Signal Processing First

Lecture 9
D-to-A Conversion
READING ASSIGNMENTS

- This Lecture:
  - Chapter 4: Sections 4-4, 4-5

- Other Reading:
  - Recitation: Section 4-3 (Strobe Demo)
  - Next Lecture: Chapter 5 (beginning)
LECTURE OBJECTIVES

- FOLDING: a type of ALIASING
- DIGITAL-to-ANALOG CONVERSION is
  - Reconstruction from samples
    - SAMPLING THEOREM applies
  - Smooth Interpolation
- Mathematical Model of D-to-A
  - SUM of SHIFTED PULSES
    - Linear Interpolation example
A-to-D
- Convert $x(t)$ to numbers stored in memory

D-to-A
- Convert $y[n]$ back to a “continuous-time” signal, $y(t)$
- $y[n]$ is called a “discrete-time” signal
SAMPLING $x(t)$

- **UNIFORM SAMPLING** at $t = nT_s$
  - **IDEAL**: $x[n] = x(nT_s)$

*Shannon Sampling Theorem*
A continuous-time signal $x(t)$ with frequencies no higher than $f_{\text{max}}$ can be reconstructed exactly from its samples $x[n] = x(nT_s)$, if the samples are taken at a rate $f_s = 1/T_s$ that is greater than $2f_{\text{max}}$. 

© 2003, JH McClellan & RW Schafer
NYQUIST RATE

- "Nyquist Rate" Sampling
  - $f_s > \text{TWICE} \text{ the HIGHEST Frequency in } x(t)$
  - "Sampling above the Nyquist rate"

- BANDLIMITED SIGNALS
  - DEF: $x(t)$ has a HIGHEST FREQUENCY COMPONENT in its SPECTRUM
  - NON-BANDLIMITED EXAMPLE
    - TRIANGLE WAVE is NOT BANDLIMITED
SPECTRUM for $x[n]$

- INCLUDE **ALL** SPECTRUM LINES
  - ALIASES
    - ADD INTEGER MULTIPLES of $2\pi$ and $-2\pi$
  - FOLDED ALIASES
    - ALIASES of NEGATIVE FREQS
- PLOT versus NORMALIZED FREQUENCY
  - i.e., DIVIDE $f_o$ by $f_s$

$$\hat{\omega} = 2\pi \frac{f}{f_s} + 2\pi \ell$$
EXAMPLE: SPECTRUM

- \( x[n] = \text{Acos}(0.2\pi n + \phi) \)
- FREQUENCIES @ \( 0.2\pi \) and \(-0.2\pi\)
- ALIASES:
  - \( \{2.2\pi, 4.2\pi, 6.2\pi, \ldots\} \) & \( \{-1.8\pi, -3.8\pi, \ldots\} \)
  - EX: \( x[n] = \text{Acos}(4.2\pi n + \phi) \)
- ALIASES of NEGATIVE FREQ:
  - \( \{1.8\pi, 3.8\pi, 5.8\pi, \ldots\} \) & \( \{-2.2\pi, -4.2\pi \ldots\} \)
SPECTRUM (MORE LINES)

\[ \hat{\omega} = 2\pi \frac{f}{f_s} \]

\[ f_s = 1 \text{kHz} \]

\[ x[n] = A\cos(2\pi(100)(n/1000) + \varphi) \]

100-Hz Cosine Wave: Sampled with \( T_s = 1 \text{ msec} \) (1000 Hz)
SPECTRUM (ALIASING CASE)

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$$f_s = 80\text{kHz}$$

$$x[n] = A\cos(2\pi(100)(n/80) + \varphi)$$

100-Hz Cosine Wave: Sampled with $$T_s = 12.5\text{ msec (80 Hz)}$$
FOLDING (a type of ALIASING)

- EXAMPLE: 3 different $x(t)$; same $x[n]$

\[
\begin{align*}
&f_s = 1000 \\
&\cos(2\pi(100)t) \rightarrow \cos[2\pi(0.1)n] \\
&\cos(2\pi(1100)t) \rightarrow \cos[2\pi(1.1)n] = \cos[2\pi(0.1)n] \\
&\cos(2\pi(900)t) \rightarrow \cos[2\pi(0.9)n] \\
&= \cos[2\pi(0.9)n - 2\pi n] = \cos[2\pi(-0.1)n] = \cos[2\pi(0.1)n]
\end{align*}
\]

- $900$ Hz “folds” to $100$ Hz when $f_s=1$kHz
Normalized Radian Frequency

\[ \hat{\omega} = \omega T_s = \frac{2\pi f}{f_s} + 2\pi l \]

Aliasing

Folded Alias

DIGITAL FREQ \( \hat{\omega} \) AGAIN
SPECTRUM (FOLDING CASE)

\[
\hat{\omega} = 2\pi \frac{f}{f_s}
\]

