MIDDLE EAST TECHNICAL UNIVERSITY

FRESHMAN CALCULUS

BOOK TWO
part two

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FRESHMAN

CALCULUS

BOOK TWO

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Sayın
Haluk Bingöl'e
sevgi ve başarı dilek-
lerimle.

B. SÜER & H. DEMİR

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CONTENTS

CHAPTER 4
DIFFERENTIATION
(FUNCTIONS OF SEVERAL VARIABLES)

4. I. FUNCTIONS, LIMIT, CONTINUITY

A. FUNCTIONS, DOMAIN, RANGE, GRAPH

Let D be a non empty subset of $\mathbb{R}^2$ and let further be not a set of finite number of points or not a curve. Then a relation from D to $\mathbb{R}$ is called a *function of two* (independent) *variables*, if to each point $P(x, y) \in D$ there is assigned a single value $z = f(x, y)$.

$z = f(x, y)$ is the *image* of $P(x, y)$ under f or the *value* of f at $P(x, y)$.

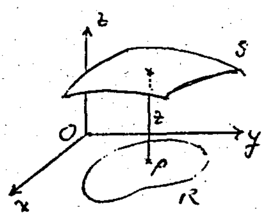
The notation for a function defined by the rule $z = f(x, y)$ is

$$f: D \to \mathbb{R}, \quad z = f(x, y)$$

where D is the *domain* of f, written Dom f or D_f. The set of all images as a subset of $\mathbb{R}$ is the *range* of f.

In the function $z = f(x, y)$, the variables x is the *abscissa* or the *first variable*, y is the *ordinate* or the *second variable*, and z is the *dependent variable*. x and y very independently of each other over D.

The graph of $z = f(x, y)$ is a surface S (any line parallel to z-axis intersects it at one point atmost).
(See Book II, Part 1, § 3. 3).



The following are familiar examples of functions given explicitly:

$V = \frac{1}{3} \pi R^2 h$ (Volume of a cone)

$E_k = \frac{1}{2} m v^2$ (kinetic energy)

$V = RI$ (voltage)

A function of two variables may also be defined implicitly as $F(x, y, z) = 0$ (by a restriction). Unless otherwise stated it is considered that z is the dependent variable. However $F(x, y, z) = 0$ may define x (or y) as a function of the other two variables.

Example 1. Which ones of the following relations are functions. If not, write a restriction to be a function.

a) $x^2+y^2+z^2 = 16$ b) $x^2+y-z^2 = 0$

c) $-2z = 2x^2+y^2$ d) $z^3 = x$

Solution.

a) This relation is not a function, since $P(x, y)$ has more than one image. A restriction for this to be a function is $z>0$. Another restriction is, for instance, the point $(0, \sqrt{7}, -3)$ lies on the surface.

b) Same as in (a). A restriction is $z>5$. Observe that y is a function of x and z.

c) This is a function, since to each pair (x, y) there is assigned a single image, namely $z = -x^2-y^2/2$.

d) It is a function: $z = f(x, y) = \sqrt[3]{x}$.

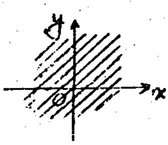
Example 2. Find and sketch the domains of the following functions:

a) $z = e^{xy}$ b) $z = \ln (xy)$

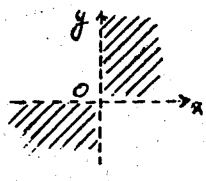
c) $z = \sqrt{1-x^2-y^2}$ d) $z = \dfrac{e^x}{1-y}$

Solution.

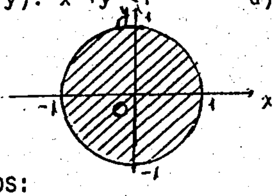
a) $D = \mathbb{R}^2$

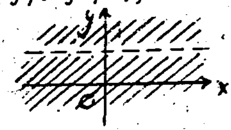


b) $D = \{(x, y): xy>0\}$



c) $D = \{(x, y): x^2+y^2<1$ d) $D = \{(x, y): y \neq 1\}$



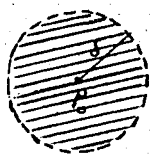


NEIGHBORHOODS:

Let $z = f(x, y)$ be a function of two variables, and $P_0(x_0, y_0)\epsilon R^2$. Then by a *δ-neighborhood* of P_0 is meant the set of all points (x, y) satisfying

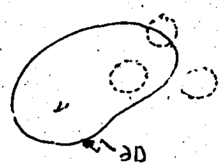
$$(x-x_0)^2 + (y-y_0)^2 < \delta^2 \qquad (\delta>0)$$

that is, the set of all points inside the circle with center P_0 and radius δ, written $N_\delta\left[P_0\right]$. This set excludes its boundary, that is, the points of the circumference. If also excludes the center P_0 one talks about the deleted neighborhood $N_\delta(P_0)$ of P_0.



δ-neighborhood of P_0

A point of D is an *interior (exterior) point,* if there exists a neighborhood all points of which are (are not) in D. A point of D is a *boundary point* if every neighborhood of that point contains interior and exterior points. The set of all boundary points of D is the *boundary*(*) of D, written ∂D which belongs or does not belong to D.



R^2 has no exterior points and no boundary. $N_\delta(P)$ has both interior and exterior points as well as boundary, but the boundary does not belong to $N_\delta(P)$.

(*) Domains will be taken closed (open) including (excluding) its boundary depending upon the function and we make no difference between the terms "domain" and "region".

Functions of more than two variables:

$$f: D \to \mathbb{R}, \qquad u = f(x, y, z)$$

represents a function of three variables, where D is a solid as a subset of $\mathbb{R}^3$, and x, y, z are the *first, second, third variable.*

Since the graph of f is a subset of $\mathbb{R}^4$ its sketching cannot be realized in $\mathbb{R}^3$.

More generally

$$f: D \to R, \qquad z = f(x_1, \ldots, x_n)$$

is a function of n variables, where $D \subseteq \mathbb{R}^n$ is the domain, and the image of the point $P(x_1, \ldots, x_n) \varepsilon \mathbb{R}^n$ is $z = f(P) \varepsilon \mathbb{R}$. The graph of f is a subset of $\mathbb{R}^{n+1}$.

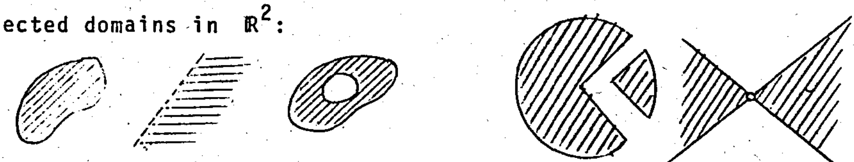
If $P_0 = (x_{1,0}, \cdots, x_{n,0}) \varepsilon D$, then the δ-neighborhood $N_\delta(P_0)$ will represent inside the sphere (hypersphere) with center at P_0 and radius δ, namely

$$(x_1 - x_{1,0})^2 + \ldots + (x_n - x_{n,0})^2 < \delta^2.$$

Connectedness of domains:

A domain is called *connected* iff any two of its interior points can be joined by a polygonal line (or by a continuous curve) lying entirely in the domain. Otherwise it is *disconnected.*

The following figures illustrate some connected and disconnected domains in $\mathbb{R}^2$:



(a) (b) (c) Connected domains

(d) (e) Disconnected domains

Two of the above connected domains, (a) and (b), are *simply connected.* A simply connected domain is a connected domain in which any closed curve Γ can be shrinked to a point in D. Observe that the connected domain (c) above is not simply connected.

This concept can be generalized to domains of functions of more than two variables.

B. Limit and Continuity

The function $f: D \to \mathbb{R}$, $z = f(x, y)$ is said to have the limit ℓ at $P_0(x_0, y_0) \varepsilon D$ if $|f(x, y) - \ell|$ can be made arbitrarily small by taking $P(x, y)$ sufficiently close to P_0, written

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = \ell \qquad (\lim_{P \to P_0} f(x, y) = \ell\ )$$

or

$$f(x, y) \to \ell \quad \text{as} \quad (x, y) \to (x_0, y_0)$$

In other words $f(x, y)$ has a limit ℓ at $P_0(x_0, y_0)$, if given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$$|f(x, y) - \ell| < \varepsilon$$

whenever

$$(x-x_0)^2 + (y-y_0)^2 < \delta^2 \quad \text{or} \quad P(x, y) \varepsilon N_\delta(P_0)$$

For a function of two (or more) variables, a point $P(x, y)$ may approach $P_0(x_0, y_0)$ in infinitely many directions. Recall that in the case of a function of a single variable the approach was only from right or left.

If any two directions give two distinct limits, it is obvious that the function has no limit. If $f(x, y)$ approach the same value ℓ independently from direction, that the value ℓ is (is not) the limit can be checked by the use of ε, δ.

It can be verified that polynomial and rational functions have limits at any point of their domain:

$$\lim_{(x,\, y)\to(x_0,\, y_0)} P(x, y) = P(x_0, y_0)$$

$$\lim_{(x,\, y)\to(x_0,\, y_0)} \frac{P(x, y)}{Q(x, y)} = \frac{P(x_0, y_0)}{Q(x_0, y_0)} \quad \text{if } \; Q(x_0, y_0) \neq 0$$

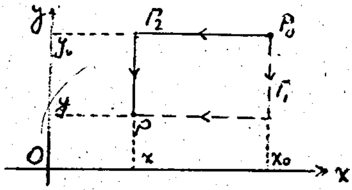
where

$$P(x, y) = \sum_{0,\,0}^{m,\,n} a_{pq}\, x^p\, y^q, \quad Q(x, y) = \sum_{0,\,0}^{r,\,s} b_{pq}\, x^p\, y^q$$

Evaluation of Limit:

A point $P(x, y)$ may approach $P_0(x_0, y_0)$ in the following ways

1)



$$\lim_{y\to y_0}\left(\lim_{x\to x_0} f(x, y)\right)$$

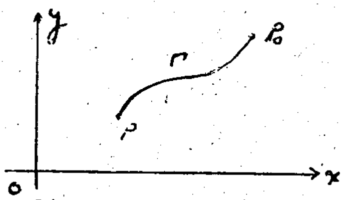
along Γ_1

$$\lim_{x\to x_0}\left(\lim_{y\to y_0} f(x, y)\right)$$

along Γ_2.

If one of these iterated limits does not exist or if both exist but unequal, the function has no limit.

2)



$$\lim_{(x,\, y)\to(x_0,\, y_0)} f(x, y)$$

along Γ: $y = f(x)$

If it does not exist the function has no limit. If exists, it is the limit of the function along that curve. To show its independence of direction, one uses the substitution $x = x_0 + r\cos\theta$, $y = y_0 + r\sin\theta$ and find limit (independent of θ) as $r \to 0$.

In the case of existence of limit along a curve one may verify it to be the limit of the function by the use of ε, δ definition.

Example 1. Evaluate

$$\lim_{(x,\ y)\to(-1,\ 3)} (x^2-y)$$

Solution. Since x^2-y is a polynomial, we have

$$\lim_{(x,\ y)\to(-1,\ 3)} (x^2-y) = 1-3 = -2$$

Example 2. Show that

$$\lim_{(x,\ y)\to(0,\ 0)} \frac{x+y}{x-y}$$

does not exist by two ways.

Solution.

1) $$\lim_{y\to 0} \left(\lim_{x\to 0} \frac{x+y}{x-y}\right) = \lim_{y\to 0} \left(\frac{y}{-y}\right) = \lim_{y\to 0} (-1) = -1$$

$$\lim_{x\to 0} \left(\lim_{y\to 0} \frac{x+y}{x-y}\right) = \lim_{x\to 0} \left(\frac{x}{x}\right) = \lim_{x\to 0} (1) = 1$$

2) $$\lim_{r\to 0} \frac{r\cos\theta + r\sin\theta}{r\cos\theta - r\sin\theta} = \lim_{r\to 0} \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} = \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$$

dependent on θ.

Example 3. Evaluate the following limit

$$\lim_{(x,\ y)\to(1,\ 1)} \frac{y \sin\pi x}{x+y-z}$$

along the curve $y = x^3$.

Solution.

$$\lim_{(x,\ y)\to(1,\ 1)} \frac{y \sin\pi x}{x+y-z} = \lim_{x \to 1} \frac{x^3 \sin\pi x}{x+x^3-2} = \left[\frac{0}{0}\right]$$

$$= \lim_{x \to 1} \frac{3x^2 \sin\pi x + x^3 \cos\pi x}{1 + 3x^2} = -\frac{\pi}{4}.$$

Example 4. If

$$f(x,\ y) = \frac{2x^2 - 3xy + y^2}{2x^2 + xy - y^2} \qquad (1)$$

show that

a) lim $f(x, y)$ at $(1, 2)$ is independent of direction,

b) the limit obtained in a) is the limit.

Solution. Setting

$$x = 1+r\cos\theta, \quad y = 2+r\sin\theta$$

in (1) and replacing $\cos\theta$, $\sin\theta$ by c, s, then (1) becomes

$$f(1+rc,\ 2+rs) = \frac{2(1+rc)^2-3(1+rc)(2+rs)+(2+rs)^2}{2(1+rc)^2 + (1+rc)(2+rs)-(2+rs)^2}$$

$$= \frac{\cancel{r}(4c+2rc^2-6c-3s-3rs+4s+rs^2)}{\cancel{r}(4c+2rc^2+s+2c+rcs-4s-rs^2)}$$

and we have

$$\lim_{r \to 0} f(1+rc,\ 2+rs) = \frac{-2c+s}{6c-3s} = -\frac{1}{3}$$

b) $$\left|f(x,\ y)-(-\tfrac{1}{3})\right| = \left|\frac{2x^2-3xy+y^2}{2x^2+xy-y^2} + \frac{1}{3}\right|$$

$$= \left|\frac{8x^2-8xy+2y^2}{2x^2+xy-y^2}\right| = 2\left|\frac{(2x-y)^2}{(2x-y)(x+y)}\right|$$

$$= \frac{2|2x - y|}{|x+y|}$$

Let $\varepsilon>0$. We need to find $\delta(\varepsilon)>0$ such that for all (x, y) in

$$(x-1)^2 + (y-2)^2 < \delta^2 \qquad (a)$$

on has $2\frac{|2x-y|}{|x+y|} < \varepsilon$

Since (a) $\Rightarrow$ $|x-1| < \delta$, $|y-2| < \delta$ or $1-\delta < x < 1+\delta$, $2-\delta < y < 2+\delta$, we have

$$2\,\frac{|2x-y|}{|x+y|} < 2\,\frac{2(1+\delta)-(2-\delta)}{|1-\delta+2-\delta|} \quad \frac{6\delta}{|3-2\delta|}.$$

Then

$$\frac{6\delta}{3-2\delta} < \varepsilon \Rightarrow \delta < \frac{3\varepsilon}{6+2\varepsilon} \quad \text{for} \quad 3-2\delta > 0$$

Any δ satisfying the inequality proves that $-1/3$ is the limit ($\delta \to 0$ as $\varepsilon \to 0$).

The concept of limit can be extended to function of more than two variables.

Example 5. Evaluate

$$\lim_{P \to P_0} \frac{x\,\ell n y + z}{xy+z-1}$$

along Γ: $r(t) = (t, e^{t-1}, t-1)$ where $P_0(1, 1, 0)$.

Solution.

$$\lim_{P \to P_0} \frac{x\,\ell n y + z}{xy+z-1} = \lim_{t \to 1} \frac{t(t-1)+t-1}{t\,e^{t-1}+t-1-1}$$

$$= \lim_{t \to 1} \frac{t^2-1}{te^{t-1}+t-2} = \lim_{t \to 1} \frac{2t}{e^{t-1}+te^{t-1}+1} = \frac{2}{3}.$$

Theorem. If the functions f, g have limits at a point P_0, then

a) $\lim_{P \to P_0} (f(P) \pm g(P)) = \lim_{P \to P_0} f(P) \pm \lim_{P \to P_0} g(P)$

b) $\lim_{P \to P_0} (f(P)\, g(P)) = \lim_{P \to P_0} f(P) \lim_{P \to P_0} g(P)$

c) $\lim_{P \to P_0} (f(P) : g(P)) = \lim_{P \to P_0} f(P) : \lim_{P \to P_0} g(P)$

(if $\lim g(P) \neq 0$).

The proof is the same as that of Theorem 2 on limits of a function of a single variable.

CONTINUITY.

A function $f: D \to \mathbb{R}$, $z = f(P)$ is said to be continuous at a point $P_0 \varepsilon D$ if

$$\lim_{P \to P_0} f(P) = f(P_0),$$

otherwise it is *discontinuous* at P_0.

If f is continuous at every point of D we call f *continuous function,* written $f \varepsilon C[D]$ or $f \varepsilon C(D)$ according as D does (does not) contain its boundary.

Any polynomial and any rational function are continuous on their domains of definitions.

Example 1. Discuss continuity of

$$u(x, y, z) = \frac{2x - y^2 + z}{z - x^2 - y^2}$$

Solution. u being a rational function we have

$$\lim_{P \to P_0} u(P) = u(P_0)$$

if $z-x^2-y^2 \neq 0$. Hence u is continuous when P_0 does not lie on the paraboloid $S: z = x^2+y^2$. There is discontinuity when $P_0 \varepsilon S$. The discontinuity may be removable along the curve of intersection of the surfaces $S: z-x^2-y^2$, $T: 2x-y^2+z = 0$.

The properties and Theorems on continuity on continuous functions are valid for function of several variables.

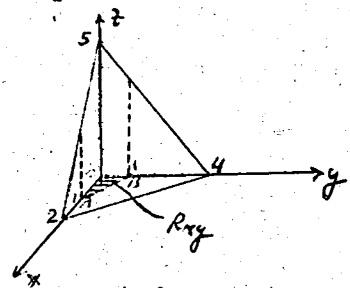
Example 2. Given the function

$$\frac{x}{2} + \frac{y}{4} + \frac{z}{5} = 1$$

defined on $R_{xy} = \left(0, 1; 0, 1-x\right)$

a) find m and M for z,

b) find points on R_{xy} such that $\mu = \frac{1}{2}$ (m+M)



Solution.

a) $z = (20-10x-5y)/4$ is max when $x=0$ and $y=0$. Then $M=5$.

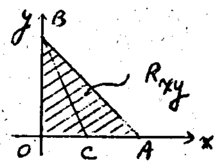
From the figure, min z occurs either at A or at B:

$z(A)=5/2$, $z(B)=15/4 \Rightarrow m=5/2$.

b) $\mu = \frac{1}{2}\left(\frac{5}{2} + 5\right) = \frac{15}{4}$

$$\Rightarrow \frac{15}{4} = \frac{20-10x-5y}{4} \Rightarrow 2x+y = 1$$

$(x, y) \varepsilon (BC)$.



In complicated cases, solution may involve solution of max min problems of functions of several variables.

EXERCISES (4. I)

1. Which ones of the following relations are functions of x and y?

a) $f(x, y) = \ell n(x^2+y^2)$ b) $x^2+y^2-z^2 = 0$

c) $x^2+y^2+10z = 0$ d) $\sin(xy) = z^2$

2. Determine and sketch the domain of definition of the following functions:

a) $z = \sqrt{\frac{x+y}{x-y}}$ b) $z = \ell n\left|x^2-y^2\right| + \ell n(xy)$

3. Same question for:

a) $z = \arcsin xy$ b) $z = \text{Argsech } \frac{x}{y}$

4. Same question for:

a) $z = \text{Argch } \frac{x}{y}$ b) $z = (x^2+y-1)^{x/y}$

5. Determine and sketch the domain of the following functions:

a) $u = \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9} - \frac{z^2}{16}}$ b) $v = \ell n\ xy+9^{\sqrt{z-x}}$

6. Same question for

a) $v = \arcsin \frac{x+y+z}{3}$ b) $v = \left(\text{Tanh}(z-x^2-y^2)\right)^{\frac{x+z}{y}}$

7. Evaluate the following limits of the given functions

a) $f(x, y) = \frac{x^2-y}{x+y}$ at $(0, 0)$ along $y=x^3$

b) $f(x, y) = \frac{xy}{x+y-1}$ at $(0, 1)$ along $y=x^2+1$

8. Same question for the functions:

a) $f(x, y, z) = \frac{xe^{y+z}}{ye^{x-z}}$ at $(0, 0, \ell n 2)$ along $r(t)=(t, 2t, \ell n(t+2))$

b) $f(x, y, z) = \frac{x \cos y \sin t}{y \sin x \cos z}$ at $(0, \pi/2, \pi/3)$ along

$r(t) = (t, \frac{\pi}{2} + t, \frac{\pi}{3} + t)$

9. Evaluate the iterated limits:

a) $\lim\limits_{y \to 0} \left(\lim\limits_{x \to 0} \frac{x+y}{x-y}\right)$ b) $\lim\limits_{x \to 0} \left(\lim\limits_{y \to 0} \frac{x+y}{x-y}\right)$

10. Evaluate the iterated limits:

a) $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \dfrac{x+y+x^2+y^2}{x+y-x^2-y^2}$ b) $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \dfrac{\sin x + \ln(1+y)}{\sin y + \ln(1+x)}$

11. Same question for

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{(y - x^2)(y - 2x^2) + \sin^5 x}{(y - x^2)(y - 2x^2) + \ln^5(1+y)}$$

12. Discuss the independence of direction:

a) $\lim\limits_{(x, y)\to(0, 0)} \dfrac{x^2+y^2}{x-y}$ b) $\lim\limits_{(x, y)\to(1, 0)} \dfrac{y \sin x}{x+y-1}$

13. Given

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & \text{when } (x, y) \neq (0, 0) \\ 0 & \text{when } (x, y) = (0, 0) \end{cases}$$

show that $f(x, 0)$ is continuous, $f(0, y)$ is continuous but $f(x, y)$ is not continuous at $(0, 0)$.

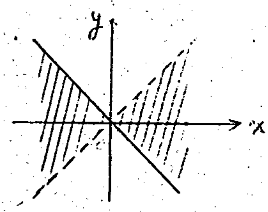
14. Discuss the continuity at the given point:

a) $f(x, y) = \dfrac{x \ln y}{y \ln x}$ at $(1, 1)$

b) $f(x, y) = \operatorname{Argth} \dfrac{y}{x}$ at $(1, 1)$

15. Find m and M for $z = x^2+y^2$ on $R_{xy} = [0, 2; 0, 2]$.

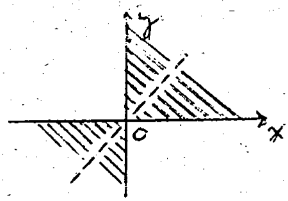
ANSWERS TO EVEN NUMBERED EXERCISES



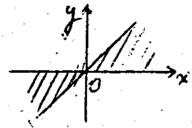
2. a) $D = \{(x, y): \frac{x+y}{x-y} > 0\} - \{(x, y): x=y\}$

$= \{(x, y): (x+y)(x-y) > 0, \; x \neq y\}$

b) $D = \{(x, y): \; \left|x^2-y^2\right| \neq 0, \quad xy > 0\}$

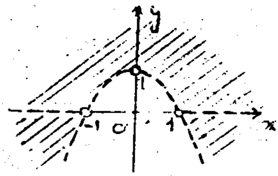


4. a) $D = \{(x, y): \; \frac{x}{y} > 1\}$



b) $D = \{(x, y): \; x^2+y-1 > 0, \quad y \neq 0\}$

$= \{(x, y): \; y > 1-x^2, \quad y \neq 0\}$



6. a) $D = \{(x, y, z): \frac{|x+y+z-4|}{3} \leq 1\} = \{(x, y, z): \; 1 \leq |x+y+z| \leq 1\}$

The region is between and on the parallel planes $x+y+z = 1$, $x+y+z = 1$

b) $D = \{(x, y, z): \; z-x^2-y^2 > 0, \quad y \neq 0\}$

The region is inside the paraboloid $z = x^2+y^2$ excluding the plane $y = 0$.

8. a) 2, b) 0

10. a) 1, b) 1

12. a) indep. b) dep.

14. a) discont., b) discont.

4. 2. DIFFERENTIATION

A. PARTIAL DERIVATIVES

Let $f: D \to R$, $z = f(x, y)$ be a function of two variables defined on the domain D (where graph is a surface S).

Let $P(x, y) \varepsilon D$.

When the second variable y is held constant, then $z = f(x, y)$ will be a function of the variable x alone, and the derivative of $z = f(x, y)$ with respect to x at $P(x, y)$, that is

$$\lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

will be called the *partial derivative* of $z = f(x, y)$ with *respect to* x at $P(x, y)$.

If above limit exists, it is denoted by any one of the symbols:

$$\frac{\partial f}{\partial x}, \quad f_x, \quad f_1, \quad f'_x, \quad f'_1, \quad D_x f; \quad D_1 f, \quad p,$$

$$\frac{\partial z}{\partial x}, \quad z_x, \quad z_1, \quad z'_x, \quad z'_1, \quad D_x z, \quad D_1 z$$

where D_x' stands for the partial derivative operator $\frac{\partial}{\partial x}$.

Similarly holding x constant instead of y at $P(x, y)$, then $z = f(x, y)$ will be a function of y alone, and we talk about

$$\lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}.$$

which is the *partial derivative* of $z = f(x, y)$ with *respect to* y at $P(x, y)$, written

$$\frac{\partial f}{\partial y}, \quad f_y, \quad f_2, \quad f'_y, \quad f'_2, \quad D_y f; \quad D_2 f, \quad q$$

$$\frac{\partial z}{\partial y}, \quad z_y, \quad z_2, \quad z'_y, \quad z'_2, \quad D_y z, \quad D_2 z$$

The notations for partial derivative at a particular point $P_0(x_0, y_0)$ are

$$\left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_0\\ y=y_0}}, \quad \frac{\partial}{\partial x} f(x_0, y_0), \quad \frac{\partial}{\partial x} f(P_0), \ \ldots \ f_x(P_0)$$

$$\left.\frac{\partial f}{\partial y}\right|_{\substack{x=x_0\\ y=y_0}}, \quad \frac{\partial}{\partial y} f(x_0, y_0), \quad \frac{\partial}{\partial y} f(P_0), \ \ldots, f_y(P_0)$$

The partial derivative operator D_x(or D_y) is linear, that is $D_x(cf+g) = c\, D_x f + D_x g$.

