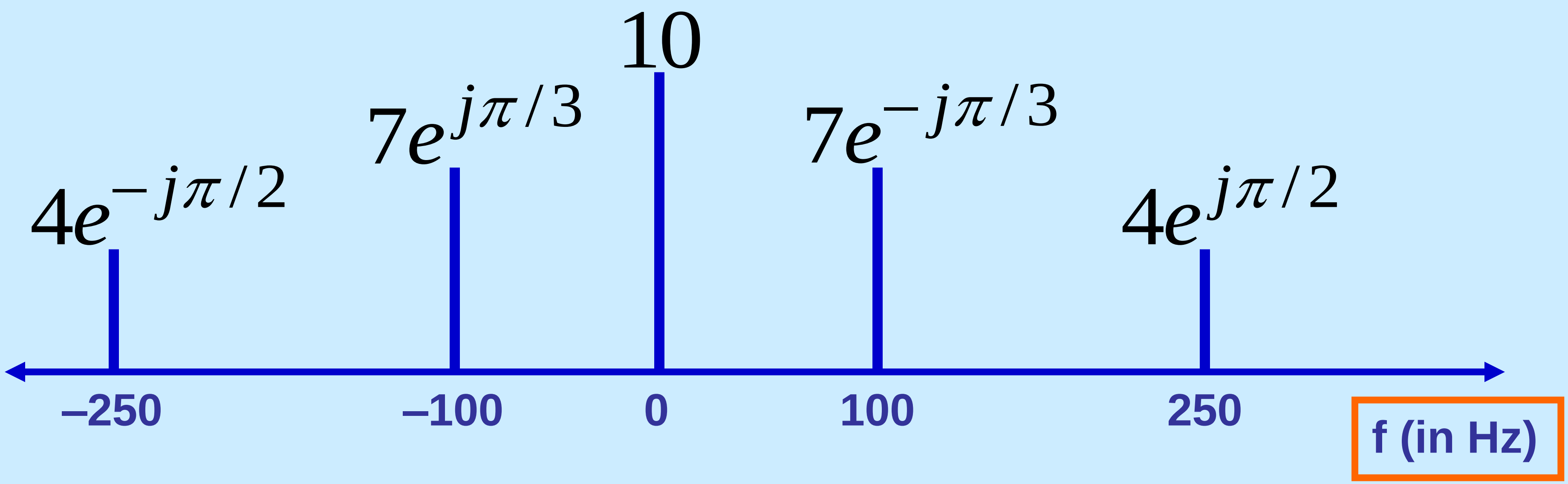


AMPLITUDE, PHASE & FREQUENCY are shown

© 2003, JH McClellan & RW Schafer

SPECTRUM ---> SINUSOID

- Add the spectrum components:



What is the formula for the signal x(t)?

© 2003, JH McClellan & RW Schafer

Gather (A,ω,ϕ) information

- Frequencies:
  - -250 Hz
  - -100 Hz
  - **0** Hz
  - 100 Hz
  - 250 Hz

- Amplitude & Phase
  - 4 $-\pi/2$
  - 7 $+\pi/3$
  - 10 **0**
  - 7 $-\pi/3$
  - 4 $+\pi/2$

Note the **conjugate phase**

DC is another name for zero-freq component

DC component always has $\phi=0$ or π (for real ***x(t)***)

© 2003, JH McClellan & RW Schafer

Add Spectrum Components-1

- Frequencies:
  - -250 Hz
  - -100 Hz
  - 0 Hz
  - 100 Hz
  - 250 Hz
- Amplitude & Phase
  - 4 $-\pi/2$
  - 7 $+\pi/3$
  - 10 0
  - 7 $-\pi/3$
  - 4 $+\pi/2$

