Signal Processing First

Lecture 8
Sampling & Aliasing
READING ASSIGNMENTS

This Lecture:
- Chap 4, Sections 4-1 and 4-2
  - Replaces Ch 4 in DSP First, pp. 83-94

Other Reading:
- Recitation: Strobe Demo (Sect 4-3)
- Next Lecture: Chap. 4 Sects. 4-4 and 4-5
LECTURE OBJECTIVES

- SAMPLING can cause ALIASING
  - **Nyquist/Shannon Sampling Theorem**
  - Sampling Rate \( f_s \) > \( 2f_{\text{max}} \) (Signal bandwidth)

- Spectrum for digital signals, \( x[n] \)
  - Normalized Frequency

\[
\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s} + 2\pi \ell
\]
PROCESSING GOALS:

- We need to change $x(t)$ into $y(t)$ for many engineering applications:
  - For example, more BASS, image deblurring, denoising, etc
System IMPLEMENTATION

- ANALOG/ELECTRONIC:
  - Circuits: resistors, capacitors, op-amps

- DIGITAL/MICROPROCESSOR
  - Convert $x(t)$ to numbers stored in memory
**SAMPLING** $x(t)$

- **SAMPLING PROCESS**
  - Convert $x(t)$ to numbers $x[n]$
  - “$n$” is an integer; $x[n]$ is a sequence of values
  - Think of “$n$” as the storage address in memory

- **UNIFORM SAMPLING at** $t = nT_s$
  - IDEAL: $x[n] = x(nT_s)$
SAMPLING RATE, $f_s$

- **SAMPLING RATE ($f_s$)**
  - $f_s = 1/T_s$
    - NUMBER of SAMPLES PER SECOND
  - $T_s = 125$ microsec $\Rightarrow f_s = 8000$ samples/sec
    - UNITS ARE HERTZ: 8000 Hz

- **UNIFORM SAMPLING at** $t = nT_s = n/f_s$
  - IDEAL: $x[n] = x(nT_s) = x(n/f_s)$

\[ x(t) \xrightarrow{A-to-D} x[n] = x(nT_s) \]
Continuous Waveform: \( x(t) = \cos(2\pi 100t) \)

Sampled Signal: \( x[n] = x(nT_s) = \cos(2\pi 100nT_s) \), with \( T_s = 0.0005 \) when \( f_s = 2 \text{ kHz} \)

Sampled Signal: \( x[n] = x(nT_s) = \cos(2\pi 100nT_s) \), with \( T_s = 0.002 \) when \( f_s = 500 \text{ Hz} \)
SAMPLING THEOREM

- HOW OFTEN?
  - DEPENDS on FREQUENCY of SINUSOID
  - ANSWERED by NYQUIST/SHANNON Theorem
  - ALSO DEPENDS on “RECONSTRUCTION”

*Shannon Sampling Theorem*

A continuous-time signal $x(t)$ with frequencies no higher than $f_{\text{max}}$ can be reconstructed exactly from its samples $x[n] = x(nT_s)$, if the samples are taken at a rate $f_s = 1/T_s$ that is greater than $2f_{\text{max}}$. 
Reconstruction? Which One?

Given the samples, draw a sinusoid through the values

\[ x[n] = \cos(0.4\pi n) \]

When \( n \) is an integer
\[ \cos(0.4\pi n) = \cos(2.4\pi n) \]
STORING DIGITAL SOUND

- $x[n]$ is a SAMPLED SINUSOID
  - A list of numbers stored in memory
- EXAMPLE: audio CD
- CD rate is 44,100 samples per second
  - 16-bit samples
  - Stereo uses 2 channels
- Number of bytes for 1 minute is
  - $2 \times \frac{16}{8} \times 60 \times 44100 = 10.584 \text{ Mbytes}$
DISCRETE-TIME SINUSOID

- Change $x(t)$ into $x[n]$  

\[ x(t) = A \cos(\omega t + \varphi) \]
\[ x[n] = x(nT_s) = A \cos(\omega nT_s + \varphi) \]
\[ x[n] = A \cos((\omega T_s)n + \varphi) \]
\[ x[n] = A \cos(\hat{\omega} n + \varphi) \]

\[ \hat{\omega} = \omega T_s = \frac{\omega}{f_s} \]

**DEFINE DIGITAL FREQUENCY**
**DIGITAL FREQUENCY**

- \( \hat{\omega} \) VARIES from 0 to \( 2\pi \), as \( f \) varies from 0 to the sampling frequency
- UNITS are radians, **not** rad/sec
- DIGITAL FREQUENCY is **NORMALIZED**

\[ \hat{\omega} = \omega T_s = \frac{2\pi f}{f_s} \]
\[ \hat{\omega} = 2\pi \frac{f}{f_s} \]

\[ f_s = 1 \text{kHz} \]

\[ x[n] = A\cos(2\pi(100)(n/1000) + \phi) \]