<u>Example 1</u>. Find the partial derivatives z_x, z_y of the functions

a) $z = 2x^2y - 3y^3$ b) $z = \text{Arctan}\,\frac{y}{x}$

<u>Solution</u>.

a) $z_x = 4xy, \quad z_y = 2x^2 - 9y^2$

b) $z_x = \dfrac{1}{1 + \dfrac{y^2}{x^2}} \cdot \left(-\dfrac{y}{x^2}\right) = -\dfrac{y}{x^2+y^2}$

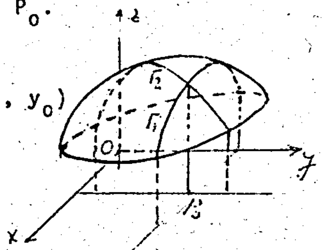
$$z_y = \frac{1}{1 + \frac{y^2}{x^2}} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$$

Geometric Interpretation.

Let $z = f(x, y)$ be a function defined on its domain D with graph S.

Since in defining f_x at $P_0(x_0, y_0) \varepsilon D$, the variable y is held constant, say y_0, then $z = f(x, y_0)$ will be a curve

Γ_1 on S with domain as the line through P_o and parallel to x-axis. Hence $f_x(P_o)$ will be the slope of Γ_1 at P_o.



Similarly $f_y(P_o)$ is the slope of a curve Γ_2 on S with domain through $P_o(x_o, y_o)$ and parallel to y-axis. Hence $f_y(P_o)$ will be the slope of Γ_2 at P_o.

HEIGHER ORDER DERIVATIVES:

The partial derivative $\frac{\partial f}{\partial x}(x, y)$, $\frac{\partial f}{\partial y}(x, y)$ of $f(x, y)$ being again functions of the same variables, they have in general partial derivatives, namely

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f\right) = \frac{\partial^2}{\partial x^2} f = D_{xx}f = f_{xx} = f_{11} = f''_{xx} = f''_{11}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f\right) = \frac{\partial^2}{\partial y \partial x} f = D_{xy}f = f_{xy} = f_{12} = f''_{xy} = f''_{12}$$

and

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f\right) = \frac{\partial^2}{\partial x \partial y} f = D_{yx}f = f_{yx} = f_{21} = f''_{yx} = f''_{21}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} f\right) = \frac{\partial^2}{\partial y^2} f = D_{yy}f = f_{yy} = f_{22} = f''_{yy} = f''_{22}$$

Under certain continuity conditions it will be proved that the mixed derivative f_{xy} and f_{yx} are equal, and f_{xx}, $f_{xy}=f_{yx}$, f_{yy} are denoted by r, s, t (or p_x, $p_y=q_x$, q_y) respectively.

The concept of partial derivative can be extended to function of more than two variables $z = f(x_1, \ldots, x_n)$. Thus

$$\frac{\partial z}{\partial x_i} = z_i = \lim_{h_i \to 0} \frac{f(\ldots, x_i+h_i, \ldots)-f(\ldots, x_i, \ldots)}{h_i}$$

is the partial cerivative of z with respect to x_i, where the variables other than x_i are hold constant.

EQUALITY OF MIXED DERIVATIVES:

Theorem. (SCHWARZ). If $z = f(x, y)$, $f_1(x, y)$, $f_2(x, y)$ and $f_{12}(x, y)$, $f_{21}(x, y)$ are continuous on D_f, then

$$f_{12}(x, y) = f_{21}(x, y)$$

Proof. Consider the expression

$$F = f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x, y)$$

at $(x, y) \varepsilon D_f$. Setting

$$\phi(t) = f(t, y+k) - f(t, y)$$
$$\psi(t) = f(x+h, t) - f(x, t)$$

F becomes

$$\phi(x+h) - \phi(x) = F = \psi(y+k) - \psi(y)$$

Since ϕ, ψ are continuous, by the MVT applied to ϕ and ψ, one has

$$h\phi'(c_1) = k\psi'(c_2)$$
$$\Rightarrow \quad h[f_1(c_1, y+k)-f_1(c_1, y)] = k[f_2(x+k, c_2)-f_2(x, c_2)]$$

with $c_1 \varepsilon \overline{(x, x+h)}$, $c_2 \varepsilon \overline{(y, y+k)}$.

Again by the MVT applied to f_1, f_2 one gets

$$hk\, f_{12}(c_1, c'_2) = kh\, f_{21}(c'_1, c_2)$$

with $c'_1 \varepsilon \overline{(x, x+h)}$, $c'_2 \varepsilon \overline{(y, y+k)}$ implying

$$f_{12}(x, y) = f_{21}(x, y)$$

as $h \to 0$, $k \to 0$. ∎

PARTIAL DERIVATIVES OF A VECTOR FUNCTION:

If

$$r(u, v) = x(u, v)i + y(u, v)j + z(u, v)k$$

with scalar components $x(u, v)$, $y(u, v)$, $z(u, v)$ admit partial derivatives, then

$$r_u = \frac{\partial r}{\partial u} = x_u\, i + y_u j + z_u k = (x_u, y_u, z_u)$$

$$r_v = \frac{\partial r}{\partial v} = x_v\, i + y_v j + z_v k = (x_v, y_v, z_v).$$

The vector r_u, r_v are tangent vectors to v-constant, u-constant curves lying on the surface defined by the position vector $r(u, v)$.

B. Total Increment, Total Differential, Total Derivative

By the *total increment* of $z = f(x_1, \dots, x_n)$ is meant the difference

$$\Delta z = f(x_1 + \Delta x_1, \dots, x_n + \Delta x_n) - f(x_1, \dots, x_n)$$

where Δx_i is the individual increment of the variable x_i.

<u>Theorem</u>. If $f(P)$ and the partial derivatives $f_1(P), \dots, f_n(P)$ are continuous, then

$$\Delta z = f_1\, \Delta x_1 + \dots + f_n\, \Delta x_n + \varepsilon_1\, \Delta x_1 + \dots + \varepsilon_n\, \Delta x_n$$

where $\varepsilon \to 0, \dots, \varepsilon_n \to 0$ as $\Delta x_1 \to 0, \dots, \Delta x_n \to 0$.

<u>Proof</u>. The proof is given for $n=2$ only.

Let $z = f(x, y)$.

$$\begin{aligned}\Delta z &= f(x + \Delta x, \ y + \Delta y) - f(x, y)\\ &= [f(x + \Delta x, y + \Delta y - f(x, y + \Delta y] + [f(x, y + \Delta y) - f(x, y)]\\ &= [\Delta x\, f_1(c_1, y + \Delta y)] + [\Delta y\, f_2(x, c_2)]\end{aligned}$$

by continuity of f with $c_1 \varepsilon \overline{(x, x+h)}$, $c_2 \varepsilon \overline{(y, y+k)}$. We have

$$\Delta z = f_1(c_1, y + \Delta y)\Delta x + f_2(x, c_2)\Delta y.$$

Setting

$$f_1(c_1, y + \Delta y) = f_1(x, y) + \varepsilon_1$$
$$f_2(x_1, c_2) = f_2(x, y) + \varepsilon_2$$

where $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$ as Δx, $\Delta y \to 0$ by continuity of f_1, f_2,

$$\Delta z = (f_1(x, y) + \varepsilon_1)\Delta x + (f_2(x, y) + \varepsilon_2)\Delta y$$
$$= f_1(P)\Delta x + f_2(P)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y. \blacksquare$$

The sum

$$f_1(P)\Delta x_1 + \ldots + f_n(P)\Delta x_n$$

in the total increment Δz is called the *principal part* of Δz.

The principal part where $\Delta x_1, \ldots, \Delta x_n$ are replaced by the differentials $dx_1, \ldots, dx_n$ is called the *total differential* dz of z:

$$dz = f_1(P)dx_1 + \ldots + f_n(P)dx_n.$$

If $x_i = x_i(t)$, then

$$\frac{dz}{dt} = f_1(P)\frac{dx_1}{dt} + \ldots + f_n(P)\frac{dx_n}{dt}$$

is the *total derivative* of z with respect to t.

The function $f: D \to \mathbb{R}$, $z = f(x, y)$ is said to be *differentiable* at $P(x, y)\varepsilon D$, if f_x (or f_y) exists at P and f_y (or f_x) exists in a neighborhood $N(P)$ of P.

If f is differentiable at every point of D, then it is said to be differentiable on D, written $f\varepsilon d(D)$.

APPROXIMATIONS:

We have

$$\Delta z = f_1(P)\Delta x_1 + \ldots + f_n(P)\Delta x_n + \sum x_i \, \Delta x_i$$

as total increment of z at $P(x_1, \ldots, x_n)$. Neglecting the second order small quantities $x_i \Delta x_i$, we get the approximate increment

$$\tilde{\Delta z} = \sum_{i=1}^{n} f_i(P)\Delta x_i$$

of z which is the principal part of the increment and is equal to the total differential

$$dz = \sum f_i(P)\, dx_i$$

when Δx_i is replaced by dx_i.

The functions used frequently are in explicit forms $z = f(x, y)$ and $u = f(x, y, z)$, and have total differentials

$$dz = f_x dx + f_y dy$$

$$du = f_x dx + f_y dy + f_z dz.$$

<u>ERROR</u>. Let $u = f(x, y, z)$ be a function having f_x, f_y, f_z. If x, y, z are measured with errors dx, dy, dz respectively, it is required to find the error made in u.

The error in u is

$$du = f_x dx + f_y dy + f_z dz$$

Then du/u and $100du/u$ will be the relative and percentage errors respectively.

$|du| = |u_c - u_m|$ is the absolute error in u, where u_c, u_m are the correct and measured values of u. Hence

$$u_c = u_m \pm du$$

<u>Example 1</u>. If the sides of a rectangle are measured as $60 \pm 0,2$ m, $80 \pm 0,3$ m what is the relative error in the area of the rectangle?

Solution. $A = xy \Rightarrow dA = y\,dx + x\,dy$. Then the maximum relative error is

$$\frac{dA}{A} = \left|\frac{dx}{x}\right| + \left|\frac{dy}{y}\right| = \frac{0{,}2}{60} + \frac{0{,}3}{80} = \frac{1{,}7}{240} = 0{,}007.$$

Example. By the use of total differential evaluate $\sqrt{(0{,}98)^2 + (2{,}01)^2 + (1{,}94)^2}$ approximately.

Solution. Consider the function

$$u(a) = \sqrt{x^2 + y^2 + z^2}$$

Let $P_0(1, 2, 2)$ for an easy evaluation. From

$$u(P) = u(P_0) + du\Big|_{P_0}$$

we get

$$u(P) = \sqrt{1 + 2^2 + 2^2} + \frac{\partial u}{\partial x}\Big|_{P_0} dx + \frac{\partial u}{\partial y}\Big|_{P_0} dy + \frac{\partial u}{\partial z}\Big|_{P_0} dz$$

$$= 3 + \frac{x}{u}\Big|_{P_0}(-0{,}02) + \frac{y}{u}\Big|_{P_0}(0{,}01) + \frac{z}{u}\Big|_{P_0}(-0{,}06)$$

$$= 3 - \frac{1}{3}(0{,}02) + \frac{2}{3}(0{,}01) - \frac{2}{3}(0{,}06)$$

$$\cong 3 - 0{,}006 + 0{,}006 - 0{,}04 = 2{,}94$$

C. Differentiation of Composite and Implicit Functions, Chain Rule in Differentiation :

A function $z = f(x_1, \ldots, x_n)$ where $x_1 = x_1(t_1, \ldots, t_p)$, $\ldots$, $x_n = x_n(t_1, \ldots, t_p)$ is a composite function or a function of functions or a chain function.

Then z is a function of the variables $t_1, \ldots, t_p$. Now question may arise as how to obtain partial derivatives of z with respect to the new variables $t_1, \ldots, t_p$.

In particular, for the function

$$z = f(x, y), \qquad x = x(\cdot, t), \qquad y = y(\cdot, t),$$

we have

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$= \frac{\partial f}{\partial x}\left[\frac{\partial x}{\partial s} ds + \frac{\partial x}{\partial t} dt\right] + \frac{\partial f}{\partial y}\left[\frac{\partial y}{\partial s} ds + \frac{\partial y}{\partial t} dt\right]$$

$$= \left[\frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}\right] ds + \left[\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\right] dt$$

If t is held constant, then $dt = 0$, and dz/ds becomes partial derivative $\partial z/\partial s$, implying

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \quad \left(= \frac{\partial f}{\partial s}\right)$$

In the same manner

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \quad \left(= \frac{\partial f}{\partial t}\right)$$

Such a rule of partial differentiation is called the *chain rule*.

For a function

$$u = f(x, y, z) \text{ with } x = x(s, t), \quad y = y(s, t), \quad z = z(s, t)$$

the chain rule is

$$\frac{\partial u}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s} \quad \left(= \frac{\partial f}{\partial s}\right)$$

$$\frac{\partial u}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s} \quad \left(= \frac{\partial f}{\partial t}\right)$$

<u>Example 1</u>. Given $z = x^2 \arctan y$ and $x = u^2 - v^2$, $y = uv$, find z_u, z_v.

<u>Solution</u>.

$$z_u = z_x x_u + z_y y_u$$

$$= (2x \arctan y)2u + \frac{x^2}{1 + y^2} v$$

$$z_v = z_x x_v + z_y y_v$$

$$= (2x \arctan u)(-2v) + \frac{x^2}{1+y^2} u$$

Example 2. Given $z = f(x^2y, x^2-y^2)$, find z_x, z_y.

Solution. Setting $u = x^2y$, $v = x^2-y^2$ we have $= f(u, v)$. Then

$$z_x = f_u u_x + f_v v_x = f_u 2xy + f_v 2x$$
$$= f_1 2xy + f_2 2x$$
$$z_y = f_u u_y + f_v v_y = f_u x^2 + f_v(-2y)$$
$$= f_1 x^2 - f_2 2y$$

Example 3. Given $f(x^2 \sin y)$, find f_x, f_y and $(xf_x)^2 + (2f_y)^2$.

Solution.

$$f_x = f' 2x \sin y, \quad f_y = f' x^2 \cos y,$$

and

$$(xf_x)^2 + (2f_y)^2 = 4x^2 f'^2.$$

Example 4. Given $u = xy \ln z$ and $x = 2r + 3s$, $y = r-s$, $z = rs$ find $u_s(r, s)$ at $(1, e)$.

Solution.

$$u_s = u_x x_s + u_y y_s + u_z z_s$$
$$= (y \ln z)3 + (x \ln z)(-1) + \frac{xy}{z} r$$
$$u_s(1, e) = (1-e)3 + (2+3)(-1) + \frac{(2+3e)(1-e)}{e} 1$$
$$= 3 - 3e - 5 + \frac{2 + e - 3e^2}{e}$$
$$= -1 - 6e + 2e^{-1}$$

Example 5. Evaluate

$$A = \left(2 \frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 (x^3 + 3x^2y)$$

Solution.

$$A = \left(2\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(2\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)(x^3 + 3x^2y)$$

$$= \left(2\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left[2\frac{\partial}{\partial x}x^3 + 6\frac{\partial}{\partial x}x^2y + \frac{\partial}{\partial y}x^3 + 3\frac{\partial}{\partial y}x^2y\right]$$

$$= \left(2\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)(6x^2 + 12xy + 3x^2)$$

$$= 2\frac{\partial}{\partial x}(9x^2 + 12xy) + \frac{\partial}{\partial y}(9x^2 + 12xy)^2)$$

$$= 2(18x + 12y) + 12x = 48x + 24y.$$

IMPLICIT DIFFERENTIATION:

Let a function be given implicitly in the form $F(x, y, z) = 0$ where z is the dependent variable. Taking total differential of each side, one gets

$$F_x dx + F_y dy + F_z \underbrace{\left(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right)}_{dz} = 0$$

$$\Rightarrow \left(F_x + F_z\frac{\partial z}{\partial x}\right)dx + \left(F_y + F_z\frac{\partial z}{\partial y}\right)dy = 0$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

If the function is given implicitly as $f(x, y) = 0$ with y as dependent variable, we have similarly

$$f_x d_x + f_y d_y = 0 \Rightarrow \frac{dy}{dx} = -\frac{f_x}{f_y}.$$

A partial derivative of an implicit function with respect to a variable can also be obtained in the way done for a function of a single variable by term-by-term differentiation with respect to that variable.

Example. Given $x \ln yz - yz \sin x = 5$

a) considering $z = z(x, y)$, find $\partial z/\partial y$,

b) considering $y = y(x, z)$, find $\partial y/\partial z$.

Solution.

a) Differentiating every term with respect to y, we get

$$x\,\frac{z + y\,z_y}{yz} - (z + y\,z_y)\sin x = 0$$

$$\Rightarrow \left(\frac{x}{z} - y\sin x\right)z_y + \left(\frac{x}{y} - z\sin x\right) = 0$$

$$\Rightarrow z_y = \frac{\frac{x}{y} - z\sin x}{y\sin x - \frac{x}{z}}$$

b) Differentiating every term with respect to z, we get

$$x\,\frac{y + z\,y_z}{yz} - (y + y_z\,z)\sin x = 0$$

$$\Rightarrow \left(\frac{x}{y} - z\sin x\right)y_z + \left(\frac{x}{z} - y\sin x\right) = 0$$

$$\Rightarrow y_z = \frac{\frac{x}{z} - y\sin x}{\frac{x}{y} - z\sin x}$$

D. Euler's Theorem on Homogeneous Functions

A non zero function $f(x_1, \ldots, x_n)$ is said to be *positively homogeneous* of degree r in the all variables $x_1, \ldots, x_n$ on D_f if for every $t > 0$ one has

$$f(t\,x_1, \ldots, t\,x_n) \equiv t^r f(x_1, \ldots, x_n) \qquad (1)$$

where the degree r is a positive or zero or a negative real number.

Example. Which ones of the following functions are (positively) homogeneous and find their degrees.

a) $f(x, y) = x^2 + 2xy - y^2$ b) $g(x, y) = 2x + y^3$

b) $h(x, y, z) = \frac{x}{y^2} - \frac{2y}{z^2} + 3\,\frac{z}{x^2}$ d) $k(x, y) = \left(\frac{x}{y}\right)^2 \ln \frac{y}{x}$

Solution.

a) $f(tx, ty) = (tx)^2 + 2(tx)(ty) - (ty)^2$

$$= t^2(x^2 + 2xy - y^2) = t^2 f(x, y)$$

f is homogeneous with degree 2

b) $g(tx, ty) = 2tx + t^3y^3 = t(2x + t^2y)$

g is non homogeneous

c) $h(tx, ty, tz) = t^{-1} h(x, y, z)$

h is homogeneous of degree -1

d) k is clearly homogeneous of degree 0

Clearly the sum (difference) of two homogeneous function of the same degree γ is homogeneous of degree γ, and the product (ratio) of two homogeneous functions of degrees μ, γ is homogeneous of degree $\mu\gamma$ (μ/γ).

Theorem. (EULER) A function $f(x_1, \ldots, x_n)$ admitting partial derivatives $f_1, \ldots, f_n$ at any point $P \varepsilon D_f$ is positively homogeneous of degree γ, if and only if

$$x_1 f_1(x_1, \ldots, x_n) + \ldots + x_n f_n(x_1, \ldots, x_n) = \gamma f(x_1, \ldots, x_n), \quad (2)$$

that is, $(1) \Leftrightarrow (2)$.

Proof. For simplicity we give it for a function of two variables $z = f(x, y)$.

Let $f(x, y)$ be homogeneous of degree γ. Then

$$f(tx, ty) = t^{\gamma} f(x, y) \quad (1)$$

holds.

Differentiating each side of (1) with respect to t, we have

$$x f_1(tx, ty) + y f_2(tx, ty) = \gamma t^{\gamma-1} f(x, y)$$

which yields (2) for $t = 1$.

Conversely, let

$$x\, f_1(x, y) + y\, f_2(x, y) = \gamma\, f(x, y) \qquad (2)$$

hold. Let $(x_0, y_0) \varepsilon D_f$, and consider the function

$$\varphi(t) = f(t\, x_0, t\, y_0) \quad \text{for} \quad t > 0.$$

We have

$$\varphi'(t) = x_0 f_1(t\, x_0, t\, y_0) + y_0 f_2(t\, x_0, t\, y_0)$$

$$\Rightarrow \quad t\,\varphi'(t) = (tx_0) f_1(tx_0, ty_0) + (ty_0) f_2(tx_0, ty_0)$$

$$= \gamma\, f(tx_0, ty_0) \quad \text{by} \quad (2)$$

$$= \gamma\, \varphi(t)$$

$$\Rightarrow \quad \frac{\varphi'(t)}{\varphi(t)} = \frac{\gamma}{t} \quad \Rightarrow \quad \int \frac{\varphi'(t)}{\varphi(t)}\, dt = \gamma \int \frac{dt}{t}$$

$$\Rightarrow \quad \ln \varphi(t) = \gamma \ln t + \ln c$$

$$\Rightarrow \quad \varphi(t) = ct^{\gamma} \quad \Rightarrow \quad c = \varphi(1) = f(x_0, y_0)$$

$$\Rightarrow \quad \varphi(t) = t^{\gamma} f(x_0, y_0)$$

$$\Rightarrow \quad f(tx_0, ty_0) = t^{\gamma}\, f(x_0, y_0) \qquad (1) \quad \blacksquare$$

<u>Example</u>. Verify EULER's Theorem for

a) $x^2 + 2xy - y^2$ b) $x^2 y + 2y^3 - \sin^3 x$

<u>Solution</u>.

a) $x\, f_x + y\, f_y = x(2x + 2y) + y(2x - 2y)$

$= 2x^2 + 4xy - 2y^2 = 2(x^2 + 2xy - y^2)$

$= 2 . f(x, y)$ (Verified)

b) $x\, f_x + y\, f_y = x(2xy - 3\sin^2 x \cos x) + y(x^2 + 6y^2)$

$= 3x^2 y + 6y^2 - 3x \sin^2 x \cos x$

$= 3(x^2 y + 2y^3 - x \sin^2 x \cos x)$

(not verified)

E. TANGENT PLANE, NORMAL LINE

Let a surface S be given by its vector equation

$$r(u, v) = x(u, v)i + y(u, v)j + z(u, v)k$$

Holding v constant $(= v_o)$, $r(u, v_o)$ represents a parametric curve Γ_1, and u-constant $(= u_o)$ a curve Γ_2 on S.

Then the partial derivative vectors $r_u(P_o)$, $r_v(P_o)$ represent tangent vectors to Γ_1, Γ_2 at P_o.

Any curve Γ on S through P_o being represented by $r(u(t), v(t))$, its tangent vector is given by

$$\frac{dr}{dt} = r_u \frac{du}{dt} + r_v \frac{dv}{dt} .$$

showing that it is a linear combination of tangent vectors r_u, r_v. This result proves the existence of a plane π at P_o which is tangent to all curves of S through P_o. This plane is the *tangent plane* of S at P_o and its normal ℓ at P_o the *normal line* ℓ of S at P_o.

A vector parallel to the normal line is therefore $r_u(P_o) \times r_v(P_o)$.

Considering now the surface S of $z = f(x, y)$ its vector equation being

$$r(x, y) = xi + yj + f(x, y)k,$$

we have

$$N = r_x \times r_y = \left.\begin{vmatrix} i & j & k \\ 1 & 0 & p \\ 0 & 1 & q \end{vmatrix}\right|_{P_o} = (-p, -q, 1)$$

where $p = z_x$, $q = z_y$. Hence we have

$$\frac{x - x_o}{z_x(P_o)} = \frac{y - y_o}{z_y(P_o)} = \frac{z - z_o}{-1}$$

as the equations of the normal line, and

$$z_x(P_o).(x-x_o) + z_y(P_o).(y-y_o)-(z-z_o) = 0$$

as that of the tangent plane at P_o.

If the surface is given by the implicit equation $F(x, y, z) = 0$, having $p = z_x = -F_x/F_z$, $q = z_y = -F_y/F_z$, we get

$$N = (p, q, -1) = \left(-\frac{F_x}{F_z}, -\frac{F_y}{F_z}, -1\right) /\!/ (F_x, F_y, F_z).$$

Then the equations become:

$$\ell: \frac{x-x_o}{F_x(P_o)} = \frac{y-y_o}{F_y(P_o)} = \frac{z-z_o}{F_z(P_o)}.$$

$$\pi: F_x(P_o)(x-x_o) + F_y(P_o)(y-y_o) + F_z(P_o)(z-z_o) = 0$$

<u>Example 1</u>. Given the paraboloid $z = 4x^2 + 9y^2$, find the equations of the normal line and tangent plane at $A(2, 1, 25)$.

<u>Solution</u>. $N = (p, q, -1)_A = (8x, 18y, -1)_A$
$= (16, 18, -1)$

$$\ell: \frac{x-2}{16} = \frac{4-1}{18} = \frac{z-25}{-1}$$

$$\pi: 16(x-2) + 18(y-1) - (z-25) = 0.$$

<u>Example 2</u>. Given the cone

$$(x-2)^2 + 4(y-1)^2 - (z+3)^2 = 0$$

a) show that all tangent planes pass through the vertex $V(2, 1, -3)$

b) find points on the cone at which normal lines are parallel to yz-plane.