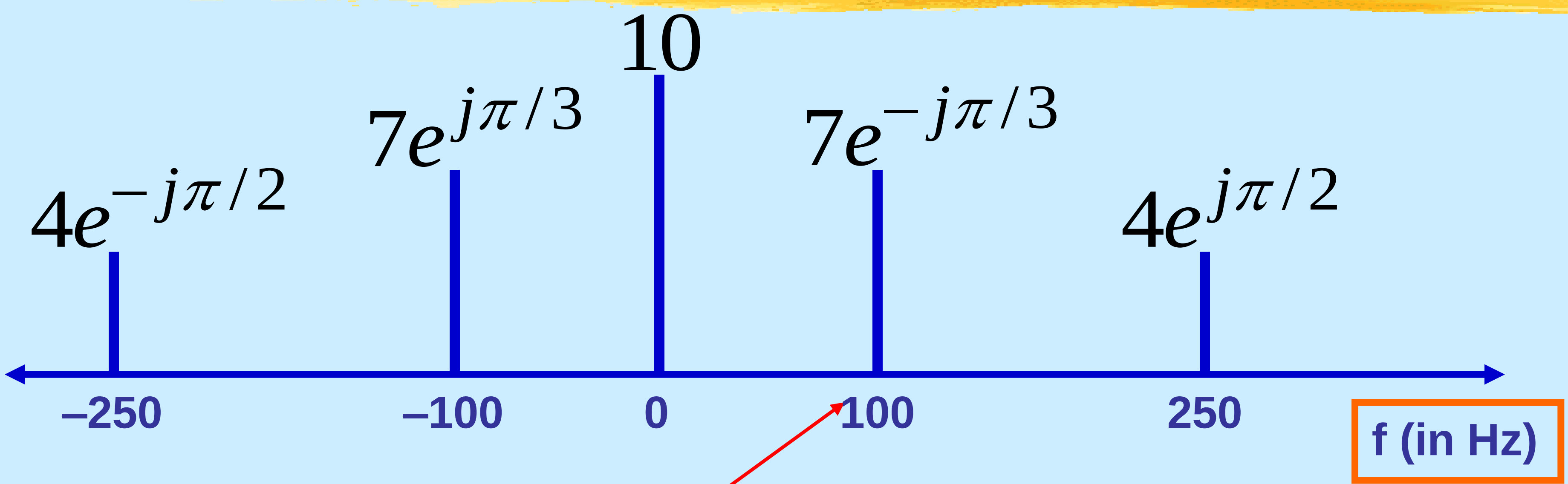
$$x(t) = 10 +$$

$$7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t}$$

$$4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

© 2003, JH McClellan & RW Schafer

Add Spectrum Components-2



$$x(t) = 10 + 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t} + 4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

© 2003, JH McClellan & RW Schafer

Simplify Components

$$x(t) = 10 +$$
$$7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t}$$
$$4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

Use Euler's Formula to get REAL sinusoids:

$$A\cos(\omega t + \varphi) = \tfrac{1}{2}Ae^{-j\varphi}e^{j\omega t} + \tfrac{1}{2}Ae^{-j\varphi}e^{-j\omega t}$$

© 2003, JH McClellan & RW Schafer

FINAL ANSWER

$$x(t) = 10 + 14\cos(2\pi(100)t - \pi/3) + 8\cos(2\pi(250)t + \pi/2)$$

So, we get the general form:

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

© 2003, JH McClellan & RW Schafer

Summary: GENERAL FORM

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

$$x(t) = X_0 + \sum_{k=1}^{N} \Re e\left\{X_k e^{j2\pi f_k t}\right\}$$

$$\Re e\{z\} = \tfrac{1}{2} z + \tfrac{1}{2} z^*$$

$$X_k = A_k e^{j\varphi_k}$$

$$\text{Frequency} = f_k$$

$$x(t) = X_0 + \sum_{k=1}^{N} \left\{\tfrac{1}{2} X_k e^{j2\pi f_k t} + \tfrac{1}{2} X_k^* e^{-j2\pi f_k t}\right\}$$

