I. CmpE 300 Homework I Answers

A. Question I

$f(n)$ is given as:

$$f(n) = n^3 log(n!) \tag{1}$$

and Stirling's approximation is as follows:

$$n! \approx \sqrt{2\pi n}\frac{n^n}{e^n} \tag{2}$$

Using Stirling's approximation, we can find an approximate value for $f(n)$.

$$f(n) \approx n^4 log(n) - n^4 log(e) + \frac{1}{2}n^3 log(n) + \frac{1}{2}n^3 log(2\pi) \tag{3}$$

Once this term is found, it is easy to verify the statements given in the question.

1) False.
   The reason is that the term $loglog(n)$ grows exponentially slower than $log(n)$. Hence, the term $n^4 loglog(n)$ cannot be an upper bound for the fastest growing term of $f(n)$, which is $n^4 log(n)$. So, formally, we cannot find constants c and n_0, such that $f(n) \le cn^4 loglog(n)$ for all $n \ge n_0$.
2) True.
   The term $n^4 log^2(n)$ grows $log(n)$ times faster than the fastest growing term of $f(n)$, which is $n^4 log(n)$. Therefore, $n^4 log^2(n)$ is an upper bound for $f(n)$. Note that this bound does not necessarily have to be tight.
3) True.
   $n^4 log(n)$ is a tight bound on $f(n)$, and we should be able to find two constants c and n_0, such that $f(n) \ge cn^4 log(n)$ for all $n \ge n_0$. Note that c should be smaller than 1, as the second fastest term of $f(n)$ is negative and the coefficient of the fastest term in $f(n)$ is 1. So, we choose c as 0.5, and n_0 as 3, as $f(3)$ for log base 2 is 68.72, whereas $cn^4 log(n)$ is 64.19. The relationship holds for larger values of n.
4) True.
   This is simple to check, since $o(n^5)$ grows much faster than $n^4 log(n)$. Therefore:
   $$\underbrace{lim}_{n\to\infty} \frac{f(n)}{n^5} = 0 \tag{4}$$
5) False.
   $O(n^4)$ is not an upper bound for $f(n)$, as the fastest growing term of $f(n)$, namely $n^4 log(n)$ grows faster than n^4. Therefore, it is not possible to find constants c and n_0 such that $f(n) \le cn^4$ for all $n \ge n_0$.

B. Question II

The code given in the question consists of five different for loops with indices i, k, l, j and m. The corresponding loops are referred here as I, K, L, J and M loops.

Let $|X|$ be the cardinality of the set of possible values the index of loop X can have. Thus, for instance, $|I| = n$. According to this definition, the following holds:

$$\begin{aligned} |I| &= n && (5)\\ |K| = |L| = |M| &= \frac{3n-i}{3} && (6)\\ |J| &= 3n - i && (7)\end{aligned}$$

So, as i increases from 0 to $3n$, k, l and m cover the rest of the values from i to $3n$ with equal proportions. Also note that the value of j is independent from l, the index if its parent loop. The following summation rules will be useful in determining the time complexity:

$$\begin{aligned} \sum_{t=1}^{N} 1 &= N && (8)\\ \sum_{t=1}^{N} t &= \frac{N(N+1)}{2} && (9)\\ \sum_{t=1}^{N} t^2 &= \frac{N(N+1)(2N+1)}{6} && (10)\end{aligned}$$

i	$\|K\|$	$\|L\|$	$\|M\|$	$\|J\|$
0	n	n	n	3n
3	n-1	n-1	n-1	3n-3
6	n-2	n-2	n-2	3n-6
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
3n-3	1	1	1	3

The following table shows the number of executions of loops for specific values of i:

The number of times K and M loops are executed can be found simply, by using the rule given in Equation 5, to be $\frac{n(n+1)}{2}$. Moreover the effects of loops K and M on $f(n)$ eliminate each other. The number of times L is executed is of no interest for this question, since we are allowed to ignore the index updates. The number of executions of J equals $|L||J|$ for each i.

i	$\|L\|\|J\|$
0	$3n^2$
3	(n-1)(3n-3)
6	(n-2)(3n-6)
$\vdots$	$\vdots$
3n-3	3

So, the number of executions of loop J can be found via formula:

$$\sum_{t=0}^{n-1}(n-t)(3n-3t) = \tag{11}$$

$$3\sum_{t=0}^{n-1}(n-t)^2 = \tag{12}$$

$$3\sum_{t=1}^{n}t^2 = \frac{n(n+1)(2n+1)}{2} \tag{13}$$

The total time complexity is the summation of execution times of $|K|$, $|J|$ and $|M|$, which is:

$$\frac{n(n+1)}{2} + \frac{n(n+1)}{2} + \frac{n(n+1)(2n+1)}{2} = \frac{n(n+1)(2n+3)}{2} \tag{14}$$

$$= \; = \Theta(n^3) \tag{15}$$

The only effect on $f(n)$ comes from the J loop. Therefore:

$$f(n) = \frac{n(n+1)(2n+1)}{2} \tag{16}$$