<u>Solution</u>. Setting

$$F(x, y, z) = (x-2)^2 + 4(y-1)^2 - (z+3)^2 = 0$$

and taking a point $P_o(x_o, y_o, z_o)$ on the cone, we have

$$(x_o-2)^2 + 4(y_o-1)^2 - (z_o+3)^2 = 0 \qquad (*)$$

Now,

$$F_x(P_o) = 2(x_o-2), \quad F_y(P_o) = 8(y_o-1), \quad F_z(P_o) = -2(z_o+3)$$

$$\pi: (x_o-2)(x-x_o) + 4(y_o-1)(y-y_o) - (z_o+3)(z-z_o) = 0$$

Setting $x = 2, \ y = 1, \ z = -3$ we have

$$-(x_o-2)^2 - 4(y_o-1)^2 + (z_o+3)^2 = 0$$

which is $(*)$.

b) From (a), $N = (x_o-2, 4(y_o-1), -(z_o+3))$.

$N \ /\!/ \ yz$-plane $\cdot\ x_o-2 = 0$

$(*) \quad 4(y_o-1)^2 - (z_o+3)^2 = 0$

$$2(y_o-1)+(z_o+3) = 0 \quad \text{or} \quad 2(y_o-1)-(z_o+3) = 0$$

which together with the equation of the cone represent the generators as yz-traces of the cones.

F. Directional Derivative

Let $z = f(x, y)$ be defined on D_f, and let Γ be a curve on xy-plane through $P(x, y) \varepsilon D_f$. Then the *directional derivative* of $f(x, y)$ along Γ at $P(x, y) \varepsilon D_f$ is defined to be

$$\lim_{\Delta s \to 0} \frac{\Delta f(P)}{\Delta s} = \frac{df}{ds}$$

which is equal to (by chain rule)

$$\frac{df(P)}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}$$

The unit tangent vector of Γ at P being

$$\left(\frac{dx}{ds}, \frac{dy}{ds}\right) = (\cos\alpha, \cos\beta),$$

we have

$$\frac{df}{ds} = f_x \cos\alpha + f_y \cos\beta$$

The directional derivative of f in the direction of any vector $a = (a_1, a_2)$ can be written as

$$f_a = \frac{df}{ds} = f_x \frac{a_1}{|a|} + f_y \frac{a_2}{|a|} ,$$

since direction cosines of $a = (a_1, a_2)$ are $a_1/|a|$, $a_2/|a|$.

This concept can be generalized to functions of n variables for a direction in n-space:

$$\frac{df}{ds} = f_1 \cos\alpha_1 + \ldots + f_n \cos\beta_n.$$

A partial derivative of a function can be considered as directional derivative in the direction of related coordinate axis.

It is to be remarked that *the directional derivative of any function at a point in any direction can be computed is two senses which are opposite in sign.*

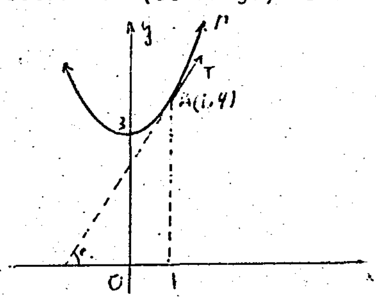
Example 1. Find the directional derivative of $z = y\,e^{x-1}$ along the curve Γ: $y = x^2+3$ at $A(1, 4)$

a) in the positive sense (of Γ)

b) in the negative sense.

Solution.

The direction of the tangent vector T (see Fig.) is in the positive sense, and that of $-T$ is in the negative sense.



$$y' = 2x\big|_A = 2 \Rightarrow \tan\alpha = 2$$

$$\cos\alpha = 1/\sqrt{5}, \quad \sin\alpha = 2/\sqrt{5}$$

since α is acute.

a) $\frac{df(A)}{ds} = y\ e^{x-1}\ \frac{1}{\sqrt{5}} + e^{x-1}\ \frac{2}{\sqrt{5}}\Big|_A = \frac{4}{\sqrt{5}} + \frac{2}{\sqrt{5}} = \frac{6}{\sqrt{5}}$

b) $\frac{df(A)}{ds} = y\ e^{x-1}\ (-\frac{1}{\sqrt{5}}) + e^{x-1}(-\frac{2}{\sqrt{5}})_A = -\frac{6}{\sqrt{5}}$

Example 2. Given $u = xy\ \ell n\ z$, find its directional derivative along the curve Γ: $x = \ell n\ t$, $y = t^2/2$, $z = 2t$ at $A(0, 1/2, 2)$ in the positive sense.

Solution. A tangent vector of Γ at A (in the positive sense of Γ) being

$$(\dot{x}, \dot{y}, \dot{z})_A = (1/t, t, 2)_A = (1, 1, 2)$$

the unit tangent vector T has components

$$\cos\alpha = 1/\sqrt{6}, \quad \cos\beta = 1/\sqrt{6}, \quad \cos\gamma = 2/\sqrt{6},$$

and

$$\frac{du(A)}{ds} = (y\ \ell n\ z)\ \frac{1}{\sqrt{6}} + (x\ \ell n\ z)\ \frac{1}{\sqrt{6}} + (\frac{xy}{z})\ \frac{2}{\sqrt{6}}\Big|_A = \frac{1}{2}\ (\ell n\ 2)\ \frac{1}{\sqrt{6}}$$

Example 3. If the temperature distribution in a room ($x\varepsilon[0, 4]$, $y\varepsilon[0, 12]$, $z\varepsilon[0, 3]$) is given by

$$T = \frac{z}{(x+1)(y+2)} \qquad \text{(in degrees)},$$

find the rate of change of T along the diagonal [OB] of the room in the sense from O to B, at the center C.

Solution.

$a = (4, 12, 3)$, $a/|a| = (4, 12, 3)/13$, $C(2, 6, 3/2)$

$$\frac{dT}{ds} = T_a = T_x\ \frac{4}{13} + T_y\ \frac{12}{13} + T_z\ \frac{3}{13}\Big|_C$$

where

$$T_x(C) = -\frac{z}{(x+1)(y+2)}\Big|_C = -1/48$$

$$T_y(C) = -\frac{z}{(x+1)(x+2)^2}\Big|_C = -1/128$$

$$T_z(C) = \frac{1}{(x+1)(y+2)}\Big|_C = 1/18$$

Then

$$T_a = -\frac{1}{48}\cdot\frac{4}{13} - \frac{1}{128}\cdot\frac{12}{13} + \frac{1}{18}\cdot\frac{3}{13} = \frac{-1}{1248} \text{ deg/unit length}$$

HEIGHER ORDER DIRECTIONAL DERIVATIVES:

The directional derivative of f at $P(x, y)$ in the direction of the vector $a_1 = (\cos\alpha_1, \cos\beta_1)$ is

$$\frac{df}{ds_1} = \cos\alpha_1 \frac{\partial f}{\partial x} + \cos\beta_1 \frac{\partial f}{\partial y}, \quad (\cos^2\alpha_1 + \cos^2\beta_1 = 1)$$

which is a function of x and y. Then it can be differentiated in the direction, say $a_2 = (\cos\alpha_2, \cos\beta_2)$, obtaining

$$\frac{d}{ds_2}\frac{df}{ds_1} = \cos\alpha_2 \frac{\partial}{\partial x}\left(\frac{df}{ds_1}\right) + \cos\beta_2 \frac{\partial}{\partial y}\left(\frac{df}{ds_1}\right).$$

Let $\cos\alpha_i = h_i$, $\cos\beta_i = k_i$. Then

$$\frac{d}{ds_2}\frac{df}{ds_1} = h_1h_2f_{11} + (h_1k_2 + h_2k_1)f_{12} + k_1k_2f_{22} = \frac{d}{ds_1}\frac{df}{ds_2}$$

If the directions are the same $(a_1 = a_2 = a = (h, k))$ the second order directional derivative is

$$\begin{aligned}\frac{d^2f}{ds^2} &= h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x\partial y} + k^2\frac{\partial^2 f}{\partial y^2}\\ &= \left(h^2\frac{\partial^2}{\partial x^2} + 2hk\frac{\partial^2}{\partial x\partial y} + k^2\frac{\partial^2}{\partial y^2}\right) f\\ &= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 F\end{aligned}$$

where $h\partial/\partial x + k\partial/\partial y$ is the directional derivative operator in the direction of (h, k) if $h^2+k^2 = 1$.

By induction one may obtain

$$\frac{d^nf}{ds^n} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f.$$

For a function of n variables we have similarly

$$\frac{d^n f}{ds^n} = \left(h_1 \frac{\partial}{\partial x_1} + \ldots + h_n \frac{\partial}{\partial x_n}\right)^n f$$

in the direction $(h_1, \ldots, h_n)$ with $(h_1^2 + \ldots + h_n^2 = 1$.

G. Del Operator ∇, and its Applications

The del operator in R^3 is the symbolic vector

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

whose components are the partial derivative operators.

It is an operator which can be applied to a scalar function f, and to a vector function F by "dot" and "cross" operations:

$$\nabla f = \text{grad } f = \text{gradient of } f,$$

$$\nabla \cdot F = \text{div } F = \text{divergence of } F,$$

$$\nabla \times F = \text{curl } F = \text{curl of } F$$

GRADIENT OF A SCALAR FUNCTIONS:

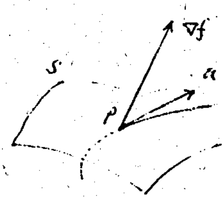
If $f(x, y, z)$ is a function having the first order partial derivatives, then

$$\text{grad } f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

which is a vector function.

The gradient vector (f_x, f_y, f_z) at P is seen to be normal to the level surface $f(x, y, z) = c$ passing through P.

The directional derivative $f_a(P)$ in the direction of a unit vector a (tangent to a curve Γ) is related to ∇f by



$$f_a = f_x \cos \alpha + f_y \cos \beta + f_z \cos \gamma = \nabla f . a$$

Therefore the directional derivative is extremum when the direction "a" is parallel to grad f, and it is zero if the vector "a" is tangent to the level surface.

Properties:

1. $\nabla(cf + g) = c\nabla f + \nabla g$ (linearity of ∇)

2. $\nabla(f g) = (\nabla f)g + f(\nabla g)$

3. $\nabla\frac{f}{g} = \frac{(\nabla f)g - f(\nabla g)}{g^2}$

Observe analogy between the roles of ∇ and D.

Example 1. Given $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, find the directional derivative in the direction of $a = (1, -2, 2)$ at $A(4, 2, 4)$.

a) as df/ds b) by $\nabla f.a/|a|$

Solution.

a) $\frac{df}{ds} = f_x \frac{1}{3} - f_y \cdot \frac{2}{3} + f_z \cdot \frac{2}{3}\Big|_A = \frac{4}{9}$, since

$f_x(A) = 4/6$, $f_y(A) = 2/6$, $f_z(A) = 4/6$

b) $\nabla f = (f_x, f_y, f_z)_A = (4/6, 2/6, 4/6)$

$a/|a| = (1, -2, 2)/3 \Rightarrow \nabla \cdot \frac{a}{|a|} = \frac{4}{9}$.

Example 2. Given $f(x, y, z) = x^2 - y^2 + 4z$ $(D_f \epsilon R^3)$, find the locus $S = \{(x, y, z): |\text{grad } f| = 6\}$.

Solution.

$\text{grad } f = (2x, -2y, 4) \Rightarrow |\text{grad } f| = 2\sqrt{x^2+y^2+4}$

$\Rightarrow \sqrt{x^2+y^2+4} = 3 \Rightarrow x^2+y^2 = 5$. Then

$S = D_f \cap \{(x, y, z): x^2+y^2 = 5\}$

$= \mathbb{R}^3 \cap \{(x, y, z): x^2+y^2 = 5\} =$

$= \{(x, y, z): x^2+y^2 = 5\}$ (a cylinder).

DIVERGENCE AND CURL OF A VECTOR FUNCTION

If $f(x, y, z) = (P, Q, R)$ is a vector function, then

$$\text{div } F = \nabla.F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (R_y - Q_z,\ P_z - R_x,\ Q_x - P_y)$$

which are scalar and vector function respectively.

Example. Given the vector functions

$$F(x, y, z) = (x^2, 2xy, xyz), \quad G(x, y, z) = (yz, z^2, xy),$$

find

a) div F, b) curl G, c) div curl G.

Solution.

a) div F = $2x + 2x + xy$

b) $\text{curl } G = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (x-2z,\ 0,\ -z)$

c) div(curl G) = 0.

Properties. For a scalar function f and vector functio F, G:

1. $\nabla .fF = (\nabla f).F + f\ \nabla.F$
2. $\nabla xfF = (\nabla f)xF + f\ \nabla xF$
3. $\nabla.(FxG) = (\nabla xF).G - F.(\nabla xG)$
4. $\nabla.\nabla f = (\nabla.\nabla)f$
5. $\nabla x\nabla f = 0$
6. $\nabla.\ \nabla xF = 0$, 7. $\nabla.r = 3$, $\nabla xr = 0$, $F.\nabla r$

where r is the position vector.

Proof of (4):

$$\nabla.\nabla f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = (\nabla.\nabla)f \quad \blacksquare$$

$\nabla^2 f$ is called the *LAPLACIAN* of f where ∇^2 is the *LAPLACE operator*. $\nabla^2 f = 0$, that is

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

is the *LAPLACE's Differential Equation*, and any solution is called a *harmonic function*.

The differential equation

$$\nabla^2 f = -4\pi$$

is the *POISSON's Differential Equation*.

Example. Given $f(x, y) = e^{ax} \cos y$, for what values of the constant "a", it is a harmonic function?

Solution.

$f_{xx} = a^2 e^{ax} \cos y, \quad f_{yy} = -e^{ax} \cos y.$

$f_{xx} + f_{yy} = (e^{ax} \cos y)(a^2-1) = 0 \Rightarrow a = \pm 1.$

H. DERIVATIVE UNDER THE INTEGRAL SIGN.

Consider the definite integral

$$F(t) = \int_a^b f(x, t)dx \qquad (a, b \text{ const})$$

where t is a parameter.

Theorem (LEIBNIZ). If

$$F(t) = \int_a^b f(x, t)dx$$

with $a \leqslant x \leqslant b$, $\alpha \leqslant t \leqslant \beta$, a, b const. and if $f(x, t)$, $f_2(x, t) \varepsilon C(D_f)$; then

$$\frac{dF(t)}{dt} = \int_a^b \frac{\partial}{\partial t} f(x, t)dx.$$

Proof.

$$\begin{aligned}
\Delta F(t) &= F(t+\Delta t)-F(t) \\
&= \int_a^b \left(f(x, t+\Delta t)-f(x, t)\right)dx \\
&= \int_a^b \Delta t\ f_2(x, \tau)dx, \qquad \tau \varepsilon (\overline{t, t+\Delta t}) \\
\frac{\Delta F}{\Delta t} &= \int_a^b f_2(x, \tau)dx \\
\frac{dF}{dt} &= \lim_{\Delta t \to 0} \int_a^b f_2(x, \tau)dx = \int_a^b f_2(x, t)dx
\end{aligned}$$

Corollary. If in

$$F(t) = \int_{a(t)}^{b(t)} f(x, t)dx$$

$a(t)$, $b(t)$ are differentiable and $f(x, t)$, $f_t(x, t)$ are continuous, then

$$\frac{dF(t)}{dt} = \int_{a(t)}^{b(t)} f_t(x, t)dt + f(b, t)b'-f(a, t)a'$$

Proof. Let

$$F(t) = G\big(a(t), b(t), t\big) = \int_{a(t)}^{b(t)} f(x, t)dx$$

Then

$$\begin{aligned}
\frac{dF(t)}{dt} &= \frac{\partial G}{\partial a}\ \frac{da}{dt} + \frac{\partial G}{\partial b}\ \frac{db}{dt} + \frac{\partial G}{\partial t}\ .1 \\
&= -f(a, t)a' + f(b, t)b' + \int_a^b f_t\ dx. \blacksquare
\end{aligned}$$

Example. Evaluate

a) $\frac{d}{dt} \int_{\pi/2}^{\pi} \frac{\cos xt}{x}\, dx$ b) $\frac{d}{dt} \int_0^{x^2} \arctan \frac{x}{y^2}\, dx$

Solution.

a) $\frac{d}{dt} \int_{\pi/2}^{\pi} \frac{\cos xt}{x}\, dx = \int_{\pi/2}^{\pi} \frac{\partial}{\partial x} \frac{\cos xt}{x}\, dx$

$= \int_{\pi/2}^{\pi} - \frac{x \sin xt}{x}\, dx = \frac{\cos xt}{t} \Big|_{x=\pi/2}^{\pi}$

$= \frac{\cos \pi t - \cos \frac{\pi}{2} t}{t}$

b) $\frac{d}{dy} \int_0^{y^2} \arctan \frac{x}{y^2}\, dx$

$= (\arctan \frac{y^2}{y^2}) 2y - (\arctan 0) 0 + \int_0^{y^2} \frac{\partial}{\partial y} \arctan \frac{x}{y^2}\, dx$

$= \frac{\pi}{2} y + \int_0^{y^2} \frac{-2xy}{x^2 + y^4}\, dx = \frac{\pi}{2} y - y \ln 2.$

EXERCISES (4. 2)

16. Evaluate

a) $\frac{\partial}{\partial x} \frac{x-y+1}{x\ y-1} \Big|_{(0,\ 0)}$ b) $\frac{\partial}{\partial y} \frac{x-y+1}{x+y-1} \Big|_{(0,\ 0)}$

17. If $u(x, y, z) = x^2 y + y^2 z + z^2 x$, verify that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = (x+y+z)^2$$

18. Verify the equality $f_{xy} = f_{yx}$ for

a) $f(x, y) = \cos xy^2$ b) $f(x, y) = \sin^2 x \cos y$

c) $f(x, y) = e^{y/x}$ d) $f(x, y) = \sqrt{x^2 + y^2}$

19. Prove

a) $f(x, y) = \ln(x^2 + y^2) + \arctan \frac{y}{x} \Rightarrow f_{xx} + f_{yy} = 0$

b) $f(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}} \Rightarrow f_{xx} + f_{yy} + f_{zz} = 0$

20. Suppose $y = xf(z) + g(z)$ defines $z = z(x, y)$. Show that

$$z_x^2 \, z_{yy} - 2 \, z_x \, z_y \, z_{xy} + z_y^2 \, z_{xy} = 0$$

21. Show that

$$u(x, y, z) = \arctan \frac{y}{z} + \arctan \frac{z}{x} + \arctan \frac{x}{y}$$

verifies

$$u_{xx} + u_{yy} + u_{zz} = 0$$

22. If $f(x, y) = x \, e^x \cos y$, then evaluate

$$\frac{\partial^4 f}{\partial x^4} + 2 \, \frac{\partial^4 f}{\partial y^2 \partial x^2} + \frac{\partial^4 f}{\partial y^4}$$

23. Find an approximate value of

$$\sqrt{(9{,}02)^2 + (11{,}99)^2}$$

24. Find approximate increments of the following functions from the point P to P':

a) $f(x, y) = x^2 + 3xy - y^2$, $P(1, 2)$, $P'(1{,}02;\ 1{,}99)$

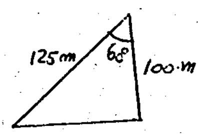
b) $f(x, y) = x^2 y + \frac{y}{x}$, $P(2, 0)$, $P'(1{,}98;\ 0{,}30)$

25. Same question for:

a) $f(x, y, z) = x \ln y + y \ln z + z \ln x$, $P(1, 1, 1)$, $P'(0{,}9;\ 0{,}8;\ 1{,}2)$

b) $f(x, y, z) = \arctan xyz$, $P(1, \frac{1}{2}, 2)$, $P'(1{,}2;\ 0{,}6;\ 1{,}7)$

26. Find the approximate error in calculating the area of a triangular piece of land, if the error in measurement the sides is $0{,}2$ m and that of the angle is 1°.



27. a) Show $z = -\arctan x - \arctan y \Rightarrow dz = -\frac{dx}{1+x^2} - \frac{dy}{1+y^2}$

b) Show $z = \arccos \frac{1-xy}{\sqrt{1+x^2}\ \sqrt{1+y^2}} \Rightarrow dz = -\frac{dx}{1+x^2} - \frac{dy}{1+y^2}$

28. Find du/dt, if

a) $u = \arctan \frac{y}{x}$ where $x = e^t - e^{-t}$, $y = e^t\ e^{-t}$

b) $u = e^x \sin yz$ where $x = t^2$, $y = t-1$, $z = 1/t$

29. Evaluate

a) $\frac{\partial}{\partial y}\ \frac{\partial}{\partial x} \sin(2xy^2 - 4x)$ b) $\frac{\partial}{\partial x}\ \frac{\partial}{\partial y} f(2x-3y,\ 2x+3y)$

30. Let $F(x+y,\ xy) = 0$ define $y = y(x)$. Show that

a) $\frac{dy}{dx} = \frac{y-\varphi(x+y)}{\varphi(x+y)-2}$ where φ is a convenient function,

b) $\frac{dy}{dx} = \frac{1-x\ \psi(xy)}{x\ \psi(xy)-1}$ where ψ is a convenient function.

31. Evaluate dy/dx for the implicit function:

a) $x^2+y^2 = \exp 2 \arctan \frac{y}{x}$ b) $\frac{x\ \ln y}{y\ \ln x} + \frac{y\ \ln x}{x\ \ln y} = 2$

c) $x(\sin \frac{y}{x} + e^{x/y}) - ky = 0$ d) $1 + xy = \ln \operatorname{Sh} xy$

32. Evaluate dy/dx, if

$$F(\arctan \frac{2x+y+1}{x+2y+1},\ \arctan \frac{x+2y+1}{2x+y+1}) = 0$$

33. Given that $z = F(x, y)$, $x = e^u \cos v$, $y = e^u \sin v$, show that

$$z_{uu} + z_{vv} = e^{2u}\ (z_{xx} + z_{yy})$$

34. If $z = f(x+2y) + g(x-2y)$, show that

$$4\ z_{xx} - z_{yy} = 0$$

35. If $xy = x+y$ defines $y = y(x)$, show

a) $y''^2 + 4y'^3 = 0$, b) $y''' + 6y'^2 = 0$

36. Show that the function $z = r - a \cos\theta$ where $r = \sqrt{x^2+y^2}$,

$\theta = \arctan \frac{y}{x}$ (a constant) satisfies

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 0$$

37. If in $H(x, y)$ one makes the substitution

$$x^2 = 2(au + bv), \quad y^2 = 2(au - bv),$$

evaluate $\partial^2 H/\partial x \partial y$ in terms of partial derivatives of H with respect to u, v.

38. Find z_{uu}, if $z = x^2 - 2xy - y^2$ where

$$x = u^2 - v^2, \quad y = 2uv$$

a) by chain rule

b) by finding z in terms of u and v.

39. Which one of the following is homogeneous and find its degree:

a) $f(x, y) = \frac{(x^2 + y^2 - 3xy)^{1/2}}{x^3 + y^3 + 2xy^2}$ b) $f(x, y) = x \ln y - y \ln x$

40. Verify EULER's Theorem for the function $f(x, y) = \ln \frac{y}{x} + \arctan \frac{x}{y}$

41. Same question for:

a) $u = \arcsin \sqrt{\frac{x^2 - y^2}{x^2 + y^2}}$

b) $u = (x + y + z)^3 - (x + y + z)^3 - (x - y + z)^3 - (x + y - z)^3$

42. Show that every line normal to the curve $z^2 = 3x^2 + 3y^2$ intersects z-axis.

43. Find the equation of the tangent plane and the equation of the normal line to the given surface at the given point:

a) $z = e^x \sin y$, $(1, \frac{\pi}{2}, e)$ b) $\sqrt{x} + \sqrt{y} + \sqrt{z} = 6$, $(4, 1, 9)$

44. Show that the sum of the squares of the intercepts of any plane tangent to the surface

$$x^{2/3} + y^{2/3} + z^{2/3} = c^{2/3}$$

is constant.

45. Find the equations of the tangent line to curve of intersection of two surfaces, at the given point:

a) $z = x^2+y^2$, $\quad z = 2x + 4y + 20$, $\quad (4, -2, 20)$

b) $z = \sqrt{25 - 16x^2}$, $\quad z = e^{xy} + 2$, $\quad (1, 0, 3)$

46. Find the point(s) on the surface $z = x^2-y^2$ at which the tangent plane is parallel to $\vec{V} = (2, 1, 3)$

47. Find the points of the surface

$$x^{2/3} + y^{2/3} + z^{2/3} = 4$$

at which the normal line is parallel to $\vec{v} = (2, 1, -1)$.

48. Show that the tangent plane of the following surfaces at any point passes through a fixed point and find this fixed point:

a) $\dfrac{(x-1)^2}{4} + \dfrac{(y-2)^2}{3} - z^2 = 0 \qquad$ b) $\dfrac{(x-\alpha)^2}{a^2} + \dfrac{(y-\beta)^2}{b^2} - \dfrac{(z-\gamma)^2}{c^2} = 0$

49. Find the equations of the tangent plane and the normal line of the following surfaces at the given point:

a) $z = \ln xy$, $\quad (1, e^2, 2) \qquad$ b) $z = x^2+2xy-y^2$, $\quad (0, 1, -1)$

50. Find the directional derivative of the following function in the given direction at the given point in the positive sense:

a) $z = x^2-2xy$ along $y = x^2$ at $(2, 4, -12)$

b) $z = x \ln \sin y$ along $y = \frac{\pi}{2} x$ at $(1/3, \pi/6, -2/3)$

51. Find the directional derivative of the following function at the given point in the given direction:

a) $w = x \ln yz$ at $A(1, 1, 1)$ toward $B(3, -1, 2)$

b) $w = e^{x^2} \sin y \cos z$ at $A(0, \pi/2, 0)$ toward $B(-1, \pi/2, 2)$

52. Same question for:

a) $f(x, y, z) = e^x \cos y + e^y \sin z$ at $P(2, 1, 0)$ toward $P'(-1,2,2)$

b) $f(x, y, z) = \ln(x^2+z^2)+e^y$ at $P(0, 0, 1)$ toward $P'(-4, 3, 2)$.