© 2003, JH McClellan & RW Schafer

Example: Synthetic Vowel

- Sum of 5 Frequency Components

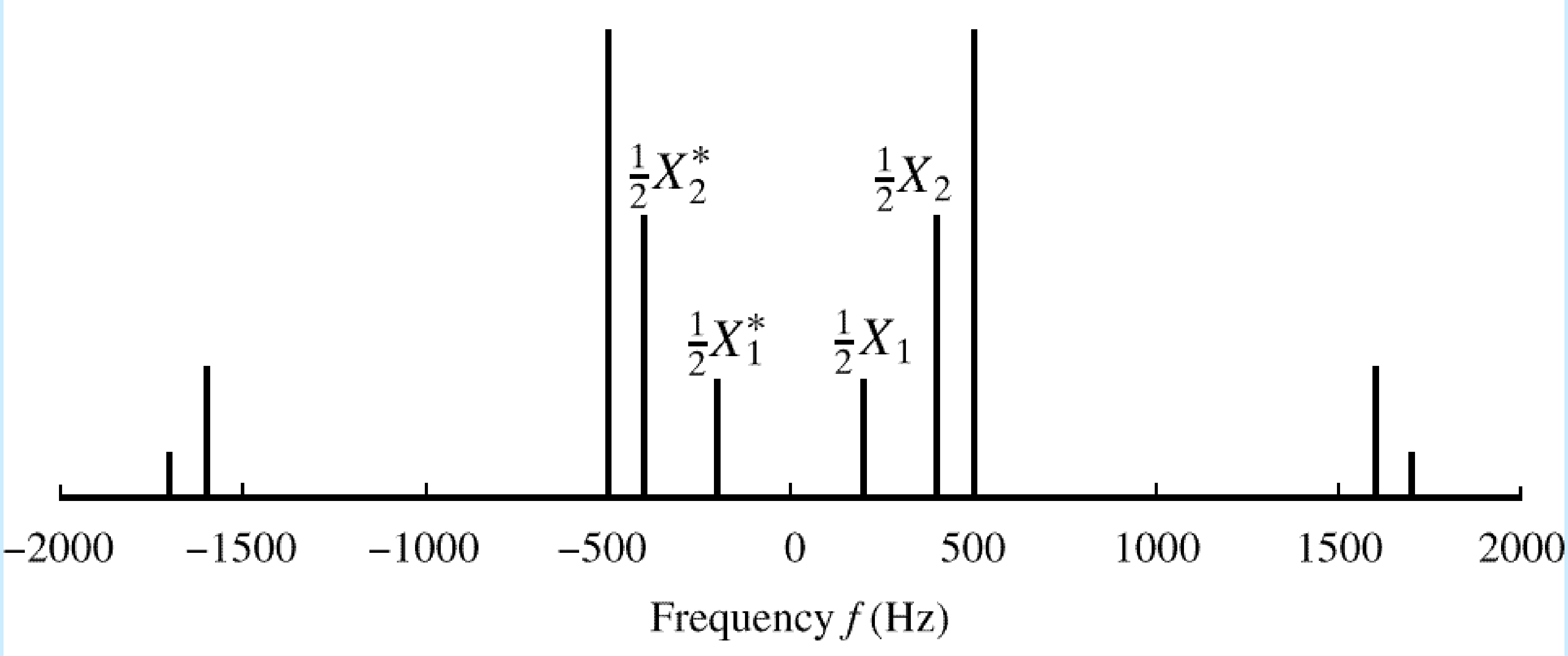
f_k (Hz)	X_k	Mag	Phase (rad)
200	$(771 + j12202)$	12,226	1.508
400	$(-8865 + j28048)$	29,416	1.876
500	$(48001 - j8995)$	48,836	-0.185
1600	$(1657 - j13520)$	13,621	-1.449
1700	$4723 + j0$	4723	0

Table 3.1: Complex amplitudes for harmonic signal that approximates the vowel sound "ah".

