Cmpe 300 - Analysis of Algorithms
Spring 2013
Assignment 1

Due Date: 01/04/2013 17:00

Question 1 (20 Points)

Consider the function $f(n) = 1 + k + k^2 + k^3 + \ldots + k^n$ where k is a positive real number. Calculate the tightest bound on the growth of this function ($f(n) = \Theta(\ldots)$) for the following cases:

(a) $k < 1$

(b) $k = 1$

(c) $k > 1$

Answer:

The sum of geometric series is written as (when $k \neq 1$):

$$f(n) = 1 + k + k^2 + k^3 + \ldots + k^n = \frac{1 - k^{n+1}}{1 - k}$$

(a) For $k < 1$,

$$\frac{1 - k^{n+1}}{1 - k} < \frac{1}{1 - k}$$

since k is positive. We can see that $(f(n) \in \Theta(1)$. We can find constants c_1, c_2 and n_0 such that

$$c_1 g(n) \leq f(n) \leq c_2 g(n)$$

where $g(n) = 1$, $c_1 = 1$, $c_2 = \frac{1}{1-k}$, and $n_0 = 1$.

(b) For $k = 1$, $f(n) \in \Theta(n)$

$$f(n) = 1 + 1 + 1^2 + 1^3 + \ldots + 1^n = n + 1$$

It is clear that $f(n) \in \Theta(n)$. We can use the following constants $c_1 = 1, c_2 = 2, n_0 = 1$ for $g(n) = n$.

(c) For $k > 1$,

$$f(n) = 1 + k + k^2 + k^3 + \ldots + k^n$$

The series grows with k^n. Hence $f(n) \in \Theta(k^n)$. We can use the constants $c_1 = 1, c_2 = k, n_0 = 1$ for $g(n) = k^n$.

Question 2 (40 Points)

Given a list $L[1:8]$ and a search element K, calculate the number of comparisons for the best, wost and average cases of searching K in the list with *Linear Search* with the following assumption: The probability of finding K at the position x is given by $p(x) = 2^{x-9}$, and the probability that K is not in the list is given by $p(0) = 2^{-8}$.

Answer:

If the search element K is at the position x, we need to make x comparisons to find it. If the search element is not in the list, we need to make maximum number of comparisons. The average number of comparisons according to the assumed probability distribution is

$$\begin{aligned} C_{avg} &= 8p(0) + \sum_{x=1}^{8} xp(x) \\ &= 8(2^{-8}) + 1(2^{-8}) + 2(2^{-7}) + 3(2^{-6}) + \ldots + 8(2^{1}) \\ &= 7.0352 \end{aligned}$$

The best and worst cases do not depend on this probability distribution. In the best case the search element is the first element, there will be 1 comparison for it. In the worst case the search element is either the last element or not in the list. In both cases we have to make 8 comparions.

Question 3 (40 Points)

Consider the given function $f(n)$ and determine whether the following case are true or false. Justify your answers formally. *(Hint: Use Stirling's Approximation)*

$$f(n) = n^2 + n\log(n!)$$

a) $f(n) \in O(n^2)$

b) $f(n) \in o(n^2 \log(n))$

c) $f(n) \in \Omega(n^2)$

d) $f(n) \in \Theta(n^2 \log(n))$

Answer:

We use Stirling's approximation to simplify the term $log(n!)$.

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

The function becomes:

$$\begin{aligned} f(n) &\approx n^2 + n\log\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right) \\ &= n^2 + n\left(+\frac{1}{2}\log(2\pi n) + n\log(n) - n\log(e)\right) \\ &= n^2 + \frac{1}{2}n\log(2\pi n) + n^2\log(n) + n^2\log(e) \\ &= n^2\log(n) + 2n^2 + \frac{1}{2}n\log(2\pi n) + \log e \end{aligned}$$

The function has 4 terms, all positive and monotonically increasing.

a) $f(n) \in O(n^2)$ **: False**

We can directly show that there is no c and n_0 such that $f(n) \leq cn^2$. Let's look at the term $n^2 \log(n)$.

$$\begin{aligned} n^2 \log(n) &\leq cn^2 \quad ,\forall n \geq n_0 \\ \log(n) &\leq c \end{aligned}$$

$log(n)$ is a monotonically increasing function and c is constant. For all $\{c, n_0\}$ pairs, there is an n value, which is greater than n_0, which makes $\log(n) > c$. We don't need to look at other terms.

b) $f(n) \in o(n^2 \log(n))$ **: False**

We use the definition of *little-oh*:

$$f(n) \in o(g(n)) \quad \rightarrow \quad \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

$$\begin{aligned} \lim_{n\to\infty} \frac{f(n)}{n^2 \log(n)} &= \lim_{n\to\infty} \frac{n^2 \log(n)}{n^2 \log(n)} + \lim_{n\to\infty} \frac{2n^2}{n^2 \log(n)} + \lim_{n\to\infty} \frac{\frac{1}{2} n \log(2\pi n)}{n^2 \log(n)} + \lim_{n\to\infty} \frac{\log e}{n^2 \log(n)} \\ &= \lim_{n\to\infty} \frac{n^2 \log(n)}{n^2 \log(n)} = 1 \end{aligned}$$

therefore, $f(n) \notin o(n^3 \log(n))$.

c) $f(n) \in \Omega(n^2)$**: True**

It is very easy to show directly. If we can find c and n_0 such that $cg(n) \leq f(n)$ for all $n \geq n_0$, then $f(n) \in \Omega(n^2)$. Consider the term $n^2 \log(n)$:

$$\begin{aligned} cn^2 &\leq n^2 \log(n) \\ c &\leq \log(n) \end{aligned}$$

Let $c = 1$. $log(n) \geq 1$ for all values of $n \geq 2$ ($n_0 = 2$). This also implies that $cg(n) \leq f(n)$ since the remaining terms of $f(n)$ are positive. The proof is completed.

d) $f(n) \in \Theta(n^2 \log(n))$**: True**

We can find c_1, c_2 and n_0 such that

$$c_1(n^2 \log(n)) \leq f(n) \leq c_2(n^2 \log(n)) \quad \forall n \geq n_0$$

The function $f(n)$ has 4 terms and with $n^2 \log(n)$ as the greatest order. Let $n_0 = 4$. Then,

$$\begin{aligned} 2n^2 &\leq n^2 \log(n) \\ \frac{1}{2} n \log(2\pi n) &\leq n^2 \log(n) \\ \log e &\leq n^2 \log(n) \end{aligned}$$

for all $n \geq n_0$. Therefore, we can let $c_2 = 6$. An obvious choice for c_1 is 1. It is clear that for all $n \geq n_0 = 4$ the condition holds. Therefore $f(n) \in \Theta(n^2 \log(n))$.