53. In which direction is the directional derivative of $f(x, y) = e^x \sin y$ at $(0, \pi/6)$ is a maximum?

54. Find directional derivative $D_a f$ at the point P when a is a unit vector:

a) $x^2y + xy\, e^x - 2xz\, e^y$; $P(1, 2, 0)$, $a = (\frac{2}{7}, -\frac{3}{7}, \frac{6}{7})$

b) $\sin xz + \cos xy$; $P(0, -1, 2)$, $a = (1, -1, 2)/\sqrt{6}$.

55. Find ∇f (gradient) at the given point

a) $x^3 - 2x^2y - xy^2 + y^3$; $P(3, -2)$

b) $\ell n(x^2 + y^2 + 1) + e^{2xy}$; $P(0, -2)$

56. Show that there is a vector u such that

$$\operatorname{div}(v \times a) = u.a$$

for any constant vector a

57. Find $\operatorname{div}(f \nabla g - g \nabla f)$

58. Find curl u. If curl $u \equiv 0$, find f such that $\nabla f = u$.

a) $u = (2x-y+3z)i + (-x+3y+2z)j + (2x+3y-z)k$

b) $u = (2xy+z^2)i + (2yz+x^2)j + (2xz+y^2)k$

59. Compute $\nabla^2 u$ for $u = f(r)$, $r = \sqrt{x^2 + y^2 + z^2}$

60. Evaluate $\nabla^2 f(x, y)$ for

a) $f(x, y) = e^x \cos y$ b) $f(x, y)\ e^x \sin y$

61. Find div U and curl U for

a) $U = e^x\, \mathrm{Sh}\, yz\, i + e^y\, \mathrm{Sh}\, zx\, j + e^z\, \mathrm{Sh}\, xy\, k$

b) $U = x\, \ell n\, yz\, i + y\, \ell n\, zx\, j + z\, \ell n\, xy\, k$

62. Find the locus of the points

a) when div $U = 0$ b) curl $U = 0$

where $U = xy\, i + yz\, j + xz\, k$

63. Given

$$z = \int_0^x e^{y^2} \cos(x-y)\,dy$$

find z_x, z_y.

64. Same question for

$$z = \int_x^{x^2} \ln xy\; dy$$

65. Show that

$$y = \frac{1}{k} \int_0^x f(\alpha) \sin k(x-\alpha)\,d\alpha$$

satisfies the relation

$$\frac{d^2y}{dx^2} + k^2 y = f(x).$$

ANSWERS TO EVEN NUMBERED EXERCISES

16. a) -2, b) 0

22. 0

24. a) 0,17, b) 1,35

26. 74,0 m^2

28. a) $-\text{Sech } 2t$, b) $\left(\frac{1}{t^2} \cos \frac{t-1}{t} + 2t \sin \frac{t-1}{t}\right) e^{t^2}$

32. $-\frac{3y+1}{3x+1}$

38. $12u^2 - 24uv - 12v^2$

46. on Γ: $4x - 2y = 3$, $z = x^2 - y^2$

48. a) $(1, 2, 0)$, b) (α, β, γ)

50. a) $-20/\sqrt{17}$, b) $\left(\frac{\pi}{\sqrt{3}} - \ln 4\right)/\sqrt{4+\pi^2}$

52. a) $(-3e^2 \cos 1 - e^2 \sin 1 + 2e)/\sqrt{14}$, b) $5/\sqrt{26}$

54. a) $(-12e^2 + 5e - 5)/7$, b) $2/\sqrt{6}$

58. a) $i+j$, b) 0, $f(x, y, z) = x^2y + y^2z + z^2x + c$

60. a) 0, b) 0

62. a) $x + y + z = 0$, b) $z = 0$

64. a) $z_x = 2x \ln x^3 - \ln x^2 + x - 1$, b) $z_y = 0$

4. 3. SOME APPLICATIONS

A. RELATED RATES

Consider the total derivative

$$\frac{df}{d\alpha} = f_1 \frac{dx_1}{d\alpha} + \ldots + f_n \frac{dx_n}{d\alpha} \qquad (1)$$

of a function of n variables with respect a variable α.

In (1) there are $n+1$ rates of change with respect to α of which n of them are independent and the remaining one is dependent. Then (1) is a relation between these rates of change. So one of them can be computed in terms of the known n rates of change.

Finding one of these rates of change this way, is a related rate problem.*

Example. Three points $P(x, 0, 0)$, $Q(0, y, 0)$, $R(0, 0, z)$ on the coordinate axes, initially at the origin, move with velocities 2, 3, 4 units/sec respectively. How fast the area $|PQR|_2$ is changing when P is at $P_0(6, 0, 0)$?

Solution. We have

$$A = |PQR|_2 = \frac{1}{2} \; |\vec{PQ} \times \vec{PR}| = \frac{1}{2} \sqrt{y^2z^2 + z^2x^2 + x^2y^2}$$

$$4A^2 = y^2z^2 + z^2x^2 + x^2y^2$$

* Since in (1) $dx_i/d\alpha$ are known, the variables x_i and (hence) f are expressible as functions of α. Then the problem is reducible to that of a function of a single variable.

$$4A\,\frac{dA}{dt} = x(y^2+z^2)\,\frac{dx}{dt} + y(z^2+x^2)\,\frac{dy}{dt} + z(x^2+y^2)\,\frac{dz}{dt}$$

When P is at $P_0(6, 0, 0)$, then Q is at $Q_0(0, 9, 0)$ and R is at $R_0(0, 0, 12)$ with $|P_0\,Q_0\,R_0|_2 = 9\sqrt{61}$. Then

$$4.9\sqrt{61}\,\frac{dA}{dt} = 6(225)2 + 9(680)3 + 12(117)4$$

$$9\sqrt{61}\,\frac{dA}{dt} = 675 + 27.45 + 12.117$$

$$\sqrt{61}\,\frac{dA}{dt} = 75 + 135 + 156 = 366$$

$$\frac{dA}{dt} = \frac{766}{\sqrt{61}} = 6\sqrt{61} \quad \text{unit}^2/\text{sec.}$$

B. Taylor's Formula and Series

<u>Theorem</u>. If $f(x, y)$ has continuous partial derivatives up to order $n+1$ in a neighborhood of $(a, b)\varepsilon D_f$, then

$$f(x, y) = f(a, b) + \sum_{k=1}^{n} \frac{1}{k!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^k f(x,y)\Big|_{(a,b)} + R_{n+1} \qquad (1)$$

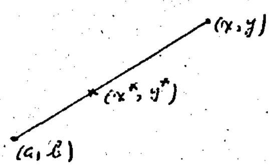
where the remainder is given by

$$R_{n+1} = \frac{1}{(n+1)!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial x}\right]^{n+1} f(x, y)_{(x^*,\ y^*)}$$

with (x^*, y^*) a point on the open segment (P_0P) joining $P_0(a, b)$ to $P(x, y)$

<u>Proof</u>. Since every point of the line segment $[P_0P]$ can be represented parametrically as

$$x = a+ht, \quad y = b+kt \qquad 0 \leqslant t \leqslant 1, \qquad (2)$$



The end points of the segment correspond to $t = 0$ and $t = 1$ (observe that h, k are direction numbers of the line segment)

Substituting (2) in $f(x, y)$ gives the function

$$F(t) = f(a+ht,\ b+kt)$$

of a single variable t. Since the TAYLOR's formula for (x, y) at (a, b) is the MACLAURIN's Formula for F(t), we have

$$F(t) = F(0) + \sum_{k=1}^{n} \frac{F^{(k)}(0)}{k!} t^k + \frac{F^{(n+1)}(t*)}{(n+1)!} t^{n+1}$$

Now

$$F(t) = f(a+ht,\ b+kt) \quad \Rightarrow \quad F(0) = f(a,\ b)$$

$$F'(t) = f_1 h + f_2 k \quad \Rightarrow \quad F'(0) = hf_1(a,\ b) + k\ f_2(a,\ b)$$

$$F''(t) = f_{11}h^2 + 2f_{12}\ hk + f_{22}k^2$$

$$\Rightarrow \quad F''(0) = h^2 f_{11}(a,\ b) + 2hkf_{12}(a,\ b) + k^2 f(a,\ b)$$

$$= \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right)^2 f(x,\ y)\Big|_{(a,\ b)}$$

$$F^{(n)}(0) = \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right)^n f(x,\ y)\Big|_{(a,\ b)}$$

and

$$F(t) = f(a,\ b) + \sum_{1}^{n} \frac{1}{k!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right)^k f(x,y)\Big|\, t^n + R_{n+1}$$

where

$$R_{n+1} = \frac{1}{(n+1)!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right)^{n+1} f(x,\ y)\Big|_{t=t*} t^{n+1},$$

recalling that $x = a+kt$, $y = b+kt$. Then for $t = 1$, having $h=x-a$, $k = y-b$ we obtain (1). ∎

The series

$$f(x,\ y) = f(a,\ b) + \sum \frac{1}{n!} \left((x-a) \frac{\partial}{\partial x} + (y-b) \frac{\partial}{\partial y}\right)^n f(x,\ y)\Big|_{(a,b)} \quad (3)$$

is called the TAYLOR's series if (x, y) at (a, b).

The TAYLOR's Series for a function of more than two variables at a point $(a_1, \ldots, a_n)$ can be obtained by similar substitution $x_i = a_i + h_i t$.

The TAYLOR's Formula written with R_1 is called the MVT for a function of two variables:

$$f(x, y) = f(a, b) + (x-a)f_x(x^*, y^*) + (y-b)f_y(x^*, y^*)$$

where (x^*, y^*) is on the open line segment joining (a, b) to (x, y). If (a, b) and (x, y) are known, one can find at least one point (x^*, y^*) satisfying the same condition.

Example 1. Given the function

$$f(x, y) = 2x^4 + 3xy^2 - y^3$$

a) obtain the TAYLOR's Formula with R_3 at $A(2, 3)$

b) evaluate $f(1,9 ; 3)$ approximately

c) show existence of (x^*, y^*) on (AB) where $B(1, 1)$.

Solution.

a) $$f(x, y) = f(2, 3) + \left[(x-2)f_x(A) + (y-3)f_y(A)\right] + \frac{1}{2}\left[(x-2)^2 f_{xx}(A) + 2(x-2)(y-3)f_{xy}(A) + (y-3)^3 f_{yy}(A)\right] + R_3$$

where

$$f(2, 3) = 59, \quad f_x(A) = 8x^3 + 3y^2\Big|_A = 9,$$

$$f_y(A) = 6xy - 3y^2\Big|_A = 9$$

$$f_{xx}(A) = 24x\Big|_A = 96, \; f_{xy}(A) = 69\Big|_A = 18, \; f_{yy}(A) = 6x - 6y\Big|_A = -6$$

Hence

$$f(x, y) = 59 + \left[91(x-2) + 9(y-3)\right] + \frac{1}{2}\left[48(x-2)^2 + 36(x-2)(y-3) - 6(y-3)^2\right] + R_3$$

b) $$f(1,9 ; 3) \cong 59 + \left[91(-0,1)\right] + \frac{1}{2}\left[48(-0,1)^2\right] =$$

$$= 59 - 9,1 + 0,24 = 50,14$$

By functional value

$f(1, 9; 3) \cong 50,4$

c) $f(1, 1) = f(2, 3) + (1-2)\ f_x(x^*, y^*) + (1-3)\ f_y(x^*, y^*)$

$$\Rightarrow \quad 4 = 59 - (8x^{*3} + 3y^{*2}) - 2(6x^*y^* - 3y^{*2})$$

$$\Rightarrow \quad 55 = 8x^{*3} + 12x^*y^* \qquad (1)$$

Since (x^*, y^*) lies on AB, we have

$$y^* = 2x^* - 1 \qquad (2)$$

and (1), (2) give

$$\varphi(x^*) = 8x^{*3} + 24x^{*2} - 12x^* - 55 = 0$$

Since Since $\varphi(1) = -35 < 0$, $\varphi(2) = 81 > 0$ hold, x^* must lie between 1 and 2.

Example 2. Given

$$f(x, y) = \text{Arctan}\,\frac{y}{x}$$

a) obtain TAYLOR's Formula with R_3 at $A(1, \sqrt{3})$

b) evaluate $f(2, 3)$

c) Show existence of (x^*, y^*) on the open line segment joining $A(1, \sqrt{3})$ to $B(\sqrt{3}, 1)$.

Solution.

a) $$f(x, y) = f(1, \sqrt{3}) + \left[(x-1)\ f_x(A) + (y-\sqrt{3})\ f_y(B)\right] + \frac{1}{2}\left[(x-1)^2\ f_{xx}(A) + 2(x-1)(y-\sqrt{3})f_{xy}(A) + (y-\sqrt{3})^2 f_{yy}(A)\right] + R_3$$

where

$$f(1, \sqrt{3}) = \frac{\pi}{3}, \quad f_x(A) = -\frac{\sqrt{3}}{4}, \quad f_y(A) = \frac{1}{4}$$

$$f_{xx}(A) = \frac{\sqrt{3}}{8}, \quad f_{xy}(A) = \frac{1}{8}, \quad f_{yy}(A) = -\frac{\sqrt{3}}{8}$$

$$f(x, y) = \frac{\pi}{3} + \left[-\frac{\sqrt{3}}{4}(x-1) + \frac{1}{4}(y-\sqrt{3})\right] + \frac{1}{2}\left[\frac{\sqrt{3}}{8}(x-1)^2 + \frac{1}{4}(x-1)(y-\sqrt{3}) - \frac{\sqrt{3}}{8}(y-\sqrt{3})^2\right] + R_3$$

b) $f(2, 3) \cong \frac{\pi}{3} + \left[-\frac{\sqrt{3}}{4} + \frac{1}{4}(3 - \sqrt{3})\right]$

$$+ \frac{1}{2}\left[\frac{\sqrt{3}}{8} + \frac{1}{4}(3 - \sqrt{3}) - \frac{\sqrt{3}}{8}(3 - \sqrt{3})^2\right] \cong 1,12$$

So, $\arctan \frac{3}{2} \cong 1,12$.

c) $f(x, y) = f(1, \sqrt{3}) + (x-1)\, f_x(x^*, y^*) + (y-\sqrt{3})\, f_y(x^*, y^*)$

$$f(\sqrt{3}, 1) = \frac{\pi}{3} + (\sqrt{3}-1)\frac{-y^*}{x^{*2}+y^{*2}} + (1-\sqrt{3})\frac{x^*}{x^{*2}+y^{*2}}$$

$$\frac{\pi}{6} = \frac{\pi}{3} - (\sqrt{3} - 1)\frac{x^*+y^*}{x^{*2}+y^{*2}} \qquad (1)$$

Since (x^*, y^*) is on AB we have

$$x^* + y^* = \sqrt{3} + 1 \qquad (2)$$

(1), (2)

$$(x^*) = x^{*2} - (\sqrt{3}+1)x^* + (2+\sqrt{3} - \frac{6}{\pi}) = 0$$

Since $(1) < 0$, $(\sqrt{3}) > 0$, then $1 < x^* < \sqrt{3}$ and have $1 < y^* < \sqrt{3}$.

C. Envelope

I. Envelope of a Family of Plane Curves

We say that the family

$$\Gamma_\lambda: \quad F(x, y, \lambda) = 0 \qquad (1)$$

admits an envelope, if a curve

$$e: \quad e(x, y) = 0$$

exists to which every curve Γ_λ of (1) is tangent.

The curve e is called the *envelope* and the points of tangency of Γ_λ with e are the characteristic point of Γ_λ.

Hence the envelope is the locus of characteristic points.

For instance a plane curve is the envelope of its tangent lines (of the family of tangent lines).

A family (1) dependent on a single parameter admits in general an envelope, and the equation of the envelope is given in the theorem:

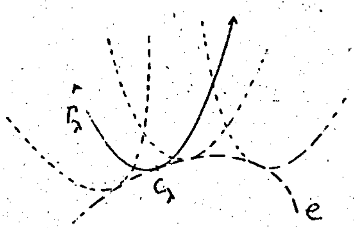
Theorem. If the family

$$\Gamma_\lambda : \; F(x, y, \lambda) = 0 \qquad (1)$$

depending on a single parameter λ admits an envelope e, the characteristic point C of Γ_λ satisfies the simultaneous equation

$$e: \; F(x, y, \lambda) = 0, \quad F_\lambda(x, y, \lambda) = 0$$

Proof. Each curve Γ_λ has at least one point of contact with the envelope (characteristic point). Let C_λ be one of these. The coordinates of C_λ depend on λ and verify $F(x, y, \lambda) = 0$ identically.



Let us express tangency of Γ_λ and the envelope at C_λ:

Slope of Γ_λ at C_λ: $-F_x/F_y$

Slope of e at C_λ: $(dy/d\lambda)/(dx/d\lambda) = 0$

$$\Rightarrow \quad -\frac{F_x}{F_y} = \frac{dy/d\lambda}{dx/d\lambda} \quad \Rightarrow \; F_x \frac{dx}{d\lambda} + F_y \frac{dy}{d\lambda} = 0$$

Using this result in the total derivative of $F(x, y, \lambda)$ with respect to λ, we have

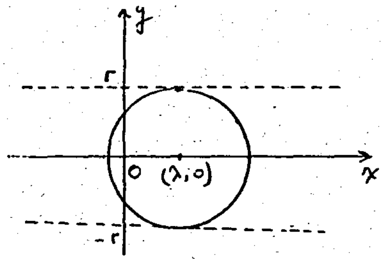
$$\underbrace{F_x \frac{dx}{d\lambda} + F_y \frac{dy}{d}}_{0} + F_\lambda = 0 \;\Rightarrow\; F_\lambda = 0. \blacksquare$$

Example 1. Find the envelope of the family of circles of constant radii r, centers on x-axis.

Solution.

Γ_λ : $F(x, y, \lambda) = (x-\lambda)^2 + y^2 - r^2 = 0$

$F_\lambda = -2(x - \lambda) = 0 \Rightarrow x = \lambda.$



Then

$$(x-\lambda)^2 + y^2 - r^2 = 0, \quad x = \lambda$$

are the parametric equation of the envelope.

Then eliminating λ, one gets

$$y^2 - r^2 = 0 \Rightarrow y = \pm r,$$

showing that the envelope consists of two lines parallel to x-axis. This envelope is the locus of heighest and lowest points (characteristic points) of the circles.

Example 2. Find the envelope of the family

$$\Gamma_\lambda \quad F(x, y, \lambda) = 2(x-\lambda)^3 - 3(y-\lambda)^2 = 0$$

Solution. The curve Γ_λ is related to Γ_0: $2x^3 - 3y^2 = 0$ by the translation (λ, λ), and Γ_0 admits a cusp[(1)] at the origin.

Since

$$F_\lambda(x, y, \lambda) = -6(x-\lambda)^2 + 6(y-\lambda) = 0$$

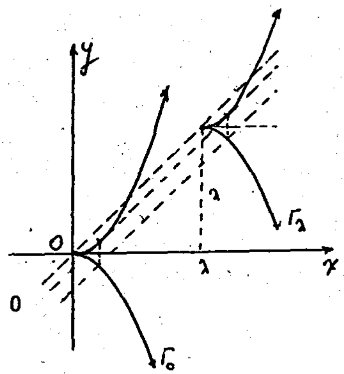
we have the envelope:

$$e: 2(x-\lambda)^3 - 3(y-\lambda)^2 = 0, \quad (x-\lambda)^2 - (y-\lambda) = 0$$

To eliminate λ, setting $y-\lambda = (x-\lambda)^2$ in the first, we have

$$2(x-\lambda)^3 - 3(x-\lambda)^4 = 0 \Rightarrow (x-\lambda)^3\left[2-3(x-\lambda)\right] = 0$$

$$\text{(i) } x-\lambda = 0, \qquad \text{(ii) } x-\lambda = \frac{2}{3}$$



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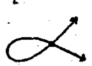
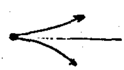
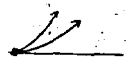
(1) A *cusp* is a double point at which two tangent lines are coincident.



A double point



A cusp
of the first kind



A cusp
of the second kind

(i) $x-\lambda = 0 \Rightarrow y-\lambda = 0 \Rightarrow y = x$

(ii) $x-\lambda = \frac{2}{3} \Rightarrow \lambda = x - \frac{2}{3}$

$$\Rightarrow 2.\frac{8}{27} - 3(y-x+\tfrac{2}{3})^2 = 0$$

$$\Rightarrow (y-x+\tfrac{2}{3})^2 = \frac{16}{81}$$

$$\Rightarrow y-x+\frac{2}{3} = \pm\frac{4}{9}$$

$$\Rightarrow y = x - \frac{2}{9} \text{ or } y = x - \frac{10}{9}$$

Thus we have obtained three equations:

$$y = x, \quad y = x - \frac{2}{9} \quad \text{and} \quad y = x - \frac{10}{9}$$

The first one, $y = x$, is the locus of cusps, the second, $y = x - \frac{2}{9}$, is the envelope, while the third is not.

<u>Corollary 1</u>. The envelope of the family

$$F(x, y, \lambda, \mu) = 0, \quad \varphi(\lambda, \mu) = 0$$

is given by

$$F = 0, \quad \varphi = 0, \quad \frac{F_\mu}{\varphi_\lambda} = \frac{F_\mu}{\varphi_\mu}$$

<u>Proof</u>. Since $\varphi(\lambda, \mu) = 0 \Rightarrow \mu = \mu(\lambda)$ theoretically, the envelope is $F(x, y, \lambda, \mu) = 0$, $F_\lambda + F_\mu \frac{d\mu}{d\lambda} = 0$.

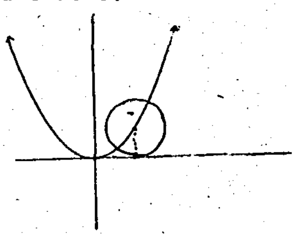
Now eliminating $d\mu/d\lambda$ between the latter and $\;_\lambda + \;_\mu \frac{d\mu}{d\lambda} = 0$ one gets $F_\lambda / \;_\lambda = F_\mu / \;_\mu$

<u>Example</u>. Find the envelope of circles centered on the parabola $y = x^2$ and tangent to x-axis.

<u>Solution</u>. Let (α, β) be the center of a circle. Then the radius being $|\beta|$ we have

$$F(x, y, \alpha, \beta) = (x-\alpha)^2 + (y-\beta)^2 - \beta^2 = 0$$

with $(\alpha, \beta) = \alpha^2-\beta = 0$ as the equation of the family.



$$e:\ F=0,\quad \varphi=0,\quad \frac{-2(x-\alpha)}{2\alpha}=\frac{-2(y-\beta)-2\beta}{-1}\quad (=2y)$$

$$\Rightarrow\quad F=0,\quad \varphi=0,\quad x-\alpha=-2\alpha y\qquad (\alpha=\frac{x}{1-2y})$$

Setting $\alpha = \frac{x}{1-2y}$ and $\beta = \frac{x^2}{(1-2y)^2}$ in $F = 0$,

we get

$$(x-\frac{x}{1-2y})^2+(y^2-2\frac{x^2}{(1-2y)^2}\,y=0$$

$$\Rightarrow\quad 4x^2y^2+y^2(1-2y)^2-2x^2y=0$$

$$\Rightarrow\quad y\left[2x^2(2y-1)+y(2y-1)^2\right]=0$$

$$\Rightarrow\quad y(2y-1)\left[2x^2-y+2y^2\right]=0,\qquad (2y-1\neq 0)$$

$$\Rightarrow\quad y=0\ \text{ or }\ x^2+(y-\tfrac{1}{4})^2=\tfrac{1}{16}$$

Hence the envelope consists of the x-axis and the circle of the radius 1/4 centered at (0, 1/4).

Solve this by eleminating one of α, β!

Corollary 2. The envelope of the family

$$\Gamma_\lambda:\ x=f(t,\lambda),\quad y=g(t,\lambda)$$

is given by

$$e:\ x=f(t,\lambda),\quad y=g(t,\lambda),\quad \frac{f_t}{g_t}=\frac{f_\lambda}{g_\lambda}$$

Proof. The characteristic point C_λ of Γ_λ is a function of λ so that $t = t(\lambda)$.

Slope of Γ_λ : $\frac{g_t}{f_t}$

Slope of e : $\frac{dy/d\lambda}{dx/d\lambda} = \frac{g_t\,t' + g_\lambda}{f_t\,t' + f_\lambda}$

Equating these one gets $f_t/g_t = f_\lambda/g_\lambda$. ∎

Example. Find the envelope of the family

$$\Gamma_\lambda: \quad x = \lambda + 3 \cos t, \quad y = 2\lambda + 3 \sin t$$

Solution.

$$e: x = \lambda + 3 \cos t, \quad y = 2\lambda + 3 \sin t, \quad \frac{-3 \sin t}{3 \cos t} = \frac{1}{2}$$

$$2x - y = 6 \cos t - 3 \sin t, \quad \tan t = -1/2$$

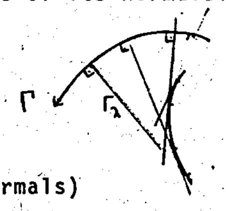
with $\cos t = \pm 2/\sqrt{5}$, $\sin t = \pm 1/\sqrt{5}$

$2x - y = \pm 3\sqrt{5}$ (two parallel lines)

EVOLUTE.