© 2003, JH McClellan & RW Schafer

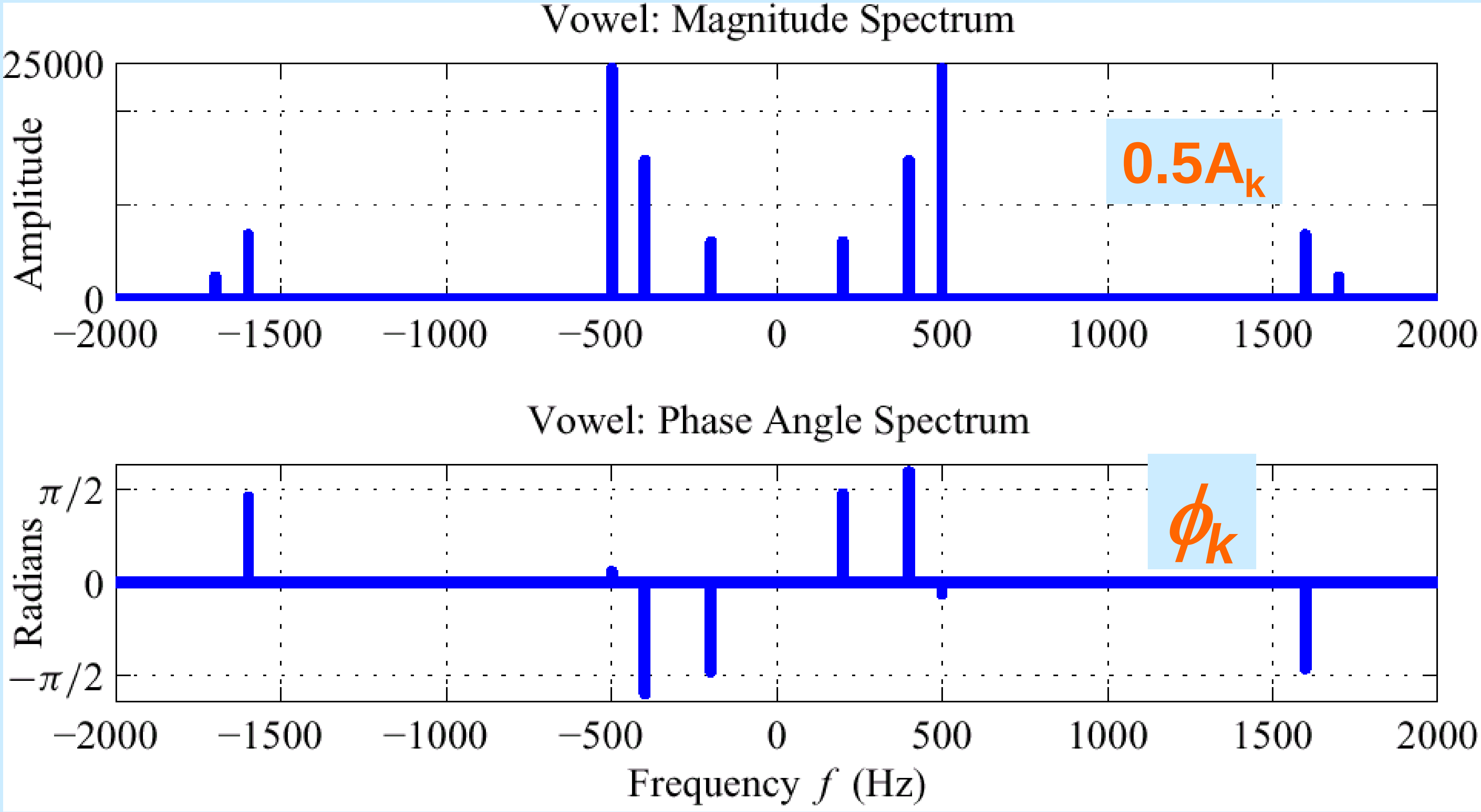
SPECTRUM of VOWEL

- Note: Spectrum has $0.5X_k$ (except X_{DC})
- Conjugates in negative frequency



© 2003, JH McClellan & RW Schafer

SPECTRUM of VOWEL (Polar Format)



