The Four-Color Theorem

Ege Onur Tağa

Boğaziçi University-CMPE220

December 11, 2019

Overview

Introduction

Theorem

Any map can be colored using no more than four colors in such a way that no two adjacent regions share the same color.

Example

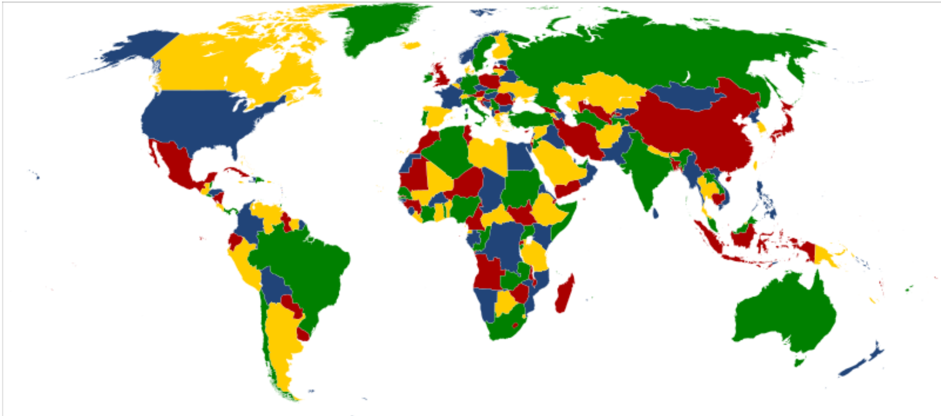


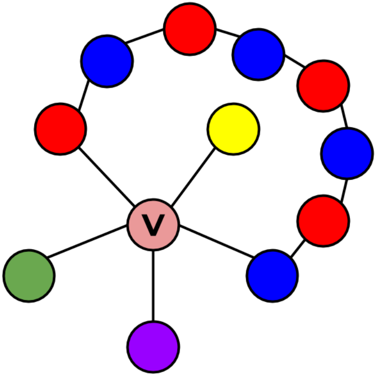
Figure: World map

It was a rainy day in London...

Being pretty bored and overwhelmed by unending chain of assignments, Francis Guthrie, a math graduate, gave up all his duties and started to color the map of Britain. He noticed that four colors sufficed. He asked his brother Frederick if it was true that any map can be colored using four colors in such a way that adjacent regions receive different colors.

Brother was unable to answer

Luckily, he was a student of Augustus De Morgan. Yet, De Morgan was unable to give an answer, however, as history shows he was instantly hooked. De Morgan tried and couldn't solve. Then he spread the problem across the mathematical world and eventually taught the problem to Cayley. Cayley taught the problem to Kempe, who used Kempe chains(it is a connected chain of points on a graph with alternating colors.) to prove the theorem(proof turned out to be flawed).



Formal Definitions in Graph Theory

Definition

A graph is said to be planar, if it can be drawn in the plane so that its edges intersect only at their ends.

Coloring

Given a graph G and K colors, assign a color to each node so that no adjacent nodes get different colors.

Definition

The minimum value of K for which such a coloring exist is the chromatic number of G.

Example

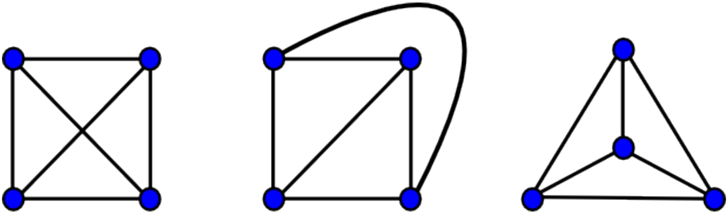


Figure: Planar graph

Example

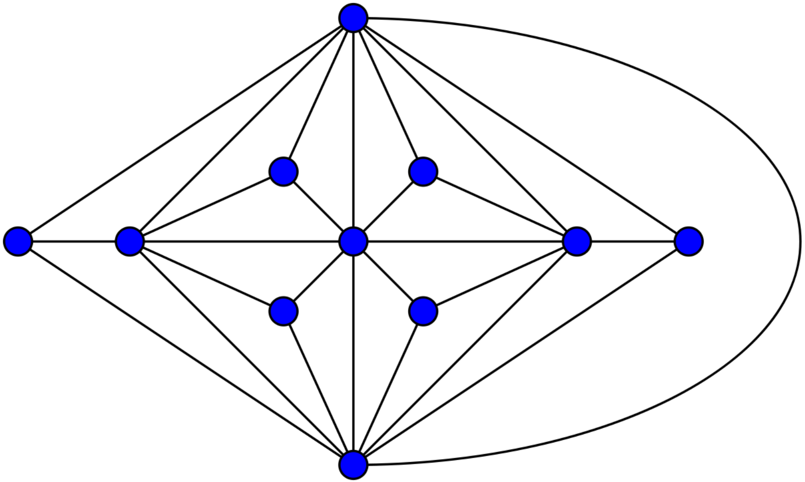


Figure: Planar graph

Formal Definitions in Graph Theory

Restatement of Four Color Theorem

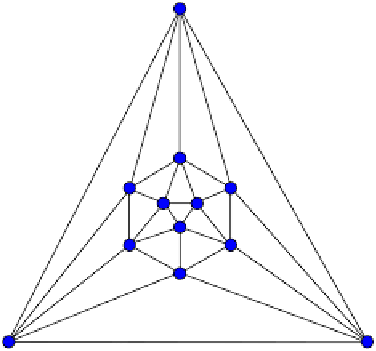
The chromatic number of a planar graph is at most 4.

Definition

Suppose that G is a graph, with edge set E, and we have a colouring function $c\colon E \to S$. If e is an edge assigned colour a, then the (a, b)-Kempe chain of G containing e is the maximal connected subset of E which contains e and whose edges are all coloured either a or b.

Five Color Theorem

Kempe's proof was flawed since in his proof, he assumed planar Graph G has a vertex connected to at most four vertices. In fact, it is not necessarily true that G has a vertex connected to at most four edges; for example, the graph corresponding to an icosahedron is planar and has every vertex connected to five others. However, his methods were used in proving five color theorem. see: Link



Icosohedral Graph

The eventual proof

The eventual proof utilized minimality: it was shown that any counterexample must contain one of a set of graphs as a subgraph, but that if any of those subgraphs were part of a counterexample, a smaller counterexample existed. This demonstrated the result by showing that there cannot be any smallest counterexample, so there cannot be any counterexample at all.

The eventual proof

Definition

An unavoidable set is a set of graphs such that any smallest counterexample to the four color theorem must contain at least one of the graphs as a subgraph.

Definition

A reducible configuration is a graph with the following property: any map containing a reducible configuration can be reduced to a smaller map, satisfying the condition that if the smaller map can be colored with 4 colors, the original map can also be colored with 4 colors.

The eventual proof

The year 1976 saw a complete solution to the Four Colour Conjecture when it was to become the Four Colour Theorem. The proof was achieved by Appel and Haken, basing their methods on reducibility using Kempe chains. They constructed an unavoidable set with around 1500 configurations. Appel and Haken used 1200 hours of computer time to work through the details of the final proof. Koch assisted Appel and Haken with the computer calculations.

Why this proof is important to us?

The Four Colour Theorem was the first major theorem to be proved using a computer, having a proof that could not be verified directly by other mathematicians. In 1996, Neil Robertson and Sanders found faster algorithm for the same problem. see: Link

Resources

- https://people.math.gatech.edu/ thomas/FC/fourcolor.htmlHistory
- https://brilliant.org/wiki/four-color-theorem/
- Bondy, JA, Graph Theory, Springer