The *evolute* of a plane curve is the envelope of its normals.

Let $y = f(x)$ be the equation of a plane curve.



$\Gamma : y = f(x)$

$\Gamma_\lambda : y - f(\lambda) = -\dfrac{1}{f'(\lambda)}(x - \lambda)$ (family of normals)

The envelope of Γ_λ is the evolute of Γ.

Theorem. The evolute of a plane curve is the locus of center of curvatures (Book II, p.241).

Proof. Differentiating the equation

$$y f'(\lambda) - f(\lambda) f'(\lambda) + x - \lambda = 0$$

of Γ_λ with respect to λ, we have

$$y f''(\lambda) - f'^2(\lambda) - f(\lambda) f''(\lambda) - 1 = 0$$

$$\Rightarrow \quad x = \lambda - f'(\lambda) \frac{1 + f'^2(\lambda)}{f''(\lambda)}, \quad y = f(\lambda) + \frac{1 + f'^2(\lambda)}{f''(\lambda)} \qquad (*)$$

Example. Find the evolute of $y^2 = x$.

Solution. Taking $y=\lambda$ as parameter, the equation of normal at (λ^2, λ) being

$$y - \lambda = -2\lambda(x - \lambda^2)$$

differentiation with respect to λ gives

$$-1 = -2x + 6\lambda^2 \qquad x = 3\lambda^2 + \frac{1}{2}$$

$$y = \lambda - 2\lambda\left(3\lambda^2 + \frac{1}{2} - \lambda^2\right) = -4\lambda^3$$

Then

$$e: \ x = 3\lambda^2 + \frac{1}{2}, \quad y = -4\lambda^3$$

is the evolute.

Obtain the same result by direct application of the formula (*)

2. Envelope of Family of Surfaces

We say that a family of surfaces admits an envelope, if there exists a surface E such that each surface of the family is tangent to E. E is the envelope.

For instance a cylinder is the envelope of its tangent planes (a tangent plane depends on a single parameter and is tangent to the cylinder along a generator), and an ellipsoid is the envelope of its tangent plane (the tangent plane depends on two parameters).

Envelope of Surfaces Dependent on one Parameter:

Theorem. The envelope of the family

$$S_\lambda : F(x, y, z, \lambda) = 0$$

is given by

$$E : F(x, y, z, \lambda) = 0, \quad F_\lambda(x, y, z, \lambda) = 0$$

Proof. Let E be the envelope of S_λ. We show that the coordinates of every point of E satisfies $F=0$ and $F_\lambda = 0$.

Let $P(x, y, z)$ be any point of the characteristic curve C along which S_λ is tangent to E. The coordinates of P are functions of λ (which fixes S_λ) and of a parameter μ which fixes P on C). $x(\lambda, \mu)$, $y(\lambda, \mu)$, $z(\lambda, \mu)$ satisfy $F = 0$ identically, and then

$$F_x\, dx + F_y\, dy + F_z\, dz + F_\lambda\, d_\lambda = 0$$

holds, where dx, dy, dz are direction numbers of the tangent line to C at P and this line being on the tangent plane, one has $F_x\, dx + F_y\, dy + F_z\, dz = 0$, and $F_\lambda = 0$ follows. ■

Corollary 1. The envelope of

$$\mathcal{S}: \quad F(x, y, z, \lambda, \mu) = 0, \quad \varphi(\lambda, \mu) = 0$$

is given by

$$\mathcal{E}: \quad F = 0, \quad = 0, \quad \frac{F_\lambda}{\varphi_\lambda} = \frac{F_\mu}{\varphi_\mu}$$

Corollary 2. The envelope of

$$\mathcal{S}: \quad F(x, y, z, \lambda, \mu, \gamma) = 0, \quad \varphi(\lambda, \mu, \gamma) = 0$$

is given by

$$\mathcal{E}: \quad F = 0, \quad \varphi = 0, \quad \frac{F_\lambda}{\varphi_\lambda} = \frac{F_\mu}{\varphi_\mu} = \frac{F_\gamma}{\varphi_\gamma}$$

Envelope of Family of Surfaces Dependent on two Parameters:

Theorem. The envelope of

$$S_{\lambda,\mu}: \quad F(x, y, z, \lambda, \mu) = 0$$

is given by

$$E: \quad F = 0, \quad F_\lambda = 0, \quad F_\mu = 0$$

Proof. The coordinates x, y, z of the characteristic point C are functions of λ, μ and satisfy $F = 0$ identically, and for the total differential

$$F_x\,dx + F_y\,dy + F_z\,dz + F_\lambda\,d_\lambda + F_\mu\,d_\mu = 0$$

holds, where the sum of the first three terms is zero, since dx, dy, dz are direction numbers of a tangent line to E at C, lying on the tangent plane.

Hence $F_\lambda\,d_\lambda + F_\mu\,d_\mu = 0$ holds for every λ, μ, implying $F_\lambda = 0$, $F_\mu = 0$.

Corollary. The envelope of

$$F(x, y, z, \lambda, \mu, \gamma) = 0, \quad \varphi(\lambda, \mu, \gamma) = 0$$

is given by

$$F = 0, \quad \varphi = 0, \quad \frac{F_\lambda}{\varphi_\lambda} = \frac{F_\mu}{\varphi_\mu} = \frac{F_\gamma}{\varphi_\gamma},$$

Example. Find the envelope of the family

a) of spheres with centers on z-axis and radii are half the third coordinate of the centers

b) of plane $4ux + 8vy - z - (2u^2 + 4v^2) = 0$

Solution.

a) S_λ : $x^2 + y^2 + (z-\lambda)^2 = (\frac{\lambda}{2})^2$

Differentiating every term with respect to λ one gets

$$0 + 0 - 2(z-\lambda) = \frac{1}{2}\lambda \Rightarrow \lambda = 4z/3$$

$$x^2 + y^2 + (z - \frac{4}{3}z)^2 = (\frac{2}{3})^2 = z^2$$

$$x^2 + y^2 - \frac{z^2}{3} = 0 \quad \text{(a cone)}$$

b) Differentiating every term with respect to u and v one gets

$$4x - 4u = 0, \quad 8y - 8v = 0$$

$$\Rightarrow \quad u = x, \quad v = y$$

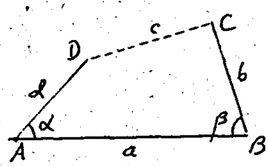
$$\Rightarrow \quad 4x^2 + 8y^2 - 2x^2 - 4y^2 \quad 0 .$$

$$z = 2x^2 + 4y^2 \quad \text{(An elliptic paraboloid).}$$

EXERCISES (4. 3)

66. A particle is moving on the line $y = x$ in xy-plane with speed 4 units/sec, and a second particle is moving on the line $x = z$ in xz-plane with speed 3 units/sec. How fast is their distance changing when the first particle is at $(2, 2, 0)$.

67. In the given figure the sides a, b, d are constant. If α increases at the rate of $\frac{\pi}{360}$ rad/sec, and β decreases at the rate of $\frac{\pi}{540}$ rad/sec, how fast



a) the side c, b) the area $|ABCD|_2$

is changing when $\alpha = \pi/2$ and $\beta = \pi/3$?

68. Expand $x^2 - xy$ about the point $(1, -2)$.

69. Same question for $e^x \cos y$ about $(0, 0)$.

70. Given the function $f(x, y) = x^2 + xy - y^2$,

   a) obtain the TAYLOR's Formula with R_2 at $A(2, 1)$

   b) find (x^*, y^*) on (AB) where $B(1, -2)$.

71. Expand $\ln(x + y)$ in powers of $x-1$ and $y-2$ up to second degree terms (with R_3)

72. Given $f(x, y) = x^3 - 2xy$

   a) obtain the TAYLOR's Formula with R_3 at $A(2, 2)$

   b) evaluate $f(2,1 \; ; \; 1,9)$ approximately

   c) find (x^*, y^*) on (AB) where $B(1, 3)$

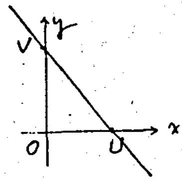
73. Find the envelope of the following family

a) $x^2 + (y-\lambda)^2 = 1$ b) $(t^2-1)x + 2ty = 1$

c) $y^2 = tx + t^2$ d) $x^2 + 2y^2 = \lambda$

e) $(x-t)^2 + y^2 = 4t$ f) $(x+\lambda)^2 + (y-\lambda)^2 = \lambda^2$

g) $y = x/m + m^2$

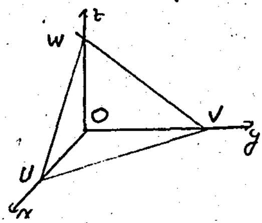


74. Show that the envelope

a) of the line UV such that

$|OU|^2 + |OV|^2 = a^2$ is

$x^{2/3} + y^{2/3} = a^{2/3}$



b) of the plane UVW such that

$|OU|^2 + |OV|^2 + |OW|^2 = a^2$ is

$x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$

75. If $z = f(x, y)$ is the envelope of

$$\pi(t): \quad z = a(t)x + b(t)y + c(t)$$

show $f_{xx}^2 = f_x^2 \, f_y^2$

76. Let a variable line ℓ intersect the axes at $(\alpha, 0)$, $(0, \beta)$. Find the envelope of ℓ under the condition:

a) $\alpha + \beta = c$ b) $\alpha\beta = c^2$

77. Find the envelope of the ellipses

$$x^2/a^2 + y^2/b^2 = 1$$

under the condition

a) $a + b = u = \text{const}$ b) $ab = u^2 = \text{const.}$

78. Show that the envelope of circles passing through the origin and having centers on the hyperbola $x^2-y^2 = a^2$ is a lemniscate.

79. If the coefficients a, b, c, p, q are functions of x and y, show that the envelope of

a) $at^2 + bt + c = 0$ is $b^2 - 4ac = 0$

b) $t^3 + pt + q = 0$ is $4p^3 + 27q^2 = 0$

c) $a \cos t + b \sin t + c = 0$ is $a^2 + b^2 = c^2$

d) $a \operatorname{Ch} t + b \operatorname{Sh} t + c = 0$ is $a^2 - b^2 = c^2$

80. Find the envelope of the family of curves:

a) $ty = 4(x + \frac{t^2}{8})$ b) $tx - \sqrt{t^2 - 9}\, y = 9$

c) $(x-\lambda)^2 + y^2 = 25$ d) $tx + \sqrt{25 - t^2}\, y = 25$

ANSWERS TO EVEN NUMBERED EXERCISES

66. 5 unit/sec.

68. $3 + \left[4(x-1)-(y+2)\right] + \frac{1}{2}\left[2(x-1)^2-2(x-1)(y+2)\right]$

70. a) $5 + 5(x-2) + R_2$, b) $(x^*, y^*) = (3/2, -1/2)$

72. a) $8(x-2)-4(y-2) + \frac{1}{2}\left[12(x-2)^2-4(x-2)(y-2)\right] + R_3$

b) 0,4 c) (13/7, 15/7)

76. a) $\sqrt{|x|} + \sqrt{|y|} = \sqrt{|c|}$, b) $xy = c^2$

80. a) $y^2 = 8x$, b) $x^2-y^2 = 9$, c) $y = \pm 5$, d) $x^2+y^2 = 25$.

4. 4. EXTREMA AND THE METHOD OF LEAST SQUARES

A. FREE EXTREMA

Let

$$f: D_f \to R_f, \quad z = f(x_1, \ldots, x_n)$$

be a function of n variables $(n \geq 2)$ with domain $D_f \subseteq R^n$ and (unknown) range $R_f \subseteq R$.

The largest (smallest) element $M(m)$ in R_f, if any, is the *global* or *absolute maximum (minimum)* of f over D_f.

If $f(P_o) \in R_f$ is the largest (smallest) in a neighborhood $N[P_o]$ of a point $P_o \varepsilon D_f$, then $f(P_o)$ is *local* or *relative maximum (minimum)* of f at P_o, and P_o is said to be an *extremum point* of f. Since no conditions are imposed on the independent variables (variables varying freely) the extrema will be called a *free extrema*.

For the global maximum (minimum) of f at a point P_o, one has

$f(P_o) \geqslant f(P)$ $(f(P_o) \leqslant f(P))$ for all $P \varepsilon D_f$, while for a local maximum (minimum) of at P_o,

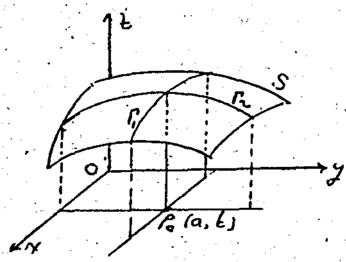
$$f(P_o) \geqslant f(P) \quad (f(P_o) \leqslant f(P)) \text{ for all } P \varepsilon N\left[P_o\right].$$

DETERMINATION OF LOCAL EXTREMA (for a function of two variables)

Let $z = f(x, y)$ be a function of two variables defined on D. If its first partial derivatives exist and both zero at $P_o(a,b)\varepsilon D$, then by the geometric interpretations of $f_x(P_o)$, $f_y(P_o)$, the point P_o is a critical point of both of the curves

Γ_1: $y = b$, $z = f(x, y)$

Γ_2: $x = a$, $z = f(x, y)$



If f has a local extrema at P_o, then f_x, f_y vanish both at P_o as stated in the following theorem.

A point P_o at which partial derivatives f_x, f_y vanish simultaneously is called a *critical point* of $f(x, y)$.

Theorem. For a function $z = f(x, y)$

1. an extrema point is a critical point (the converse is not true)

2. a critical point P_o

   a) is an extremum point if

$$\Delta = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}_{P_o} > 0 \quad \begin{cases} \text{max when } f_{11} \text{ (or } f_{22}) < 0 \\ \text{min when } f_{11} \text{ (or } f_{22}) > 0 \end{cases}$$

   b) a saddle point if

$$\Delta = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}_{P_o} < 0$$

   c) is uncertain in type if $\Delta = 0$.

Proof.

1. Let f have a relative maximum point $P_o(a, b)$ so that $f(P)-f(P_o) < 0$ holds for all $P(x, y)$ in a neighborhood $N(P_o)$ of P_o. Then, from TAYLOR's Formula,

$$f(P)-f(P_o) = h\ f_1(P_o) + k\ f_2(P_o) + R_2 < 0$$

$$\Rightarrow \qquad h\ f_1(P_o) + k\ f_2(P_o) < 0$$

since R_2, involving heigher powers of $h = x-a$, $k = y-b$, is negligible in $N(P_o)$.

Now,

$$\left.\begin{array}{l} h = k > 0 \;\Rightarrow\; f_1(P_o) + f_2(P_o) < 0 \\ h = k < 0 \;\Rightarrow\; f_1(P_o) + f_2(P_o) > 0 \end{array}\right\} \Rightarrow f_1(P_o) = 0, \quad f_2(P_o) = 0$$

The same result is obtained for relation minimum.

2. At a critical point the TAYLOR's Formula becomes

$$f(P)-f(P_0) = \frac{1}{2}\left[k^2\ f_{11}(P_0) + 2hk\ f_{12}(P_0) + k^2\ f_{22}(P_0)\right] + R_3$$

Neglecting R_3 in $N(P_0)$, the sign of $f(P)-f(P_0)$ is that of

$$\varphi(h,\ k) = h^2 f_{11}(P_0) + 2hkf_{12}(P_0) + k^2\ f_{22}(P_0)$$

where one of $h = x-a$, $k = y-b$, say k is not zero for $P \neq P_0$.
For an extremum at P_0 the sign of $\varphi(h,\ k)$ or of

$$\frac{1}{k^2}\ \varphi(h,\ k) = f_{11}(P_0)\left(\frac{h}{k}\right)^2 + 2\ f_{12}(P_0)\left(\frac{h}{k}\right) + f_{22}(P_0)$$

must not change in sign. This is the case if the discriminant δ of this quadratic (in h/k) is negative:

$$\delta = f_{12}{}^2 - f_{11}\ f_{22} < 0 \qquad \left(\Rightarrow f_{11}\ f_{12} > 0\right)$$

or that if

$$\Delta = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}_{P} > 0$$

$\varphi(h,\ k)$ being positive when $\Delta > 0$ and $f_{11}(P_0) > 0$ $(f_{11}(P_0) < 0)$ there is a local maximum (minimum).
(Note that for $\Delta > 0$, the signs of f_{11}, f_{22} at P_0 are the same).

b. If $\delta > 0$ $(\Delta < 0)$, then the sign of $\varphi(h,\ k)$ changes in $N(P_0)$. Such a point is called a *saddle point* of f.

c. When $\Delta = 0$ at P_0 one needs an investigation of heigher order derivatives.

Example. Given

$$f(x,\ y) = 3x^2 + 3y^2 - 3x^2y - y^3 - 4,$$

find and classify all critical points

Solution.

a) $f_1 = 6x-6xy = 0 \Rightarrow x(1-y) = 0 \Rightarrow x=0$ or $y=1$

$f_2 = 6y-3x^2-3y^2 = 0 \Rightarrow 2y - x^2 - y^2 = 0$

$x = 0 \Rightarrow 2y-0-y^2 = 0 \Rightarrow y=0$ or $y=2$

$y = 1 \Rightarrow 2-x^2-1 = 0 \Rightarrow x=1$ or $x=-1$

There are four critical points, namely

$A(0, 0)$, $B(0, 2)$, $C(1, 1)$, $D(-1, 1)$

b) $f_{11} = 6-6y$, $f_{12} = -6x$, $f_{22} = 6-6y$

$$\Delta(A) = \begin{vmatrix} 6 & 0 \\ 0 & 6 \end{vmatrix} > 0, \quad f_{11}(A) = 6 > 0, \quad \text{rel.min.}$$

$$\Delta(B) = \begin{vmatrix} -6 & 0 \\ 0 & -6 \end{vmatrix} > 0, \quad f_{11}(B) < 0, \quad \text{rel.max.}$$

$$\Delta(C) = \begin{vmatrix} 0 & -6 \\ -6 & 0 \end{vmatrix} < 0, \quad \text{a saddle point}$$

$$\Delta(D) = \begin{vmatrix} 0 & 6 \\ 6 & 0 \end{vmatrix} < 0 \quad \text{a saddle point.}$$

For a function of three variables we state the following theorem without proof:

Theorem. For a function $u = f(x, y, z)$.

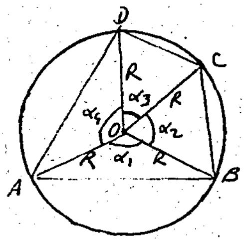
1. an extremum point is a critical point,

2. a critical point P_0 is a

a) min.point of $f_{11} > 0$, $\begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} > 0$, $\begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} > 0$

b) max. point if $f_{11} < 0$, $\begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} > 0$ $\begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} < 0$

Example. Find the quadrangle of largest area inscribed in a circle of radius R.



Solution. Referring to the figure, the area S is the sum of the areas of four triangles:

$$2S = R^2(\sin\alpha_1 + \sin\alpha_2 + \sin\alpha_3 + \sin\alpha_4)$$

$$f(\alpha_1, \alpha_2, \alpha_3) = \frac{2S}{R^2} = \sin\alpha_1 + \sin\alpha_2 + \sin\alpha_3 - \sin(\alpha_1+\alpha_2+\alpha_3)$$

since $\sum \alpha_i = 2\pi$. Then

$$f_1 = \cos\alpha_1 - \cos(\alpha_1 + \alpha_2 + \alpha_3) = 0$$
$$f_2 = \cos\alpha_2 - \cos(\alpha_1 + \alpha_2 + \alpha_3) = 0 \qquad (*)$$
$$f_3 = \cos\alpha_3 - \cos(\alpha_1 + \alpha_2 + \alpha_3) = 0$$

which is verified is particular for $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{\pi}{2}$.

Now we test the point $P_o(\pi/2, \pi/2, \pi/2)$ for extrema:

$$f_{ii} = -\sin\alpha_i + \sin\sum\alpha_k \implies f_{ii}(P_o) = -2$$
$$f_{ij} = \sin\sum\alpha_k \implies f_{ij}(P_o) = -1 \quad (\text{for } i \neq j)$$

$$f_{ii}(P_o) = -2 < 0, \quad \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}_{P_o} = \begin{vmatrix} -2 & -1 \\ -1 & -2 \end{vmatrix} = 3 > 0,$$

$$\begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} = \begin{vmatrix} -2 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \end{vmatrix} = -4 < 0$$

Then P_0 is a relative maximum point giving $f(P_0) = 4$ which is the largest value of $2S/R^2$.

Test extrema for the point $P_1(\pi, 0, 0)$ satisfying (*).

B- CONSTRAINED EXTREMA (with side condition(s))

If it is required to find the extrema of a function $z = f(x, y)$ under a condition $g(x, y) = 0$, the extrema is called the *constrained extrema* or *extrema with a side condition* (side function).

If the function is of three variables $u = f(x, y, z)$ it may be required to find its extrema under one side condition $g(x, y, z) = 0$, or two side conditions $g(x, y, z) = 0$, $h(x, y, z) = 0$:

The constrained extrema problem for

$$z = f(x_1 , \ldots , x_n)$$

under the side conditions

$$g_1 = 0 , \ldots , g_k = 0 \qquad (1 < k < n-1)$$

is reduced by LAGRANGE to the free extrema problem for a function which is a linear combination of $f, g_1 , \ldots , g_k$.

Consider a function

$$u = f(x, y, z)$$

with a side condition $g(x, y, z) = 0$ which defines z as a function of x and y. Then u becomes a function of two variables, $u(x, y)$. Hence the necessary conditions for an extrema for $u(x, y)$ are

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \frac{\partial z}{dx} = 0$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}\underbrace{(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy)}_{dz} = 0$$

$$\Rightarrow \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0 \qquad (a)$$

Now, $g = 0$ gives

$$\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz = 0 \qquad (b)$$

Multiplying (b) by a scalar λ and adding to (a) we get

$$(\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x})dx + (\frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y})dy + (\frac{\partial f}{\partial z} + \lambda \frac{\partial g}{\partial z})dz = 0 \qquad (c)$$

which is satisfied by the extremal points of u under the side condition $g = 0$. Then if one chooses λ such that

$$f_x + \lambda g_x = 0, \quad f_y + \lambda g_y = 0, \quad f_z + \lambda g_z = 0, \quad g = 0$$

we obtain the critical points of the function

$$F(x, y, z) = f(x, y, z) + \lambda g(x, y, z).$$

Thus this a constrained extrema is reduced a free extrema of the linear combination $f + \lambda g$ of the original function and the side function.

It can be shown that the extremal points of

$$z = f(x_1, \ldots, x_n)$$

under the side conditions.

$$g_1(x, \ldots, x_n) = 0, \ldots, g_k(x_1, \ldots, x_n) = 0 \qquad (1 \leq k \leq n-1)$$

are the critical points of the function

$$F(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_k)$$
$$= f + \lambda_1 g_1 + \ldots + \lambda_k g_k$$

of $n + \kappa$ variables where λ_i's are called the LAGRANGE multipliers.

Example 1. Find the minimum distance of the point $A(0, 5)$

from the parabola, $y = x^2$.

Solution. If $P(x, y)$ is a point on $y = x^2$ then the function to be minimized is $\sqrt{x^2 + (y-5)^2}$ or its square $f(x, y) = x^2 + (y-5)^2$ where x, y satisfy $g(x, y) = x^2 - y = 0$ as the side condition.

Forming

$$F(x, y, \lambda) = x^2 + (y-5)^2 + \lambda(x^2-y),$$

we have

$$F_1 = 2x + 2\lambda x = 0 \Rightarrow (1+\lambda)x = 0 \Rightarrow x \neq 0 \text{ or } \lambda = -1$$
$$F_2 = 2(y-5) - \lambda = 0 \Rightarrow \lambda = 2y-10$$
$$F_3 = x^2 - y = 0 \quad \text{(side cond.)}$$

$$x = 0 \Rightarrow y = 0 \Rightarrow (0, 0)$$
$$\lambda = -1 \Rightarrow -1 = 2y-10 \Rightarrow y = 9/2 \Rightarrow x^2 = \frac{9}{2} \Rightarrow x = \pm 3/\sqrt{2}$$

Then we have three critical points $0(0, 0)$, $B(3/\sqrt{2}, 9/2)$, $C(-3/\sqrt{2}, 9/2)$.

$$d(A, B) = d(A, C) = \sqrt{\frac{9}{2} + \left(\frac{9}{2} - 5\right)^2} = \sqrt{\frac{9}{2} + \frac{1}{4}} = \sqrt{19}/2 \quad \text{(min)}$$
$$d(A, 0) = 5 \text{ (max)}.$$

Example 2. Find the shortest distance of the origin from the line of intersection of the planes $x + 2y - z = 3$, $2x + y + z = 1$.