Vowel Waveform (sum of all 5 components)

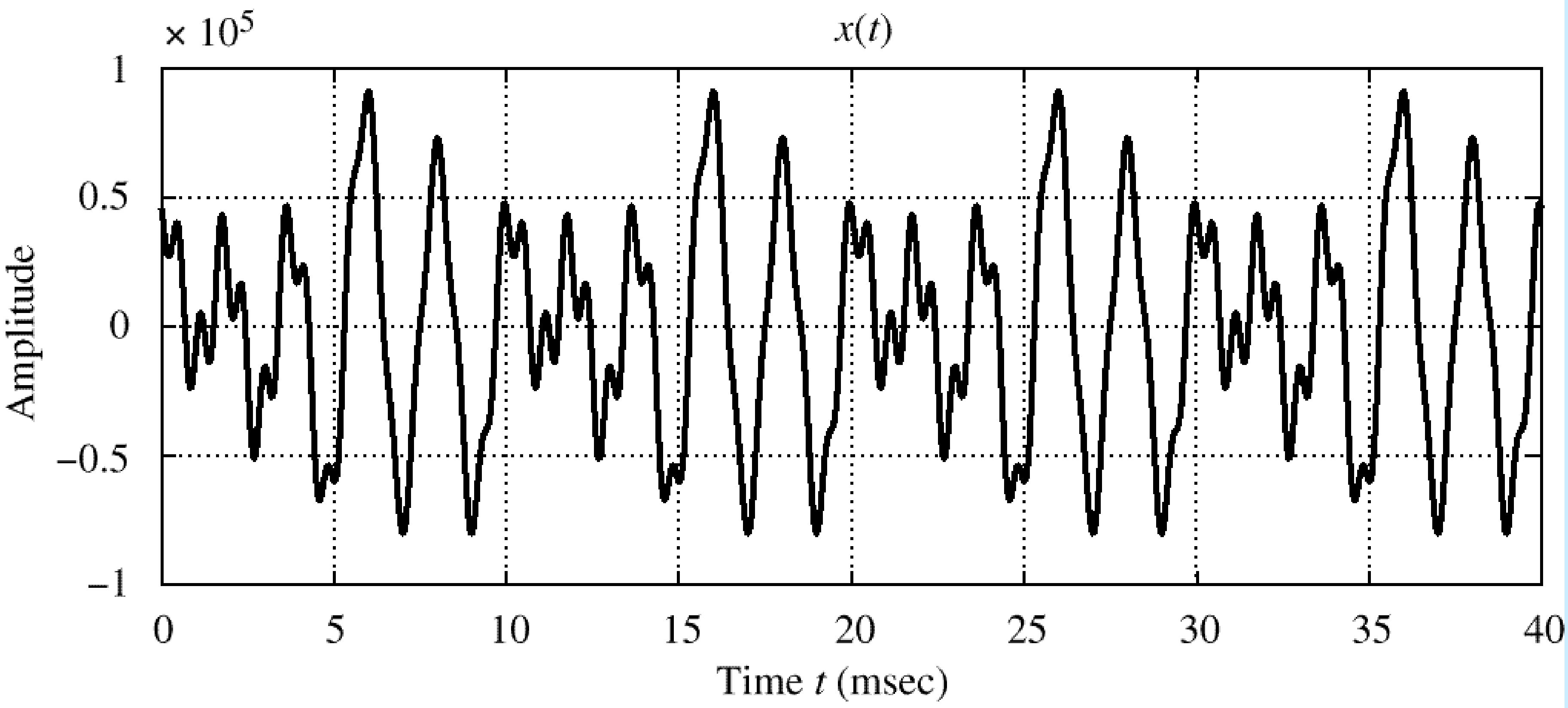


Figure 3.11 Sum of all of the terms in (3.3.4). Note that the period is 10 msec, which equals $1/f_0$.

© 2003, JH McClellan & RW Schafer