Cmpe 300 - Analysis of Algorithms
Fall 2010
Assignment 1 Answers

Question 1

Part a

There are 4 loops in the function with indexes i, j, k, and l, let's call these loops I, J, K and L respectively. Clearly loop I executes n times. Let's make a table for the number of executions of the loops for all values of index i, (ignoring the conditional statement):

i	$\|J\|$	$\|K\|$	$\|L\|$	$\|J\|\|K\|$
0	n	$\lfloor \log(n) \rfloor + 1$	n	$n(\lfloor \log(n) \rfloor + 1)$
1	$n-1$	$\lfloor \log(n) \rfloor + 1$	$n-1$	$(n-1)(\lfloor \log(n) \rfloor + 1)$
2	$n-2$	$\lfloor \log(n) \rfloor + 1$	$n-2$	$(n-2)(\lfloor \log(n) \rfloor + 1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n-1	1	$\lfloor \log(n) \rfloor + 1$	1	$\lfloor \log(n) \rfloor + 1$

Whenever the $x_i = 1$, loops J and K are executed, otherwise loop L is executed. It can be seen that whenever the $x_i = 1$, the algorithm executes more operations. Therefore for the best case analysis the input X is a stream of all zeros, whereas for the worst case, X is a stream of all ones.

Best Case:

Input X is:

$$X = [\; 1 \;\; 1 \;\; \dots \;\; 1 \;]$$

The number of executions:

$$\begin{aligned} B(n) &= \sum_{i=0}^{n-1}(n-i) = n^2 - \sum_{i=0}^{n-1} i = n^2 - \frac{n(n-1)}{2} \\ &= \frac{n(n+1)}{2} = \frac{1}{2}n^2 + \frac{1}{2}n \\ &\in \Theta(n^2) \end{aligned}$$

Worst Case:

Input X is:

$$X = [\ 0 \quad 0 \quad \dots \quad 0\ ]$$

The number of executions:

$$\begin{aligned}
W(n) &= \sum_{i=0}^{n-1}(n-i)(\lfloor\log(n)\rfloor+1) = (\log(n)+1)\sum_{i=0}^{n-1}(n-i) = \frac{n(n+1)(\lfloor\log(n)\rfloor+1)}{2} \\
&= \frac{1}{2}n^2(\lfloor\log(n)\rfloor+1) + \frac{1}{2}n(\lfloor\log(n)\rfloor+1) \\
&\in \Theta(n^2\log(n))
\end{aligned}$$

Average Case:

We assume that the input is uniform. Which means $p(x_i = 0) = \frac{1}{2}$ and $p(x_i = 1) = \frac{1}{2}$. Then, the average number of executions per step becomes:

$$\frac{1}{2}(n-i) + \frac{1}{2}(n-i)(\lfloor\log(n)\rfloor+1)$$

Therefore,

$$\begin{aligned}
A(n) &= \sum_{i=0}^{n-1}\frac{1}{2}((n-i)+(n-i)(\lfloor\log(n)\rfloor+1)) \\
&= \frac{1}{2}\left(\frac{n(n+1)}{2} + \frac{n(n+1)(\lfloor\log(n)\rfloor+1)}{2}\right) \\
&= \frac{1}{4}(n^2+n)(\lfloor\log(n)\rfloor+2) \\
&\in \Theta(n^2\log(n))
\end{aligned}$$

Question 2

We use Stirling's approximation to simplify the term $log(2n!)$.

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

The function becomes:

$$\begin{aligned}
f(n) &\approx n^2\log\left(2\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right) + 3n^3 + \sqrt{n} \\
&= n^2\left(\log 2 + \frac{1}{2}\log(2\pi n) + n\log(n) - n\log(e)\right) + 3n^3 + \sqrt{n} \\
&= n^3\log(n) + 3n^3 + n^3\log(e) + \frac{1}{2}n^2\log(2\pi n) + n^2 + \sqrt{n}
\end{aligned}$$

The function has 6 terms, all positive and monotonically increasing.

a) $f(n) \in O(n^3)$ **: False**

We can directly show that there is no c and n_0 such that $f(n) \leq cn^3$. Let's look at the term $n^3 \log(n)$.

$$\begin{aligned} n^3 \log(n) &\leq cn^3 \quad , \forall n \geq n_0 \\ \log(n) &\leq c \end{aligned}$$

$log(n)$ is a monotonically increasing function and c is constant. For all $\{c, n_0\}$ pairs, there is an n value, which is greater than n_0, which makes $\log(n) > c$. We don't need to look at other terms.

b) $f(n) \in o(n^3 \log(n))$ **: False**

We use the definition of *little-oh*:

$$f(n) \in o(g(n)) \quad \rightarrow \quad \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

$$\begin{aligned} \lim_{n\to\infty} \frac{f(n)}{n^3 \log(n)} &= \lim_{n\to\infty} \frac{n^3 \log(n)}{n^3 \log(n)} + \lim_{n\to\infty} \frac{3n^3}{n^3 \log(n)} + \lim_{n\to\infty} \frac{n^3 \log(e)}{n^3 \log(n)} \\ &+ \lim_{n\to\infty} \frac{\frac{1}{2}n^2 \log(2\pi n)}{n^3 \log(n)} + \lim_{n\to\infty} \frac{n^2}{n^3 \log(n)} + \lim_{n\to\infty} \frac{\sqrt{n}}{n^3 \log(n)} \\ &= \lim_{n\to\infty} \frac{n^3 \log(n)}{n^3 \log(n)} = 1 \end{aligned}$$

therefore, $f(n) \notin o(n^3 \log(n))$.

c) $f(n) \in \Theta(n^3 \log(n))$**: True**

We can find c_1, c_2 and n_0 such that

$$c_1(n^3 \log(n)) \leq f(n) \leq c_2(n^3 \log(n)) \quad \forall n \geq n_0$$

The function $f(n)$ has 6 terms and with $n^3 \log(n)$ as the greatest order. Let $n_0 = 8$. Then,

$$\begin{aligned} 3n^3 &\leq n^3 \log(n) \\ n^3 \log(e) &\leq n^3 \log(n) \\ \frac{1}{2}n^2 \log(2\pi n) &\leq n^3 \log(n) \\ n^2 &\leq n^3 \log(n) \\ \sqrt{n} &\leq n^3 \log(n) \end{aligned}$$

for all $n \geq n_0$. Therefore, we can let $c_2 = 6$. An obvious choice for c_1 is 1. It is clear that for all $n \geq n_0 = 8$ the condition holds. Therefore $f(n) \in \Theta(n^3 \log(n))$.

d) $f(n) \in \Omega(n^3)$**: True**

It is very easy to show directly. If we can find c and n_0 such that $cg(n) \leq f(n)$ for all $n \geq n_0$, then $f(n) \in \Omega(n^3)$. Consider the term $n^3 \log(n)$:

$$\begin{aligned} cn^3 &\leq n^3 \log(n) \\ c &\leq n^3 \log(n) \end{aligned}$$

Let $c = 1$. $log(n) \geq 1$ for all values of $n \geq 2$ ($n_0 = 2$). This also implies that $cg(n) \leq f(n)$ since the remaining terms of $f(n)$ are positive. The proof is completed.

Remark 1: You cannot just give numbers to c_1, c_2 and n_0. You have to justify why you choose them.

Remark 2: You can not prove that the necessary condition holds for a single assignment of constants. You have show that **for all** $n > n_0$ the condition holds.