Solution. The function to minimized being $\sqrt{x^2 + y^2 + z^2}$ or its square $f(x, y, z) = x^2 + y^2 + z^2$ with side conditions $x + 2y - z - 3 = 0$, $2x + y + z - 1 = 0$, we consider the function

$$F(x, y, z, \lambda, \mu) = x^2 + y^2 + z^2 + \lambda(x + 2y - z - 1) + \mu(2x + y + z - 3)$$

$$F_1 = 2x + \lambda + 2\mu = 0$$
$$F_2 = 2y + 2\lambda + \mu = 0$$
$$F_3 = 2z - \lambda + \mu = 0$$
$$\left.\begin{aligned} F_\lambda &= x + 2y - z - 1 = 0 \\ F_\mu &= 2x + y + z - 1 = 0 \end{aligned}\right\} \text{side conditions}$$

Setting

$$2x = -\lambda - 2\mu, \quad 2y = -2\lambda - \mu. \quad 2z = \lambda - \mu$$

in the side conditions

$$2x + 4y - 2z = 3, \qquad 4x + 2y + 2z = 2$$

we have

$$\begin{array}{l} -2\lambda - \mu = 2 \\ -3\lambda - 6\mu = 2 \end{array} \Rightarrow \lambda = -10/9, \quad \mu = 2/9$$

$$\Rightarrow P(x = 1/3, \quad y = 1, \quad z = -2/3)$$
$$\Rightarrow d(0,P) = \sqrt{1/9 + 1 + 4/9} = \sqrt{14}/3$$

ABSOLUTE EXTREMA (GLOBAL EXTREMA)

Let f be a function with (largest) domain D_f. Suppose f is restricted to a region R contained in D_f $(R \subseteq D_f)$.

We recall that the symbols $[R]$, (R) denote regions with or or without boundary ∂R, if any.

The *absolute maximum* M *(the absolute minimum m)* of f over R is the maximum (minimum) of the set of values f(R).

These extrema are obtained by free extrema of f for (R), and by constrained one for [R] if ∂R exists, with the consideration of values of f on the corners of the boundary. (observe similarity between the corners of the boundary and the end points of an interval for a function of a single variable). In the case of existence of boundary ∂R, since f may have local extrema on ∂R given by some number of relations $g_i = 0$, there is constrained extrema with side conditions $g_i = 0$.

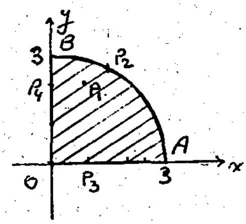
<u>Example.</u> Given the paraboloid

$$z = x^2 + y^2 - 2x - 4y + 6,$$

find the heighest and lowest points of the surface over the region

$$R = \{(x, y): \; x^2 + y^2 \leqslant 9, \quad x \geqslant 0, \quad y \geqslant 0\}.$$

Solution. The regior R is as shown in the figure where the boundary consists of a quarter of circular arc $(\widehat{AB})$ and two line segments (OA), (OB) with the corners O, A, B.



Free extrema in (R):

$$\left.\begin{array}{l} z_x = 2x-2 = 0 \\ z_y = 2y-4 = 0 \end{array}\right\} \Rightarrow P_1(1, 2) \varepsilon (R)$$

$$\underline{z(P_1) = 1} \quad \text{(a min)}$$

Contrained extrema

for $(\widehat{AB})$: $g(x, y) = x^2+y^2-9 = 0, \quad x > 0, \quad y > 0$

$$F = x^2+y^2-2x-2y+6+ \lambda(x^2+y^2-9)$$

$$\left.\begin{array}{l} F_x = 2x-2+2\lambda x = 0 \\ F_y = 2y-4+2\lambda y = 0 \end{array}\right\} \Rightarrow \lambda = \frac{1-x}{x} = \frac{2-y}{y} \Rightarrow y = 2x$$

$$F_\lambda = x^2+y^2-9 = 0 \quad \Rightarrow \quad x^2 + \pm(2x)^2 = 9 \Rightarrow x = \pm \frac{3}{\sqrt{5}}, \quad y = \pm 6/\sqrt{5}$$

$$P_2(3/\sqrt{5} \quad 6/\sqrt{5}) \varepsilon \widehat{AB}$$

$$\underline{z(P_2) = 15-6\sqrt{5}} \quad (\cong 1{,}72)$$

for (OA): $y = 0, \quad 0 < x < 3$

$$z(x, 0) = x^2-2x+6 \qquad z'_x = 2x-2 = 0 \Rightarrow x = 1$$

$$P_3(1, 0) \Rightarrow \underline{z(P_3) = 5}$$

for (OB): $x = 0, \quad 0 < y < 3$

$$z(0, y) = y^2-4y+6 \Rightarrow z'_y = 2y-4 = 0 \Rightarrow y = 2$$

$$P_4(0, 2) \Rightarrow \underline{z(P_4) = 2}$$

At the corners:

$$\underline{z(0) = 6}, \quad \underline{z(A) = 9}, \quad \underline{z(B) = 3}$$

The set of obtained values, in increasing order, being

$$\{1,\ 15-6\sqrt{5},\ 2,\ 3,\ 5,\ 6,\ 9\}$$

we have $M = 9$, $m = 1$ at the corner $A(0, 3)$ and the interior point $P_1(1, 2)$ respectively.

C. The Method of Least Squares (MLS)

1. Let $q_1, \dots, q_n$ be n *direct* measured values of a imagnitude, and let Q represent the sum of squares of their deviations from a number q:

$$Q = \sum_{i=1}^{n} (q_i - q)^2 \qquad (1)$$

The value of q making Q minimum, is the best value of the measurement, since Q involves squares of deviations which are non negative. This best value $\overline{q}$ of $q_1, \dots, q_n$ is shown to be the arithmetic of the measured values:

$$\frac{dQ}{dq} = -2 \sum_{i=1}^{n} (q_i-q) = 0 \Rightarrow \sum_{i=1}^{n} q_i - nq = 0$$

$$\Rightarrow \overline{q} = \frac{q_1 + \dots + q_n}{n},$$

$$\frac{d^2Q}{dq^2} = 2n > 0 \Rightarrow Q \text{ is minimum for } \overline{q}.$$

This process of finding $\overline{q}$ from (1) is known as *the method of least squares* (MLS)

2. Let $q_1, \dots, q_n$ be n values of $q = f(x)$ for n direct measurements of $x_1, \dots, x_n$, where f is a known function. In this case q_i's become *indirect* measurements of q.

It can be shown that the best value $\overline{q}$ of q is $\overline{q} = f(\overline{x})$ where $\overline{x}$ and $\overline{y}$ are the arithmetic means of x_i's and q_i's respectively.

<u>Example</u>. The length ℓ of a simple pendulum is measured as 24,8; 25,1; 25,0; 24,8; 25,0; 25,1; 24,7; 25,1 cm. Then

a) find the best length $\overline{\ell}$

b) find the best period.

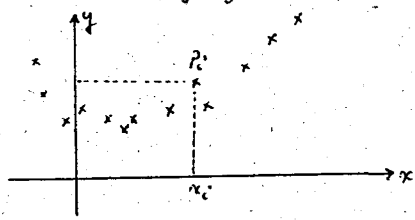
<u>Solution</u>.

a) $\overline{\ell} = (\Sigma \ell_i)/8 = 24,95$ cm,

b) $T = \pi\sqrt{\ell/g} = \pi\sqrt{24,95/g} = 4,99\pi/\sqrt{g}$.

3. Let y be related to x by an unknown function f, and let $y_1, \dots, y_n$ be the measured values of y corresponding to a set of seclected values $x_1, \dots, x_n$ of x.

When one plots the points $P_i(x_i, y_i)$ on a rectangular coordinate system Oxy, we obtain a distribution of y against x.



Now the problem is to determine the best function giving this distribution.

The solution of the problem involves the following steps:

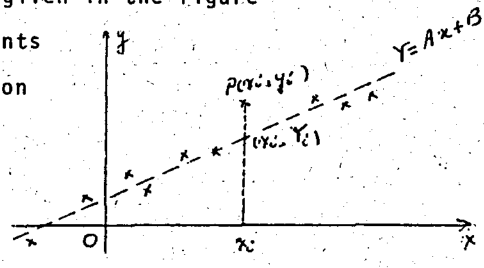
(i) from the distribution guess the type of the function as linear, quadratic, exponential, ... ,

(ii) write the general form of the function,

(iii) by the use of the MLS, determine the unknown parameters.

Let the distribution be as given in the figure

(i) We observe that the points $P_i(x_i, y_i)$ lie nearly on a straight line



(ii) The general equation is

$$Y = Ax + B$$

(iii) By the MLS, we minimize

$$Q(A, B) = \sum_{i=1}^{n} (y_i - Ax_i - B)^2$$

in which A, B are the parameters:

$$\frac{\partial Q}{\partial A} = -2 \sum_{i=1}^{n} (y_i - Ax_i - B)x_i = 0$$

$$\frac{\partial Q}{\partial B} = -2 \sum (y_i - Ax_i - B) = 0$$

$$\Rightarrow \begin{cases} (\Sigma x_i)A + n\,B = \Sigma y_i \\ (\Sigma x_i^2)A + (\Sigma x_i)B = \Sigma x_i y_i \end{cases} \qquad (1)$$

$$A = \begin{vmatrix} \Sigma y_i & n \\ \Sigma x_i y_i & \Sigma x_i \end{vmatrix} / \Delta, \quad B = \begin{vmatrix} \Sigma x_i & \Sigma y_i \\ \Sigma x_i^2 & \Sigma x_i y_i \end{vmatrix} / \Delta$$

where

$$\Delta = \begin{vmatrix} \Sigma x_i & n \\ \Sigma x_i^2 & \Sigma x_i \end{vmatrix}$$

The obtained equation $Y = Ax + B$ is called *the line of best fit* of the given date, and its determinantal equation is

$$\begin{vmatrix} x & y & 1 \\ \Sigma x_i & \Sigma y_i & n \\ \Sigma x_i^2 & \Sigma x_i y_i & \Sigma x_i \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} x & y & 1 \\ \overline{x} & \overline{y} & 1 \\ \Sigma x_i^2 & \Sigma x_i y_i & \Sigma x_i \end{vmatrix} = 0$$

The second row in the last equation show that the line of best fit passes through the point $P(\overline{x}, \overline{y})$.

Observe that the first equation in (1) for parameters can be obtained practically from $y_i = Ax_i + B$ by summation; and the second by summation after multiplying by x_i.

Example. Given the data

x	-1	0	1	2	3	4
y	2	1	0	-3	-5	-8

find the equation of the line of best fit.

Solution.

x	y	x^2	xy
-1	2	1	-2
0	1	0	0
1	0	1	0
2	-3	4	-6
3	-5	9	-15
4	-8	16	-32
9	-13	31	-55

$$\Rightarrow \quad \begin{aligned} 9A + 6B &= -13 \\ 31A + 9B &= -55 \end{aligned}$$

$$\Rightarrow \quad A = -\frac{213}{105}, \quad B = \frac{92}{105}$$

$$\Rightarrow \quad y = -\frac{213}{105}x + \frac{92}{105}$$

$$= -2{,}028\,x + 0{,}876$$

Where there are more than one variable, say two variables, in the case of linear approximation the general linear equation is

$$z = Ax + By + C,$$

and by the MLS one may obtain the following equations for parameters:

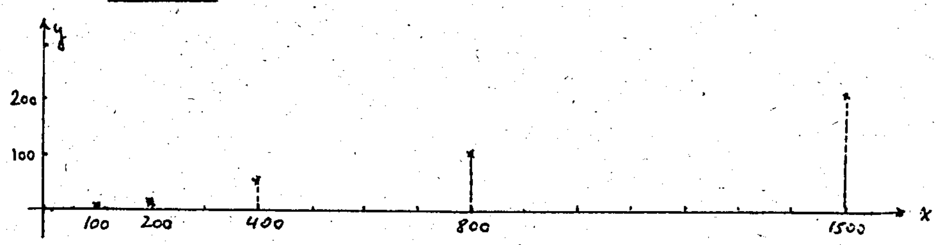
$$\Sigma z_i = A\ \Sigma x_i + B\ \Sigma y_i + C\ \Sigma 1 \qquad (\Sigma 1 = n)$$
$$\Sigma x_i z_i = A\ \Sigma x_i^2 + B\ \Sigma x_i y_i + C \Sigma x_i$$
$$\Sigma y_i z_i = A\,\Sigma\, x_i y_i + B\ \Sigma y_i^2 + C \Sigma y_i$$

The non linear cases can be reduced to linear case by the use of some transformations.

Example. Plot the data for Olympic rununing events given in the following table. Then find the function of the form $y = k\ x^r$ that best fits the data

distance (x)	y = time (for men)
100 m dash	10 = 10 sec
200 m dash	20 = 20 sec
400 m run	44,9 = 44,9 sec
800 m run	1.45,7 = 105,7 sec
1500 m run	3.35,6 = 215,6 sec

Solution.



Taking the common logarithm of both sides of $y = k\ x^r$, we get the linear equation

$$\log y = \log k + r \log x$$

in $\log y$ and $\log x$. Then setting

$$u = \log x, \qquad v = \log y, \qquad \delta = \log k$$

we have

$$v = s + r u$$

Then, from

u	v	u^2	uv
2	1	4	2
2,30	1,30	5,29	2,99
2,60	1,64	6,76	4,26
2,90	2,02	8,41	5,86
3,18	2,33	10,11	7,41

the equation for parameters r and s are

$$12{,}98\,r + 5s = 8{,}29$$

$$34{,}57\,r + 12{,}98s = 22{,}52$$

$$r = 1{,}14 \qquad s = -1{,}31 = \log k$$

$$\log k = -1{,}31 = \bar{2}{,}69 \qquad k = 0{,}05$$

$$y = 0{,}05\,x^{1{,}14}$$

Example 2. Given the date

z	x	y
1	0	0
2	1	1
1	0	1
3	1	0
4	2	0

find the plane of best fit.

Solution. General equation is

$$z = Ax + By + C$$

Then the equations for parameters become

$$\Sigma z_i = A\,\Sigma x_i + B\,\Sigma y_i + C\,\Sigma 1$$

$$\Sigma z_i x_i = A\,\Sigma x_i^2 + B\,\Sigma x_i y_i + C\,\Sigma x_i$$

$$\Sigma z_i y_i = A\,\Sigma x_i y_i + B\,\Sigma y_i^2 + C\,\Sigma y_i$$

From

z	x	y	xy	xz	yz	x^2	y^2
1	0	0	0	0	0	0	0
2	1	1	1	2	2	1	1
1	0	1	0	0	1	0	1
3	1	0	0	3	0	1	0
4	2	0	0	8	0	4	0
11	4	2	1	13	3	6	2

we have

$$4A + 2B + 5C = 11$$

$$6A + B + 4C = 13$$

$$A + 2B + 2C = 3$$

$$\Rightarrow \quad A = 21/15, \quad B = -7/15, \quad C = 19/15$$

$$\Rightarrow \quad z = \frac{21}{15}x - \frac{7}{15}y + \frac{19}{15}.$$

EXERCISES (4. 4)

81. Find the critical points of

$$f(x, y, z) = x^2 + y^2 - 2z^2 + 3x + y - z - 2$$

82. Examine the critical points of

$$f(x, y) = xy(x + y - 3)$$

83. Same question for

a) $x^2 + 2xy + 2y^2 + 4x$ b) $4xy - x^4 - y^4$

84. Discuss the following functions for extrema:

a) $x^2 + xy + y^2 - 6y + 2$ b) $3x - 3y - y^2 + 2xy - 2y^2$

85. Test the following for local extrema:

a) $2x^2 + y^2 + 4x - 4y - 3$ b) $8^{2/3} - x^{2/3} - y^{2/3}$

86. Show that the maximum value of

$$(ax + by + c)^2/(x^2 + y^2 + 1) \quad \text{is} \quad a^2 + b^2 + c^2$$

87. Prove for $x > 0$, $y > 0$, $z > 0$:

$$\sqrt{xy} \leqslant \frac{x+y}{2} \leqslant \sqrt{\frac{x^2 + y^2}{2}}$$

by maximuma or minima

88. Find the maximum volume of the tetrahedron in the first octant having faces on the coordinate planes and on the tangent plane to the surface $xyz = a^3$.

89. Find the minimum distance of the origin from the cone

$$z^2 = (x-2)^2 + (y-1)^2.$$

90. Find the maximum distance of $(1, 1, 0)$ from the cone:

$$x^2 + y^2 - z^2 = 0$$

91. Determine the slopes of the axes of the ellipse $5x^2 + 6xy + y^2 = 8$ by maxima, minima.

92. Find the dimensions of a rectangular parallelepiped of largest volume which has three faces on the coordinate planes and on the plane $x + y/2 + z/3 = 1$

93. Write the number 120 as the sum of three positive numbers x, y, z such that xy^2z^3 is a minimum.

94. Find the shortest distance between the lines

$$\ell: \ x = y - 3 = z, \qquad \ell': \ x = y/2 = z/3$$

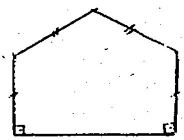
95. Find the distance of $A(1, 1, 1)$ from the line

$$\ell: \ \frac{x-1}{2} = \frac{y-2}{2} = \frac{z+1}{1}$$

96. Find the maximum volume of a rectangular box without top if its area is $12\ dm^2$.

97. Prove that the maximum area of a simple quadrilateral ABCD with given sides occurs when it is inscribed in a circle.

98. ABCDE is pentagon with a given perimeter 25. Find the maximum area.



99. Find the absolute maximum and minimum of $z = \exp(x^2 + y^2)$ over the region $\{(x, y): x^2 + (y-1)^2 \leqslant 1\}$.

100. A circular plate $(x^2 + y^2 \leqslant 4)$ is heated so that the temperature distribution is $T(x, y) = x^2 + y^2 - x$. Locate the coldest and hottest points of the plate.

101. The temperature at a point $P(x, y, z)$ in 3-space is given by $T = x + y - 2z$. Then find the lowest and heighest temperature

a) on the sphere

b) on the circle $x^2 + y^2 + z^2 = 4, \quad z = x + y + 1$

102. Find a triangle of maximum (minimum) area with given circumcircle (incircle).

103. Show that the centroid of a triangle has the property that sum of squares of its distances from the vertices is minimum.

104. Find the distance between the lines

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z}{2}, \qquad \frac{x-3}{2} = \frac{y}{1} = \frac{z-1}{-2}$$

by the use of

a) extrema, b) analytic method, c) vectors.

105. Find the best volume of the sphere from the following measurements of the diameter in cm:

3,63; 3,59; 3,61; 3,57; 3,60.

106. If F is the pull required to lift a weight W by means of a pullyy-block, find a linear law of the form $F = aW + b$ connecting F and W, using the following data:

W	25	35	50	60
F	6	8	11	13

107. In the following table y is the weight of potassium bromide which will dissolve in 100 g of water at temperature x:

x	0	10	20	30	40	50	60	70	80	90	100
y	53,5	59,5	65,2	70,6	75,5	80,2	85,5	90,0	95,0	99,2	104,0

Find linear equation of the best fit to this data.

108. If R is the resistance to the motion of a train at the speed V, find the curve of best fit of the form $R = bV^2 + a$ under the data:

V(km/k)	10	20	30	40	50
R(kg/ton)	8	10	15	21	30

109. Find the function of best fit of the form $y = ax^b$ for the following data for the olympic swimming events:

x (in meters)	100	200	400	1500
y (in sec)	52,2	115,2	249,0	998,9

110. Find the equation of the plane of best fit to the following data:

x	0	1	1	0	2	0	2
y	0	0	1	1	0	2	1
z	2	3	3	1	6	0	5

ANSWERS TO EVEN NUMBERED EXERCISES

82. A saddle point at $A(0, 0)$, min at $B(1, 1)$

84. a) min at $(-2, 4)$, b) max at $(1/2, -1)$

88. The tetrahedron has constant volume $9a^3/2$

90. $(1/2, 1/2, 1/\sqrt{2})$, $d = 1$

92. $\frac{1}{3} \times \frac{2}{3} \times 1$

94. $\sqrt{6}$

96. $V = 2 \times 2 \times 1 = 4\ dm^3$

98. $|ABCDE|_2 = (2-\sqrt{3})s^2$

100. Coldest at $A(1/2, 0)$, hottest at $B(-2, 0)$

102. Equilateral (equilateral)

104. $d = 3$

106. $f(w) = 0{,}24\ w + 1{,}1$

108. $R = 0{,}009^2\ v^2 + 6{,}7$

A SUMMARY

(CHAPTER 4)

4. 1. *δ-Neighborhood* of P_0: $N_\delta\left[P_0\right] = \{P: \left|P-P_0\right| < \delta\}$

limit: $\lim\limits_{P \to P_0} f(P) = \ell \;\; \Leftrightarrow$ to a given $\varepsilon > 0$ there is a $N_\delta(P_0)$ s.t

$$|f(P)-\ell| < \varepsilon \quad \text{for all} \quad P \in N_\delta(P_0)$$

Continuity: f is cont. at $P_0 \Leftrightarrow \lim\limits_{P \to P_0} f(P) = f(P_0)$

4. 2. *Partial derivative:* For $f(x_1, \ldots, x_n)$

$$\frac{\partial f}{\partial x_i} = \lim_{h_i \to 0} \frac{f(\ldots, x_i + h_i, \ldots) - f(\ldots, x_i, \ldots)}{h_i} = f_i$$

<u>Theorem</u> (SCHWARZ). $f, f_1, f_2, f_{12}, f_{21} \varepsilon C(D) \Rightarrow f_{12} = f_{21}$

Total increment: $\Delta f(P) = \underbrace{\Sigma f_i(P)\Delta x_i}_{\text{principal part}} + \Delta\varepsilon_i \Delta x_i$

$(\varepsilon_i \to 0$ with $\Delta x_i \to 0)$.

Total differential: $df(P) = \Sigma f_i(P) dx_i$

Total derivative: $\frac{df(P(t))}{dt} = \Sigma f_i(P(t)) \frac{dx}{dt}$

Implicit differentiation:

$$F(x, y) = 0 \Rightarrow dy/dx = -Fx/Fy$$

$$F(x, y, z) = 0 \Rightarrow z_x = -Fx/Fz, \quad z_y = -Fy/Fz$$

Tangent plane at P_o:

$F(x, y, z) = 0$: $F_x(P_0)(x-x_0) + F_y(P_0)(y-y_0) + F_z(P_0)(z-z_0) = 0$

$z = f(x, y)$: $f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) - (z-z_0) = 0$

Normal line at P_o:

$$F(x, y, z) = 0 : \frac{x-x_0}{F_x(P_0)} = \frac{y-y_0}{F_y(P_0)} = \frac{z-z_0}{F_z(P_0)}.$$

for

$$z = f(x, y) : \frac{x-x_0}{f_x(P_0)} = \frac{y-y_0}{f_y(P_0)} = \frac{z-z_0}{1}.$$

Directional derivative:

$$\frac{df(P_0)}{ds} = f_T = \Sigma f_i(P_0) \cos \alpha_i \quad (T = \Sigma(\cos \alpha_i) e_i, \ |T| = 1)$$

where T is a unit vector defining the direction and sense.

$$f_T = \nabla f . T \quad (\nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z})$$

Gradient divergence and; Laplacian

grad $f = \nabla f = f_x\, i + f_y\, j + f_z\, k$ (f is a scalar function)

div $F = \nabla . F = P_x + Q_y + R_z$

$$\text{anl } F = \nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} \quad (F = P\,i + Q\,j + R\,k)$$

$$\Delta^2 f = f_{xx} + f_{yy} + f_{zz}$$

Derivative under the integral sign:

$$\frac{d}{dt} \int_{a(t)}^{b(t)} f(x, t)dx = f(b, t)b' - f(a, t)a' + \int_a^b f_t(x, t)dx$$

4. 3. *TAYLOR's Formula:*

$$f(x, y) = f(a, b) + \sum_{k=1}^{n} \frac{1}{k!} \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^k f(x,y)\Big|_{(a,b)} + R_{n+1}$$

when the remainder is given by

$$R_{n+1} = \frac{1}{(n+1)} \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^{n+1} f(x, y)\Big|_{(x^*, y^*)}$$

with $(x^*, y^*) \varepsilon (P_oP)$, $P_o(a, b)$, $P(x, y)$.

TAYLOR's Series;

$$f(x, y) = f(a, b) + \sum_{k=1}^{\infty} \frac{1}{k!} \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^n f(x,y)\Big|_{(a,b)}$$

Envelope:

Family	Envelope
$F(x, y, \lambda) = 0$	$F = 0, \quad F_\lambda = 0$
$F(x, y, z, \lambda) = 0$	$F = 0, \quad F_\lambda = 0$
$F(x, y, z, \lambda, \mu) = 0$	$F = 0, \quad F_\lambda = 0, \quad F_\mu = 0$

Evolute of a plane curve: is the envelope of it nomels or the locus of centers of curvature.

4\. 4. *Critical points* of $f(x_1, \ldots, x_n)$ are the solution points of the system $f_1 = 0, \ldots, f_n = 0$.

For a function of two variables $f(x, y)$ a critical point is a

rel.max. if $\Delta > 0$ and $f_{11} < 0$

rel.min. if $\Delta > 0$ and $f_{11} > 0$ with $\Delta = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}$

saddle pot. if $\Delta < 0$

Global or absolute maximum (minimum) of a function f over D_f is the largest (smallest) element $u(m)$ in R_f

Constrained extrema. for $z = f(x_1, \ldots, x_n)$ under k side conditions $g_1 = 0, \ldots, g_k = 0$ $(1 < k < n-1)$ is reduced to free extrema for $F = f + \lambda_1 g_1 + \ldots + \lambda_k g_k$.

The method of least squares: If $q_1, \ldots, q_n$ are direct measured values of a maguitude, the their arithmetic we $\overline{q}$ is the least value of the maguitude.