100-Hz Cosine Wave: Sampled with \( T_s = 1 \text{ msec} \) (1000 Hz)
\[ \hat{X} = 2 \frac{f}{f_s} \]

\[ f_s = 100 \text{ Hz} \]

\[ X^n = A \cos(2\pi(100)(n/100) + \varphi) \]

100-Hz Cosine Wave: Sampled with \( T_s = 10 \text{ msec} \) (100 Hz)

\[ x[n] \] is zero frequency???
The REST of the STORY

- Spectrum of $x[n]$ has more than one line for each complex exponential
  - Called **ALIASING**
  - **MANY SPECTRAL LINES**

- SPECTRUM is PERIODIC with period $= 2\pi$
  - Because

$$A \cos(\hat{\omega}n + j) = A \cos((\hat{\omega} + 2\pi)n + j)$$
ALIASING DERIVATION

- Other Frequencies give the same

\[ x_1(t) = \cos(400\pi t) \quad \text{sampled at } f_s = 1000\text{Hz} \]
\[ x_1[n] = \cos(400\pi \frac{n}{1000}) = \cos(0.4\pi n) \]
\[ x_2(t) = \cos(2400\pi t) \quad \text{sampled at } f_s = 1000\text{Hz} \]
\[ x_2[n] = \cos(2400\pi \frac{n}{1000}) = \cos(2.4\pi n) \]
\[ x_2[n] = \cos(2.4\pi n) = \cos(0.4\pi n + 2\pi n) = \cos(0.4\pi n) \]

\[ \Rightarrow x_2[n] = x_1[n] \]

\[ 2400\pi - 400\pi = 2\pi(1000) \]
ALIASING DERIVATION–2

- Other Frequencies give the same

If \( x(t) = A \cos(2\pi(f + \frac{n}{f_s})t + \varphi) \)
and we want : \( x[n] = A\cos(\hat{\omega}n + \varphi) \)

then: \( \hat{\omega} = \frac{2\pi(f + \frac{n}{f_s})}{f_s} = \frac{2\pi f}{f_s} + \frac{2\pi n}{f_s} \)

\( \hat{\omega} = \omega T_s = \frac{2\pi f}{f_s} + 2\pi \ell \)
ALIASING CONCLUSIONS

- ADDING $f_s$ or $2f_s$ or $-f_s$ to the FREQ of $x(t)$ gives exactly the same $x[n]$
  - The samples, $x[n] = x(n/ f_s)$ are EXACTLY THE SAME VALUES

- GIVEN $x[n]$, WE CAN’T DISTINGUISH $f_o$ FROM $(f_o + f_s)$ or $(f_o + 2f_s)$
DIGITAL FREQUENCY

Normalized Radian Frequency

\[ \hat{\omega} = \omega T_s = \frac{2\pi f}{f_s} + 2\pi \ell \]

Normalized Cyclic Frequency

\[ \hat{f} = \frac{\hat{\omega}}{2\pi} = \frac{f T_s}{f_s} = \frac{f}{f_s} \]
SPECTRUM for $x[n]$

- PLOT versus NORMALIZED FREQUENCY
- INCLUDE ALL SPECTRUM LINES
  - ALIASES
    - ADD MULTIPLES of $2\pi$
    - SUBTRACT MULTIPLES of $2\pi$
  - FOLDED ALIASES
    - (to be discussed later)
    - ALIASES of NEGATIVE FREQS
\[ \hat{\omega} = 2\pi \frac{f}{f_s} \]

\[ f_s = 1\, \text{kHz} \]

\[ x[n] = A\cos(2\pi(100)(n/1000) + \phi) \]

100-Hz Cosine Wave: Sampled with \( T_s = 1 \, \text{msec} \) (1000 Hz)
SPECTRUM (ALIASING CASE)

\[ \hat{\omega} = 2\pi \frac{f}{f_s} \]

\[ f_s = 80 \text{Hz} \]

\[ x[n] = A\cos(2\pi(100)(n/80) + \varphi) \]

100-Hz Cosine Wave: Sampled with \( T_s = 12.5 \text{ msec (80 Hz)} \)
SAMPLING GUI (con2dis)
\[ \hat{w} = 2 \frac{f}{f_s} \]

\[ f_s = 125 \text{Hz} \]

\[ x[n] = A\cos(2\pi(100)(n/125) + \phi) \]

100-Hz Cosine Wave: Sampled with \( T_s = 8 \text{ msec} \) (125 Hz)