\(f_s = 125\text{Hz}
\]

\[
x[n] = A\cos(2\pi(100)(n/125) + \varphi)
\]

100-Hz Cosine Wave: Sampled with \(T_s = 8\text{ msec (125 Hz)}\)
FREQUENCY DOMAINS

\[ x(t) \rightarrow A\text{-to-}D \rightarrow x[n] \rightarrow y[n] \rightarrow D\text{-to-}A \rightarrow y(t) \]

\[ \hat{\omega} = 2\pi \frac{f}{f_s} + 2\pi \ell \]

\[ f = \frac{\hat{\omega}}{2\pi} f_s \]
DEMOS from CHAPTER 4

- CD-ROM DEMOS
- SAMPLING DEMO (con2dis GUI)
  - Different Sampling Rates
    - Aliasing of a Sinusoid
- STROBE DEMO
  - Synthetic vs. Real
  - Television SAMPLES at 30 fps
- Sampling & Reconstruction
SAMPLING GUI (con2dis)
D-to-A Reconstruction

- Create continuous $y(t)$ from $y[n]$
  - **IDEAL**
    - If you have formula for $y[n]$
    - Replace $n$ in $y[n]$ with $f_s t$
    - $y[n] = A \cos(0.2\pi n + \phi)$ with $f_s = 8000$ Hz
    - $y(t) = A \cos(2\pi(800)t + \phi)$
D-to-A is AMBIGUOUS!

- ALIASING
  - Given $y[n]$, which $y(t)$ do we pick? ? ?
  - INFINITE NUMBER of $y(t)$
  - PASSING THRU THE SAMPLES, $y[n]$
  - D-to-A RECONSTRUCTION MUST CHOOSE ONE OUTPUT

- RECONSTRUCT THE SMOOUEST ONE
  - THE LOWEST FREQ, if $y[n] = \text{sinusoid}$
SPECTRUM (ALIASING CASE)

\[ \hat{\omega} = \frac{2\pi f}{f_s} \]

\[ f_s = 80\text{Hz} \]

\[ x[n] = A\cos(2\pi(100)(n/80) + \phi) \]

100-Hz Cosine Wave: Sampled with \( T_s = 12.5 \text{ msec} \) (80 Hz)
Reconstruction (D-to-A)

- CONVERT STREAM of NUMBERS to $x(t)$
- “CONNECT THE DOTS”
- INTERPOLATION

INTUITIVE, conveys the idea
SAMPLE & HOLD DEVICE

- CONVERT $y[n]$ to $y(t)$
  - $y[k]$ should be the value of $y(t)$ at $t = kT_s$
  - Make $y(t)$ equal to $y[k]$ for $kT_s - 0.5T_s < t < kT_s + 0.5T_s$

**STAIR-STEP APPROXIMATION**

![Diagram of stair-step approximation](image-url)
SQUARE PULSE CASE

Sampling and Zero-Order Reconstruction: \( f_0 = 83, f_s = 200 \)

Original and Reconstructed Waveforms
OVER-SAMPLING CASE

Sampling and Zero-Order Reconstruction: $f_0 = 83$, $f_s = 800$

Original and Reconstructions Waveforms
MATH MODEL for D-to-A

\[ y(t) = \sum_{n=-\infty}^{\infty} y[n] p(t - nT_s) \]

SQUARE PULSE:

\[ p(t) = \begin{cases} 
1 & -\frac{1}{2}T_s < t \leq \frac{1}{2}T_s \\
0 & \text{otherwise}
\end{cases} \]
EXPAND the SUMMATION

\[ \sum_{n=-\infty}^{\infty} y[n]p(t - nT_s) = \]

\[ ... + y[0]p(t) + y[1]p(t - T_s) + y[2]p(t - 2T_s) + ... \]

- SUM of SHIFTED PULSES \( p(t-nT_s) \)
  - “WEIGHTED” by \( y[n] \)
  - CENTERED at \( t=nT_s \)
  - SPACED by \( T_s \)
- RESTORES “REAL TIME”
Figure 4.17 Four different pulses for D-to-C conversion. The sampling period is $T_s = 0.005$, i.e., $f_s = 200$ Hz. Note that the duration of each pulse is approximately one or two times $T_s$. 
TRIANGULAR PULSE (2X)
OPTIMAL PULSE?

CALLED “BANDLIMITED INTERPOLATION”

\[ p(t) = \sin \left( \frac{\pi t}{T_s} \right) \quad \text{for} \quad -\infty < t < \infty \]

\[ p(t) = 0 \quad \text{for} \quad t = \pm T_s, \pm 2T_s, \ldots \]