The line of best fit to a linear distribution y_i against x_i is

$$\begin{vmatrix} x & y & 1 \\ \Sigma x_i & \Sigma y_i & \Sigma x \\ \Sigma x_i^2 & \Sigma x_i y_i & \Sigma x_i \end{vmatrix} = 0$$

MISCELLANEOUS EXERCISES

111\. Which ones of the following relations are functions of x, y?

a) $z = e^{xy}$ b) $(x-h)^2 + (y-k)^2 - (z-\ell)^2 = 0$

c) $z^2 = xy$ d) $\tan z = x^2 + y^2$

112\. Find the domain D and sketch it:

a) $z = (y-x^2)^{\ln x}$ b) $z = (\arcsin x)^{\sqrt{x-y^2}}$

113. Find and sketch the largest possible domains of the following functions:

a) $z = (xy)^{\sqrt{x+y}}$ b) $z = \ln(x+y)(x^2 + y^2 - 4)$

114. Same question for:

a) $z = \sqrt{\frac{x+2}{y-3}}$ b) $z = \arcsin\,(x^2 + y^2)$

115. Same question for:

a) $u = \sqrt{z^2 - x^2 - y^2}$ b) $u = \arccos \frac{x^2 + y^2 + z^2}{4}$

116. Show that the following limits do not exist:

a) $\lim\limits_{(x,\, y)\to(1,\, 0)} \frac{x^2-2x-y^2+1}{x^2-2x+1+y^2}$ b) $\lim\limits_{(x,\, y)\to(2,\, 1)} \frac{x-y-1}{x+y-3}$

117. Show that the following limits are independ. of direction:

a) $\lim\limits_{(x,y)\to(0,\, 0)} \frac{x^2 + y^2}{xy + 4x}$ b) $\lim\limits_{(x,y)\to(1,\, 2)} \frac{x^2 + y^2 - 5}{xy - 2}$

118. Find the limit of the following function along the given curve at the indicated point:

a) $\frac{x^2 - xy}{x+y}$, Γ: $y = x$, $(0, 0)$

b) $\frac{y - \sin x}{x}$, Γ: $y = x^2$, $(0, 0)$

119. Find $\lim\limits_{(x,y)\to(0,0)} \frac{\sin xy}{\cos \left[\!\left[(x + \frac{\pi}{2})(y+1)\right]\!\right]}$ along $y = x$

120. Evaluate $\lim\limits_{(x,y)\to(0,0)} \frac{e^{x^2} + y - e}{e^{x+y} - e}$ along $y = x^2$

121. Two functions F and G are of one variable and z is of two variables. They are related by

$$\Big[F(x) + G(y)\Big]^2\, e^{z(x,\, y)} = 2\, F'(x)\, G'(y)$$

whenever $F(x) + G(y) \neq 0$. Show that the mixed derivative z_{21} is never zero.

122. Show $f_{xy}\, f_{yx} = f_{xx}\, f_{yy}$ for $f(x,\, y) = \sqrt{\frac{x^4 + y^4}{x^2 + y^2}}$

123. Prove

$$\left.\begin{array}{l} F(x, y, u, v) = 0 \\ G(x, y, u, v) = 0 \end{array}\right\} \Rightarrow u_x\, y_u + v_x\, y_v = 0$$

124. If $u(x, y)$, $v(x, y)$ are defined by

$$u-v^2-x^3+3y = 0, \quad u+v-y^2-2x = 0$$

find $\partial u/\partial x$.

125. If $V = V(x, y)$, where $x = e^r \cos\theta$, $y = e^r \operatorname{Sin}\theta$, then show that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = e^{-2r} \left(\frac{\partial^2 V}{\partial r^2} + \frac{\partial^2 V}{\partial n^2}\right)$$

126. Show that

$V(x, t) = f(x+ct) + g(x-ct)$ satisfies $\dfrac{\partial^2 V}{\partial t^2} = c^2 \dfrac{\partial^2 V}{\partial t^2}$

127. Show

$$f(x, y, z) = 0 \Rightarrow \frac{\partial z}{\partial x}\,\frac{\partial x}{\partial z} = 1, \quad \frac{\partial x}{\partial y}\,\frac{\partial y}{\partial z}\,\frac{\partial z}{\partial x} = 1$$

(but in general $\partial z/\partial x \cdot \partial x/\partial z \neq 1$)

128 Find $\partial w/\partial x$ and $\partial w/\partial y$ if

$$w = u/v \text{ and } x = u+v, \quad y = 3u+2v$$

129. If $u(x, y)$, $v(x, y)$ are defined by

$$u^2+v^2+y^2-2x = 0, \quad u^3+v^3+3y-x^3 = 0,$$

then find u_x, v_x and u_y, v_y.

130. Find $\partial z/\partial x$, $\partial z/\partial y$ for

a) $xz^2 + xy^2z - yz^2 = 5$, b) $xz^3 - yz + 3xy = 0$

131. Verify $f_{xy} = f_{yx}$ for

a) $f(x, y) = \cos(x, y)$ b) $f(x, y) = (\cos x^2)\cos y$

c) $f(x, y) = e^{x/y}$ d) $f(x, y) = (x^2-2y)^2 + \sqrt{xy}$

132. Show that the following functions are harmonic:

a) $\ln(x^2+y^2) + \arctan \frac{y}{x}$ b) $1/\sqrt{x^2+y^2+z^2}$

133. Given $x^2-y^2+z^2 = 1$, find

a) p, q b) r, s, t

134. If the dimensions of a rectangular right prizm have relative errors 0,03; 0,02; 0,04 respectively, what is the maximum relative error made in its volume?

135. A baseball (of ellipsoid shape) has initial size axbxc ($a = 18$, $b = 10.25$, $c = 8,4$ cm). In blowing up a, b, c increase 2, 1,5, 1,2 mm/sec respectively. How fast the volume increases when $a = 19$ cm?

136. A triangle has two sides 50 m and 70 m long. The angle between them is 30^0. If the possible errors are 1/2 % in the measurements of the sides, and 0,5 degree in that of the angle, find the maximum percentage error in the measurement of the area.

137. The diameter and the altitude of a right circular cone are found by measurements to be 8,0 and 12,5 cm respectively, with possible errors of 0,05 cm in each measurements. Find the maximum possible approximate and relative error in the computed volume.

138. Find the total differentials of

a) $u = \sqrt{x^2 + y^2 + z^2}$ b) $v = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$

c) $z = \frac{xy}{x + y}$ d) $z = \ln \tan(x^2+y^2)$

139. Find y', y'', y''' if $x^3+y^3-3axy = 0$

140. Given $1 + xy = \ln(\text{Sh}\, xy)$, find dy/dx.

141. If $z = f(x-zt)$, show $z_t + zz_x = 0$

142. What becomes the equation

$$(x+y)\frac{\partial z}{\partial x} + (x-y)\frac{\partial z}{\partial y} = 0$$

under the substitution.

$$x^2 - y^2 - 2xy = u, \quad y = v$$

143. Suppose $f(x, y, z) = 0$, $g(x, y, z)$ define $y = y(x)$ and $z = z(x)$. Evaluate y', z'

144. Verify EULER's Theorem for:

a) $f(x, y) = e^{x/y}$ b) $g(x, y) = \dfrac{x^2 + y^2}{x^2 - y^2}$

c) $F(x, y) = \sqrt{x+y}/y$ d) $G(x, y) = \arcsin(y/x)$

145. Find the locus of points on the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ at which tangent plane has equal intercepts.

146. Find the points of the paraboloid $z = x^2 - y^2$ at which the normal line is parallel to the line $x/4 = y/6 = z/1$.

147. Show that the surfaces described by $2z = -1+x^2+y^2$ and $2z = 1-x^2-y^2$ are orthogonal to each other at every point of intersection.

148. Show that the surfaces described by $z^2+25 = 2x^2+2y^2$ and $5z = x^2+y^2$ are tangent to each other at $(4, 3, 5)$.

149. Find the angle of intersection of the surfaces $x^2y + z = 3$ and $x \ln z - y^2 = -4$ at $(-1, 2, 1)$.

150. Find the direction numbers of the line tangent to the curve of intersection of the surface $z = 4x^2 + y^2$ and the plane $y = x$ at $(1, 1, 5)$

151. Show that the curve

$$x = t^2, \quad y = 3t, \quad z = 2t^2$$

pierces the surface $2x^2 + y^2 + z^2 = 15$ at right angle at the point $(1, 3, 2)$.

152. Find the directional derivative of

$$f(x, y, z) = \sin xy + \sin xz + \sin yz$$

at $P(1, 0, \pi)$ in the direction of the curve of intersection of the surfaces

$$z = e^{x+y-1} + \pi \quad \text{and} \quad z^2 = \pi^2 x^2 + y^2$$

153. In which direction the directional derivative of $f(x, y) = \sin xy$ at $(2, \pi/6)$ is maximum?

154. Find curl U. If it is identically zero find scalar function f such that $\nabla f = U$

a) $U = e^x(\sin y \cos z\, i + \cos y \cos z\, j - \sin y \sin z\, k)$

b) $U = (x+2y-z)i + (x-y+z)j + (-x+y+2z)k$

155. Find the directional derivative of

a) $f(x, y) = x^2 y + \sin xy$ at $A(1, \pi/2)$ along OA

b) $g(x, y) = x^3 y + e^{xy}$ at $A(1, 1)$ along $y = \frac{1}{\sqrt{3}} x + \frac{2}{\sqrt{3}}$

156. Show that

$$\lim_{t \to 0} \frac{1}{t} \left\{ f(x + t \cos\alpha, \; y + t \cos\beta, \; z + t \cos\gamma) - f(x, y, z) \right\}$$

is the directional derivative of $f(x, y, z)$ at $P(x, y, z)$ along the direction $(\cos\alpha, \cos\beta, \cos\gamma)$

157. Find the direction of the maximum rate of change of the function given below:

a) $z = x^2 + y^2$ at $(1, 2)$ b) $z = x^2 y - x^3$ at $(1, 2)$

158. Find the directional derivative of the following function at the given point in the indicated direction:

a) $z = x^2y + y^3$, $A(1, 1)$, $\theta = \pi/6$

b) $z = x^2 + y^2 + z^2$, $B(2, 2, 1)$, $(1, 2, 2)$

159. Find the directional derivative of $u = x^2 + y^2 - z^2$ at $(3, 4, 5)$ along the curve

$$\Gamma: \quad 2x^2 + 2y^2 - z^2 = 25, \quad x^2 + y^2 = z^2$$

160. Find the gradient of

a) $x \sin yz + y \sin zx + z \sin xy$ at $(\pi/2, 1, 0)$

b) $x\, e^{yz} + y\, e^{zx} + z\, e^{xy}$ at $(\ell n\, 2, 0, 1)$

161. Find

a) $\dfrac{d}{dt} \displaystyle\int_0^{\pi} (1-t \cos x)^2 dx$ b) $\dfrac{d}{dt} \displaystyle\int_0^{t^2} \arctan \frac{x}{t}\, dx$

c) $\dfrac{d}{du} \displaystyle\int_0^{u} \tan(x-u)dx$

162. Evaluate

a) $\displaystyle\int_0^1 \frac{x^t - 1}{\ell n\, x}\, dx$ b) $\displaystyle\int_0^{\pi} \ell n(1-2t \cos x + t^2)dx$

by differentiating under the integral sign.

163. Show

$$y(x) = \int_0^x f(t) \sin(x-t)dt \qquad y'' + y = f(x)$$

164. Find the evolute of the

a) parabola $y^2 = 2ax$ b) ellipse $x^2/a^2 + y^2/b^2 = 1$

165. Show that the envelope of the trajectories of propectiles fired from a gun at various angles with the same initial veloeity is

$$y = \frac{1}{2g} v_o^2 - \frac{g}{2v_o^2} x^2$$

166. Find the envelope of circles passing through a fixed point and intercepting on a given line a segment of constant length 2ℓ.

167. Find the parametric equation of the envelope of reflected rays when the incident ones are perpendicular to the diameter $[AB]$ of a mirror in the shape of a semicircle.

168. Find the critical points of

a) $z = 2x^2+2xy+y^2-6x$ b) $z = x^4-x^2y^2+y^4+4y^2-6x^2$

169. Find and identify the critical points of:

a) $z = 3y^4-4y^3x+x^6$ b) $z = 8x^3+y^3-3xy$

170. Find the distance between the lines

$$\frac{x}{1} = \frac{y}{2} = \frac{z+2}{2} \quad \text{and} \quad \frac{x-1}{2} = \frac{y+1}{1} = \frac{z-3}{-2}$$

171. Find the absolute maximum and minimum values of

a) $f(x, y) = x^2+y^2-xy-y$ in the unit square $\{(x,y): -1\leqslant x\leqslant 1, -1\leqslant y\leqslant 1\}$

b) $f(x, y) = 8x^3+y^3-3xy$ in the first quadrant.

172. Same question for $z = x^2-2xy+y^3-2y$ over the square $\{(x, y): |x|+|y| = 1\}$

173. Find the point on the plane $x+2y+3z = 4$ closest to origin in two ways:

a) analytically b) by extrema

174. Find the maximum valume of a rectangular box having three faces on the coordinate planes and one vertex on the plane $x/5 + y/4 + z/3 = 1$.

175. Find the maximum volume of a rectangular box that is inscribed in the ellipsoid $x^2/9 + y^2/4 + z^2 = 1$

176. Find the maximum value of xyz with the condition $x+3y+2z = 9$

177. Find the equation of the plane through $(1, 2, 3)$ and making the smallest tetrahedron with the coordinate planes; in the first octant.

178. Find the point on $4-z = x^2+y^2$ which tangent plane at that point makes the smallest tetrahedron with the coordinate planes.

179. Find the equation of the line of best fit to the data

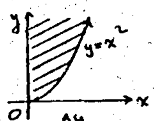
x	-1	0	1	2	3	4
y	2	2	4	5	6	8

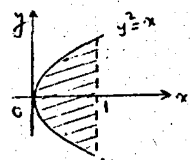
180. Find the equation of the parabola of best fit to the following data

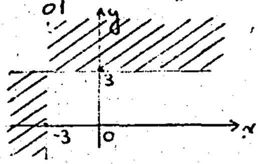
x	0	1	2	3	4
y	3	2	-1	-6	-13

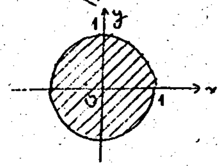
in the form $y = Ax^2 + Bx + C$.

ANSWERS TO EVEN NUMBERED EXERCISES

112. a) 

b) 

114. a) 

b) 

118. a) 0, b) -1

120. 2

124. $u_x = (3x^2 + 4)/4u$

128. $\frac{\partial w}{\partial x} = -\frac{y}{(3x-y)^2}, \quad \frac{\partial w}{\partial y} = \frac{x}{(3x-y)^2}$

130. a) $z_x = \dfrac{z^2+y^2z}{2yz-y^2x-2xz}$, $z_y = \dfrac{z^2-2xyt}{2yz-y^2x-2xz}$

b) $z_x = -\dfrac{z^3+3y}{3xz^2-y}$, $z_y = \dfrac{z-3x}{3xz^2-y}$

134. $dV/V = 0{,}09$

136. 2,51

138. a) $du = \dfrac{xdx + ydy + zdz}{(x^2+y+z^2)^{1/2}}$ b) $dv = -\dfrac{xdx+ydy+zdz}{(x^2+y^2+z^2)^{3/2}}$

c) $dz = \dfrac{y^2dx + x^2dy}{(x+y)^2}$ d) $dz = 2\,\dfrac{\operatorname{Sec}^2(x^2+y^2)}{\tan(x^2+y^2)}\,(xdx + ydy)$

140. $-y/x$

142. $\partial z/\partial v = 0$

146. $(-2, 3, -5)$

150. 1, 1, 1/10

152. $\pm(1+\pi^2)/\sqrt{2}(1-\pi+\pi^2)$

154. a) 0, $f = e^x \sin y \cos z + c$, b) $-k$

158. 16/3

160. a) $(1+\pi)k$, b) $i + (2\ln 2 + 2)i + k$

162. a) $1/(1+t)$, b) 0 if $t^2<1$, and $\pi \ln t^2$ if $t^2>1$.

164. a) $8(x-a)^3 = 27ay^2$

b) $\dfrac{x^{2/3}}{(c/a)^{2/3}} + \dfrac{y^{2/3}}{(c/b)^{2/3}} = 1$

166. Taking fixed point at the origin, and the line as $x = a$:
$ax(x^2+y^2) + (a^2+\ell^2)x^2 - 3a^2y^2 = 0$

168. a) (3, -3) min, b) ($\pm\sqrt{3}$, 0) min

170. 3

172. u = 1 at (-1, 0), (1, 0), (0, -1)

m = -1 at (0, 1)

174. V = 20/9

176. 9/2

178. (1, 1, 2)

180. $y = -x^2 + 3$

CHAPTER 5

INTEGRATION
(FUNCTIONS OF SEVERAL VARIABLES)

5. I. LINE INTEGRALS (CURVILINEAR INTEGRALS)

A. DEFINITIONS

Let

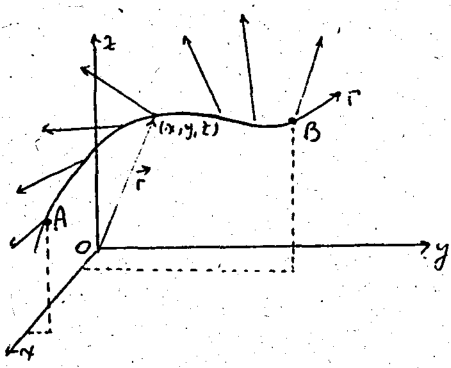
$$F(x, y, z) = P(x, y, z)i + Q(x, y, z)j + R(x, y, z)k$$

be a vector function on a domain $D \in R^3$. By F, to each point (x, y, z) of D, there is assigned a single vector $Pi + Qj + Rk$. The set of all these vectors are said to form a *vector field* F.

Let Γ be a curve lying entirely in D. Then the (indefinite) integral

$$\int_{\Gamma} Pdx + Qdy + Rdz \qquad (1)$$

is called a *line integral (curvilinear integral)* along Γ (the point (x, y, z) varies on Γ), where Γ is considered as *path* of the point (x, y, z).



If the integral is taken from a point A to a point B on Γ, (1) becomes a definite integral denoted by

$$\int_{A\Gamma}^{B} Pdx + Qdy + Rdz$$

The notation

$$\oint Pdx + Qdy + Rdz$$

indicates that Γ is a closed curve and the integral is taken for a complete revolution on Γ in one of two senses.

The line integral (1) can be written as

$$\int_\Gamma F \, . \, dr \qquad (2)$$

where $r(x, y, z)$ is the position vector representing Γ.

Clearly the integrals of functions of a single variable are line integrals.

The integral

$$\int_\Gamma f(x, y, z) ds \qquad (3)$$

is a line integral along a curve where elementary arc length is ds and (x, y, z) varies on Γ.

B. Evaluation and Applications

Let

$$I = \int_\Gamma Pdx + Qdy + Rdz$$

be an indefinite line integral. The curve Γ may be given in one of the two forms:

(i) Γ: $x = x(t)$, $y = y(t)$, $z = z(t)$ parametric form

(ii) Γ: $\phi(x, y, z) = 0$, $\psi(x, y, z) = 0$ simultaneous form (as intersection of two surfaces.)

In the first case (i), the integral I becomes an integral of a function of the single variable t with respect to t:

$$\begin{aligned} I(t) = \int P(x(t), y(t), z(t)) \dot{x} \, dt \\ + Q(x(t), y(t), z(t)) \dot{y} \, dt \\ + R(x(t), y(t), z(t)) \dot{z} \, dt = \int f(t) \, dt \end{aligned}$$

In the second case (ii), since there are two equations with three unknowns, one of them can be taken as a parameter and the two others are expressible in terms of this parameter, the integral reduces to the case (i).

When the line integral is a definite one on Γ from $A(t = \alpha)$

to $B(t = \beta)$, it becomes the definite integral $\int_{\alpha}^{\beta} f(t)dt$.

Example 1. Evaluate

$$I = \int_{\Gamma} xy\ dx + yz\ dz + xz\ dz$$

from $A(0, 0, 1)$ to $B(2, 4, 3)$ where

a) Γ: $r(t) = ti + t^2 j + (t+1)k$

b) Γ: $2x-y = 0$, $x-2z+4 = 0$

Solution. Having

$$x(t) = t, \quad y(t) = t^2, \quad z(t) = t+1 \quad \text{and}$$

$$dx = dt, \quad dy = 2t\ dt, \quad dz = dt$$

on Γ, and A and B corresponding to $t = 0$ and $t = 2$, "I" becomes

$$I = \int_0^2 t^3 dt + (t^3+t^2)2tdt + (t^2+t)dt$$

$$= \int_0^2 (2t^4 + 3t^3 + t^2 + t)dt = 442/15.$$

b) Γ: $2x-y = 0$, $x-2z+4 = 0$. Taking z as parameter we get $x = 2z-4$, $y = 4z-8$, $z = z$ and

$$dx = 2dz, \quad dy = 4dz, \quad dz = dz$$

and

$$I = \int_2^3 (8z^2-32z+32)2dz + (4z^2-8z)4dz + (2z^2-4z)dz$$

$$= \int_2^3 (34z^2 - 100z + 64)dz = -200/3.$$

Observe that the values of the line integral between the same two points but along two distinct paths are distinct.

Example 2. Evaluate

$$\int_{\Gamma} (e^x + y + z^2)dx$$

where

a) Γ: x-axis from $O(0, 0, 0)$ to $A(2, 0, 0)$

b) Γ: y-axis from $O(0, 0, 0)$ to $B(0, 2, 0)$

c) Γ: $x=y=z$ from $O(0, 0, 0)$ to $C(2, 2, 2)$

<u>Solution</u>.

a) Γ: $x = x, \quad z = 0, \quad (0 < x < 2); \quad dx = dx$

$$\int_0^2 e^x dx = e^2 - 1$$

b) Γ: $x = 0, \quad y = y, \quad z = 0 \quad (0 < y < 2), \quad dx = 0$

$$\int_0^2 1.0 = 0$$

c) Γ: $x = y = z. \quad (0 < x < 2), \quad dx = dx$

$$\int_0^2 (e^x + x + x^2)dx = e^2 + 11/3$$

<u>Example 3</u>. Evaluate

$$I = \int_\Gamma (x^2 y + z)ds$$

where Γ is the circular helix

$$x = \cos t, \quad y = \sin t, \quad z = \sqrt{3}t, \quad t \varepsilon [0, \pi/2]$$

<u>Solution</u>.

$$ds = \sqrt{x^2 + y^2 + z^2}\, dt = \sqrt{1 + 3}\, dt = 2\, dt$$

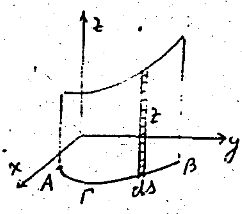
$$\delta = \int_0^{\pi/2} (\cos^2 t \sin t + \sqrt{3}t)2\, dt$$

$$= -\frac{\cos^3 t}{3} + \frac{\sqrt{3}}{2} t^2 \Big|_0^{\pi/2} = \frac{1}{3} + \frac{\sqrt{3}}{8} \pi^2$$

GEOMETRIC APPLICATIONS:

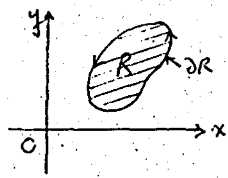
1. $\int_{A\,\Gamma}^{B} ds = |AB|$

   where Γ is a space curve and $|AB|$ is the arc length of Γ from A to B



2. $\int_{A\,\Gamma}^{B} z\, ds = S \quad (z = f(x, y) > 0)$ ....

   where Γ: $g(x, y) = 0$, $z = 0$, and S is the area of the right cylinder $g(x, y) = 0$, above $z = 0$, below the surface $z = f(x, y)$



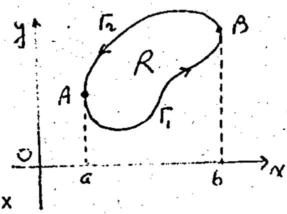
3. $\oint_\Gamma x\, dy = -\oint y\, dx = |R|$ where R is a plane region with a closed boundary ∂R

<u>Proof</u>.

1) and 2) are obvious from the definitions.

3) is proved as follows:

Consider the lower and upper curves Γ_1, Γ_2 of the region R taken as a normal region R_{xy}.



$$\oint_\partial y\, dx = \int_{\Gamma_1} y\, dx + \int_{\Gamma_2} y\, dx$$

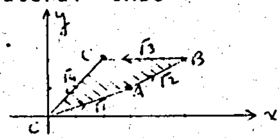
$$= \int_a^b y_1(x)dx + \int_b^a y_2(x)dx$$

$$= \int_a^b (y_1 - y_2)dx = -\int_a^b (y_2 - y_1)6x = -|R|.$$

For the proof of the other part, consider the left and right curves for the region R taken as a normal region R_{yx}.

If the region is not normal it can be split up into normal regions.

Example 1. Compute the area of the quadrilateral OABC shown in the figure, by the use of line integral.



Solution.

$$|R| = \oint_\Gamma xdy = \int_{\Gamma_1} xdy + \int_{\Gamma_2} xdy + \int_{\Gamma_3} xdy + \int_{\Gamma_4} xdy$$

where

Γ_1: $x = 3y$, $0 \leq y \leq 1$; Γ_2: $x = 3y+1$, $1 \leq y \leq 2$

Γ_3: $y = 2$, $2 \leq x \leq 5$; Γ_4: $x = y$, $0 \leq y \leq 2$

Then

$$|R| = \int_0^1 3y\,dy + \int_1^2 (2y+1)dy + \int_2^2 0 + \int_2^0 y\,dy = 7/2$$

Example 2. Find the area of the right cylinder $x^{2/3} + y^{2/3} = 1$ bounded below by the plane $z = 0$ and above by the paraboloid $z = x^2 + y^2$ in the first octant.

Solution. Since $x^{2/3} + y^{2/3} = 1$ has the parametric form

$$x = \cos^3\theta, \qquad y = \sin^3\theta$$

we have

$$S = \int_\Gamma zds = \int_0^{\pi/2} (\cos^6\theta + \sin^6\theta)3\cos\theta\sin\theta\,d\theta$$

$$= -3\int_0^{\pi/2} \cos^7\theta\,d\cos\theta + 3\int_0^{\pi/2} \sin^7\theta\,d\sin\theta$$

$$= -\frac{3}{8}(\sin^8\theta - \cos^8\theta)\Big|_0^{\pi/2} = 3/4$$

Physical Applications

1. Work: The work done by the force $F = (P, Q, R)$ when its point of application moves on a curve

Γ: $r(t) = x(t)i + y(t)j + z(t)k$ from $A = r(\alpha)$ to $B = r(\beta)$ is

$$W = \int_{A\,\Gamma}^{B} F.dr = \int_{A\,\Gamma}^{B} Pdx + Qdy + Rdz$$

2. <u>Mass</u>: Let a wire be given in the shape of the curve Γ: $r(t) = x(t)i + y(t)j + z(t)k$, $t \varepsilon (\alpha, \beta)$ with end points $A(t = \alpha)$, $B(t = \beta)$.

Let $\delta(x, y, z)$ be the density of the wire per unit length. From $dm = \delta\, ds$, one gets

$$m = \int_{A\,\Gamma}^{B} \delta\, ds$$

as the mass of the wire

3. <u>Moments</u>: The *moment* of a particle with mass $dm = \delta\, ds$ with respect to the coordinate planes, coordinate axes and origin are defined by

$x\, dm$, $y\, dm$, $z\, dm$; $\sqrt{y^2+z^2}\, dm$, $\sqrt{z^2+x^2}\, dm$, $\sqrt{x^2+y^2}\, dm$ and $\sqrt{x^2 + y^2 + z^2}\, dm$.

Hence the corresponding moments of a wire in the shape of Γ are obtained by integrations:

$$M_{yz} = \int_\Gamma x\, \delta ds, \quad M_{zx} = \int_\Gamma y\, \delta\, ds, \quad M_{xy} = \int_\Gamma z\, \delta\, ds$$

$$M_{ox} = \int_\Gamma \sqrt{y^2+z^2}\, \delta\, ds, \quad M_{oy} = \int_\Gamma \sqrt{z^2+x^2}\, \delta\, ds,$$

$$M_{oz} = \int_\Gamma \sqrt{x^2+y^2}\, \delta\, ds \qquad M_o = \int_\Gamma \sqrt{x^2+y^2+z^2}\, \delta\, ds$$

4. <u>Center of mass</u> (Center of gravity):

The center of gravity of a wire in the shape of Γ is the point $G(\bar{x}, \bar{y}, \bar{z})$ with mass $m = \int \delta\, ds$ whose moments with respect to coordinate planes are the moments of the wire with respect these coordinate planes:

$$m\,\overline{x} = M_{yz}, \quad m\,\overline{y} = M_{zx}, \quad m\,\overline{z} = M_{xy}$$

Hence

$$\overline{x} = \frac{M_{yz}}{m} = \frac{\int_\Gamma x\,\delta\,ds}{\int_\Gamma \delta\,ds}$$

$$\overline{y} = \frac{M_{zx}}{m} = \frac{\int_\Gamma y\,\delta\,ds}{\int \delta\,ds}$$

$$\overline{z} = \frac{M_{xy}}{m} = \frac{\int_\Gamma z\,\delta\,ds}{\int \delta\,ds}$$

The center of gravity may or may not be on the wire.

If the density is constant (if the wire is homogeneous) G is referred to as the *centroid* of the curve Γ.

5. Moments of inertia (second moments)

These are the expressions obtained from the moments by replacing the (directed) distances by their squares:

$$I_{yz} = \int_\Gamma x^2\,\delta\,ds, \quad I_{zx} = \int_\Gamma y^2\,\delta\,ds, \quad I_{xy} = \int_\Gamma z^2\,\delta\,ds$$

$$I_{ox} = \int_\Gamma (y^2+z^2)\delta\,ds, \qquad I_{oy} = \int_\Gamma (z^2+x^2)\delta\,ds,$$

$$I_{oz} = \int_\Gamma (x^2+y^2)\delta\,ds \qquad I_o = \int_\Gamma (x^2+y^2+z^2)\delta\,ds.$$

Observe that

$$I_{ox} = I_{xy} + I_{xz}, \quad I_{oy} = I_{yx} + I_{yz}, \quad I_{oz} = I_{zx} + I_{zy},$$

$$I_o = I_{yz} + I_{zx} + I_{xy}.$$

The radius of gyration: When the mass of an object is concentreted at a point and this point-mass produces the same moment of inertia with respect to the line ℓ, the distance ς of the point from ℓ is called the radius of gyration of the wire with respect to ℓ.

$$\rho^2 m = I_\ell \qquad \rho^2 = I_\ell / m.$$

Example 1. Find the work done by the force

$$F = \frac{y}{z} i + \frac{z}{x} j + \frac{x}{y} k$$

when its point of application moves on the path

$$\Gamma: \; x = t, \quad y = t^2, \quad z = t^3$$

from $A(1, 1, 1)$ to $B(3, 9, 27)$.

Solution.

$$W = \int_{A\,\Gamma}^{B} F.dr = \int_{A\,\Gamma}^{B} \frac{y}{z} dx + \frac{z}{x} dy + \frac{x}{y} dz$$

$$= \int_1^3 \frac{1}{t} dt + t^2.2tdt + \frac{1}{t} . 3t^2 dt$$

$$= \ell n\, t + \frac{1}{2} t^4 + \frac{3}{2} t^2 \Big|_1^3 = \ell n 3 + 52.$$

Example 2. Given a wire in the shape of the circular helix

$$\Gamma: \; r(t) = (\cos t, \; \sin t, \; 2t) \quad 0 \leq t \leq 2\pi$$

with density proportional to the cote z.

a) find its mass m

b) find its center of gravity G

c) find its moment of inertia I_{oz} and radius of gyration

Solution.

a) Since $\delta = kz = 2kt$ (k is the constant of proportionality)

$$m = \int_\Gamma \delta \, ds = \int_0^{2\pi} 2kt \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt$$

$$= 2k \int_0^{2\pi} t \sqrt{1 + 4}\, dt = 2k\sqrt{5}.\, 2\pi^2 = \underline{4k\sqrt{5}\, \pi^2}$$

b) $M_{xy} = \int_0^{2\pi} 2t.\, 2kt.\, \sqrt{5}\, dt = 4k\sqrt{5} \int_0^{2\pi} t^2 dt = \frac{4k\sqrt{5}}{3} .\, 8\pi^3$

$$M_{xz} = \int_0^{2\pi} (\sin t)2kt\sqrt{5}\,dt = 2k\sqrt{5}\int_0^{2\pi} t\sin t\,dt = -4k\pi\sqrt{5}$$

$$M_{yz} = \int_0^{2\pi} (\cos t)2kt\sqrt{5}\,dt = 2k\sqrt{5}\int_0^{2\pi} t\cos t\,dt = 0$$

$$\bar{x} = 0, \quad \bar{y} = -1/\pi, \quad \bar{z} = 8\pi/3$$

c) $I_{oz} = \int_\Gamma (x^2+y^2).kz\,ds = 2k\sqrt{5}\int_0^{2\pi} t\,dt = 4k\sqrt{5}\pi^2,$

$$\bar{\rho}^2 = I_{oz}/m = 1 \rightarrow \bar{\rho} = 1$$

Example 3. Given a wire in the shape of the curve Γ: $(\ln t, \frac{t^2}{2}, \sqrt{2}t)$, $t\in[1, 4]$, with density proportional to the y-coordinate, find

a) length, b) mass, c) average density

of the wire.

Solution.

a) $s = \int_\Gamma ds = \int_1^4 \sqrt{(\frac{1}{t})^2 + t^2 + 2}\,dt$

$$= \int_1^4 (\frac{1}{t} + t)dt = \ln 4 + \frac{15}{2}$$

b) $m = \int_\Gamma \delta\,ds = \int_1^4 k\,\frac{t^2}{2}\,(\frac{1}{t} + t)dt = \frac{285}{8}\,k$

c) $\bar{\delta} = \frac{m}{s} = \frac{285\,k}{4(2\ln 4 + 15)}$

B. INDEPENDENCE OF PATH

Let

$$\int_{A\,\Gamma}^{B} Pdx + Qdy + Rdz$$

be a line integral. If the values of this line integral along any path from A to B lying in D_F are the same, then the line integral is said to be *independent of path*. In this case the value of the line integral depends only on the end points A and B, and can be denoted by

$$\int_{A}^{B} Pdx + Qdy + Rdz$$

<u>Theorem</u>. Suppose that $u(x, y, z)$ is a scalar function having continuous first order partial derivatives on a connected domain D with

$$\frac{\partial u}{\partial x} = P, \quad \frac{\partial u}{\partial y} = Q. \quad \frac{\partial u}{\partial z} = R \quad (\text{or} \quad u = F) \tag{1}$$

Then

$$\int_{A\,\Gamma}^{B} Pdx + Qdy + Rdz = \int_{A\,\Gamma}^{B} u.dr = u(B) - u(A)$$

for any piecewise smooth path Γ.

<u>Proof.</u>

Let

$$\Gamma: r(t) = x(t)i + y(t)j + z(t)k$$

$$\int_{A\,\Gamma}^{B} u.dr = \int_{A\,\Gamma}^{B} \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy + \frac{\partial u}{\partial z}\,dz$$

$$= \int_{A\,\Gamma}^{B} \left(\frac{\partial u}{\partial x}\;\frac{dx}{dt} + \frac{\partial u}{\partial y}\;\frac{dy}{dt} + \frac{\partial u}{\partial z}\;\frac{dz}{dt}\right)dt$$

$$= \int_{\alpha}^{\beta} \frac{d}{dt}\; u(x(t),\, y(t),\, z(t))dt$$

$$= u(x(t),\, y(t),\, z(t))\Big|_{\alpha}^{\beta} = u(B) - u(A).$$

where $r(\alpha) = A$, $r(\beta) = B$. ∎

From (1), we get

$$u_{xy} = P_y,\ u_{xz} = P_z,\ u_{yx} = Q_x,\ u_{yz} = Q_z,\ u_{zx} = R_x,\ u_{zy} = R_y$$

If these derivatives exist and continuous, the equality of mixed derivatives gives

$$P_y = Q_x, \quad P_z = R_x, \quad Q_z = R_y \tag{2}$$

which are the conditions for independence of path.

Observe that

$$\text{curl } F = 0$$

gives (2). In that case $Pdx + Qdy + Rdz$ is said to be *exact (total) differential form.*

The line integral

$$\int_{A\,\Gamma}^{B} P(x,\, y)dx + Q(x,\, y)dy$$

in the plane is independent of path if

$$P_y = Q_x$$

obviously.

Note that when a line integral is independent of path, the evaluation may be done by the use of one of the following ways:

1) by choosing a simpler path

2) by determining a primitive function u.

<u>Example</u>. Evaluate

$$I = \int (e^{x} + z)dx + \ln(1+z)dy + (\frac{y}{R+Z} + x)dz$$

along the path

$$\Gamma: \quad r(t) = t^{2}i + t^{3}j + tk$$

from $0(0, 0, 0)$ to $A(1, 1, 2)$, testing for independence of paths, and if so, choose a simpler path.

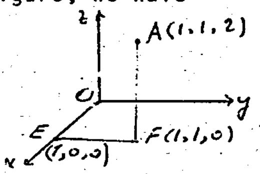
<u>Solution.</u>

$$P = e^{x} + z, \quad Q = \ln(1+z), \quad R = \frac{y}{1+z} + x$$

$$P_y = 0 \quad Q_x, \quad P_z = 1 = R_x, \quad Q_z = \frac{1}{1+z} = R_y$$

independence of path.

Then cloosing the path shown in the Figure, we have



$$\int_{(OE)} F.dr = \int_0^1 e^{x}dx = e-1$$

$$\int_{(EF)} F.dr = \int_0^1 \ln 1 \, dy = 0$$

$$\int_{(FA)} F.dr = \int_0^2 (\frac{1}{1+z} + 1)dz = \ln(1+z) + z \Big|_0^2 = \ln 3 + 2$$

$$I = e + \ln 3 + 1$$

Evaluate the same integral following the given path Γ.

If one likes to use the second way, determinations of the primitive function is necessary.

DETERMINATION OF THE PRIMITIVE:

$$\frac{\partial u}{\partial x} = P, \quad \frac{\partial u}{\partial y} = Q, \quad \frac{\partial u}{\partial z} = R$$

$$\frac{\partial u}{\partial x} = P \qquad u = \int P(x, y, z)\partial x + \phi(y, z)$$

$$= V(x, y, z) + \phi(y, z)$$

Since integration is taken with respect to x alone (in this integration y and z are considered as constant) we have a constant integration as function of y and z.

$$\frac{\partial u}{\partial y} = Q = V_y + \phi_y(y, z)$$

$$\Rightarrow \phi_y(y, z) = Q - V_y$$

$$\Rightarrow \phi(y, z) = u - V + \psi(z)$$

$$u = \phi(y, z) + V - \psi(z)$$

$$\frac{\partial u}{\partial z} = R = \phi_z(y, z) + V_z - \psi'(z)$$

$$\Rightarrow \psi'(z) = \phi_z(y, z) + V_z - R$$ (the right hand side is a function of z alone, say $h(z)$)

$$\Rightarrow \psi(z) = \int h(z)dz$$

Then

$$u = \phi(y, z) + V - \psi(z)$$

Example. Given

$$I = \int_{A\,\Gamma}^{B} (\arctan y - 2x)dx + \left(\frac{x}{1+y^2} + 2y\right)dy,$$

a) test for independence of path

b) if so, evaluate by the use of a primitive function from $A(1, 1)$ to $B(2, \sqrt{3})$

Solution.

a) $P = \arctan y - 2x, \qquad Q = \dfrac{x}{1+y^2} + 2y$

$$P_y = \frac{1}{1+y^2} = Q_x \qquad \text{(ind. of path (exactness))}$$

b) $u = \int (\arctan y - 2x)\partial x = x \arctan y - x^2 + \phi(y)$

$$u_y = \frac{x}{1+y^2} + \phi'(y) = \frac{x}{1+y^2} + 2y$$

$$\phi(y) = y^2 \qquad u = x \arctan y - x^2 + y^2$$

Then,

$$I = u(B) - u(A) = 5\pi/12-1$$

Example. Evaluate the line integral

$$I = \int_0^B (e^x + z)dx + \ln(1+z)dy + (\frac{y}{1+z} + x)dz$$

from $0(0, 0, 0)$ to $B(1, 1, 2)$ by the use of exactness and primitive function.

Solution.

$$P = e^x + z, \qquad Q = \ln(1+z), \qquad R = \frac{y}{1+z} + x$$

$$\frac{\partial u}{\partial x} = P = e^x + z \Rightarrow u = \int (e^x + z)\partial x$$
$$= e^x + zx + \phi(y, z)$$

$$\frac{\partial u}{\partial y} = Q = 0 + 0 + \phi_y(y, z)$$
$$\Rightarrow \phi_y(y, z) = Q = \ln(1+z)$$
$$\Rightarrow \phi(y, z) = \int \ln(1+z)\partial y = y\ln(1+z) + \psi(z)$$
$$u = e^x + zx + y\ln(1+z) + \psi(z)$$

$$\frac{\partial u}{\partial z} = R = 0 + x + \frac{y}{1+z} + \psi'(z)$$
$$\Rightarrow \psi'(z) = 0 \Rightarrow \psi(z) = c$$
$$u = e^x + zx + y\ln(1+z) + c$$

Then

$$I = u(B) - u(0) = e + \ln 3 + 1$$

This is the previously solved problem by the choice of a simple path. Observe that the results are the same.

In the independence of path of a line integral along a closed curve $\Gamma(= \partial R)$, the line integral is zero if the primitive and the integrands P, Q, R are continuous in the interior of Γ.

EXERCISES (5. I)

1. Evaluate

$$\int_\Gamma \frac{-y}{x\sqrt{x^2 - y^2}}\,dx + \frac{1}{\sqrt{x^2 - y^2}}\,dy$$

where Γ is the arc of $x^2 - y^2 = 9$ from $(3, 0)$ to $(5, 4)$.

2. Evaluate $\int_{O\,\Gamma}^{B} (x+y)dx + (x-y)dy$ where Γ consists of the line segment from $O(0, 0)$ to $A(2, 0)$ and that from $A(2, 0)$ to $B(2, 1)$.

3. Evaluate $\int_\Gamma (x^2-2y)dx + (2x+y^2)dy$, Γ being the arc of curve $y^2 = 4x-1$ from $A(1/2, -1)$ to $B(5/4, 2)$.

4. Evaluate $\int_\Gamma (x^2-2y)dx + (2x+y^2)dy$, Γ being the line segment from $A(1/2, -1)$ to $B(5/4, 2)$.

5. Evaluate $\int y\,dx + (x^2+y^2)dy$ where Γ consists of the line segment from $(-2, 0)$ to $(0, 0)$ followed by that from $(0, 0)$ to $(0, 2)$.

6. Evaluate the line integral

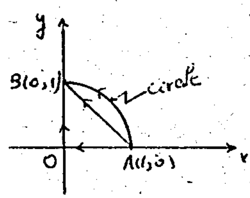
$$\int_C \frac{x^2}{\sqrt{x^2 - y^2}}\,dx \quad \frac{2y}{4x^2 + y^2}\,dy$$

where C is the arc of $y = x^2/2$ from $(0, 0)$ to $(2, 2)$.

7. Evaluate

$$\int_\Gamma \frac{x\,dy - y\,dx}{x + y}$$

along



a) AOB b) $[AB]$ c) $\widehat{AB}$

8. Show that the area of the region bounded by the evolute $x = \frac{c^2}{a}\cos^3\theta$, $y = -\frac{c^2}{b}\sin^3\theta$ of the ellipse $x = a\cos\theta$, $y = b\sin\theta$ is

equal to $\frac{3}{8}\pi\frac{c^4}{ab}$

9. Evaluate

$$\int_\Gamma y^2\sqrt{1+\cos^2 x}\,\sin^3 x\,ds$$

where Γ is the arc of $y = \sin x$ from origin to $(\pi/2, 1)$.

10. Evaluate $\int_\Gamma \sin x\,dy + \cos y\,dx$ where

$$\Gamma:\ x = t^2+3, \quad y = 2t^2-1, \quad 0 \leq t \leq 2$$

11. Evaluate $\int_\Gamma \sqrt{x+3y}\,ds$, Γ being the line segment from $(0, 0)$ to $(3, 9)$.

12. Find the area of the right cylinder with directrix $x = t^2+3$, $y = 2t^2-1$, $t\in[0, 2]$ bounded by the surfaces $z = 0$ and $z = x^2+y^2$.

13. Find the mass of a wire in the shape of $y = (\cos 2x)/2$, $(0 \leq x \leq \ln 2)$, if the density per unit length is given by $\delta = xy$.

14. Evaluate

$$\int_\Gamma z\,dx + x\,dy + y\,dz$$

where Γ is the circular helix $x = a\cos\theta$, $y = a\sin\theta$, $z = a\theta$ from $A(\theta = 0)$ to $B(\theta = 2\pi)$.

15. Evaluate

$$\int (y^2+z^2)dx + (z^2+x^2)dy + (x^2+y^2)dz$$

along the curve

$$\Gamma:\ x^2+y^2+z^2 = 4ax, \quad x^2+y^2 = 2ax \quad (a>0,\ z>0)$$

(Hint: set $z = 2a\cos t$).

16. Evaluate $\int_\Gamma F.dr$ where $F = (yz, zx, xy)$ and Γ is the intersection of the unit sphere centered at 0 and the plane $z = 0$, revolved once in the positive sense.

17. A particle moves along the straight line in R^3 joining $A(1, 2, 1)$ to $B(7, 4, 4)$, subject to the force

$$F = \frac{x\,i + y\,j + z\,k}{(x^2+y^2+z^2)^{3/2}}$$

Find the work done.

18. Evaluate

$$\int_\Gamma \frac{x\,dx + y\,dy + z\,dz}{x^2 + y^2 + z^2}$$

where Γ is the arc

$x = 2t$, $y = 2t+1$, $z = t^2+1$ from $(0, 1, 1)$ to $(2, 3, 2)$

19. Evaluate $\int_\Gamma e^{y/x}dx + \sin x dy + \frac{1}{2} xy\,dz$, Γ being $x = t$, $y = t^2$, $z = t^3$, $t\varepsilon(2, 3)$.

20. Evaluate $\int_\Gamma xy\,dx + x^2z\,dy + xyz\,dz$ where
Γ: $x = e^t$, $y = e^{-t}$, $z = t^2$, $t\varepsilon(0, 1)$.

21. Evaluate $\int_\Gamma xyz\,dx + \ln(x+y+z)dy + zdz$ along the perimeter of the triangle ABC where $A(0, 0, 1)$, $B(0, 1, 0)$, $C(1, 0, 0)$ $(A \to B \to C \to A)$

22. Evaluate

$$\int_\Gamma (x^2+y^2)dx + xyz\,dy + (x+y+z)dz$$

where Γ is the perimeter of the rectangle with vertices $A(0, 0, 1)$, $B(1, 0, 1)$, $C(1, 1, 1)$, $D(0, 1, 1)$ in the positive sense.

23. Show that the integrand is exact differential and evaluate the integral

$$\int_{(2, 1)}^{(4, 2)} (x^2+2y)dx + (2y+2x)dy$$

24. Same question for

$$\int_{(0,\ 2)}^{(5,\ 1)} \left(\frac{2x^2y}{1+x^2} + 3\right)dx + \left[2y\ \ell n(1+x^2)-2\right]dy$$

25. Same question for

$$\int_{(1,\ 1)}^{(3,\ 2)} (3y^2+4xy-2x^2)dy + (2y^2-4xy-3x^2)dx$$

ANSWERS TO EVEN NUMBERED EXERCISES

2. 7/2

4. 1319/64

6. $4 + 2\ \ell n(5/4)$

10. $2(\cos 3 - \cos 7) + \frac{1}{2}(\sin 7 + \sin 1)$

12. $220\sqrt{5}$

14. $3\pi\ a^2$

16. 0

18. $\frac{1}{2}\ \ell n\ \frac{17}{2}$

20. 1/2

22. 1/2

24. $16 + \ell n\ 26$.

5. 2. DOUBLE INTEGRALS

A. DEFINITIONS AND PROPERTIES

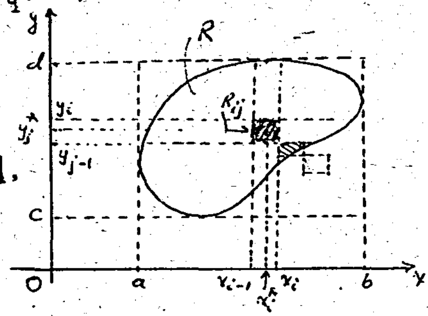
Historically the double integrals arose in an effort to formulate the volume of the solid under a surface $z = f(x, y) \geq 0$ over a bounded plane region R.

Omitting the above restriction $z > 0$, the double integral of a function $f(x, y)$ continuous over a bounded region is defined as follows:

Let the region R be inscribed in the smallest rectangular region $[a, b] \times [c, d]$, and let the intervals $[a, b]$, $[c, d]$ be partioned into subintervals $[x_{i-1}, x_i]$, $[y_{j-1}, y_j]$ by points

$$x_0(= a), \ \ldots\ , x_i\ , \ldots\ , x_p(= b)$$

$$y_0(= c), \ \ldots\ , y_j\ , \ldots\ , y_q(= d)$$



Consider the cross product

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$$

which is a subregion of $[a, b] \times [c, d]$, with area

$$|R_{ij}| = \Delta x_i \Delta y_j = \Delta y_j \Delta x_i = \Delta A_{ij}$$

where

$$\Delta x_i = x_i - x_{i-1}, \quad \Delta y_j \quad y_j - y_{j-1}$$

Letting $(x_i{}^*, y_j{}^*) \varepsilon R_{ij}$, and defining $f(x, y)$ to be zero when $(x_i{}^*, y_j{}^*) \notin R$, consider the double sum

$$\sum_{j=1}^{q} \sum_{i=1}^{p} f(x_i{}^*, y_j{}^*) \Delta A_{ij}.$$

If $m = \min f$, $M = \max f$ over R and if $m_{ij} = \min f$, $M_{ij} = \max f$ over R_{ij}, we have

$$m\ A < \Sigma\Sigma f(x_i^*,\ y_j^*)\Delta A_{ij} < M\ A$$

when $A = |R|$.

It is proved in Advanced Calculus that, for a continuous function f over R

$$\lim_{\substack{(p,\ q)\to(\infty,\ \infty) \\ \eta \to 0}} \sum_{j=1}^{q} \sum_{i=1}^{p} f(x_i^*,\ y_j^*)\Delta A_{ij},$$

where $\eta = \max\ (\Delta x_i,\ \Delta y_j)$ for all i, j, exists for any rectangular or arbitrary partition of R, and the limit is denoted by

$$\iint_R f(x,\ y)dA,$$

called the (definite) *double integral* of $f(x, y)$ over R, and f is said to be *integrable* over R.

<u>Properties</u>. Let f, g be continuous functions of two variables. Then

1. $\iint_R dA = |R| = A$
2. $\iint_R f\ dA =$ volume if $f(x,\ y) \geqslant 0$
3. $\iint_R (f+g)dA = \iint_R f\ dA + \iint_R g\ dA$
4. $\iint_R c\ f\ dA = c \iint_R f\ dA$ (c is a const)
5. $\iint_{R_1} f\ dA + \iint_{R_2} f\ dA = \iint_{R_1 \cup R_2} f\ dA$ (if $R_1 \cap R_2 = \phi$)
6. $m\ A \leqslant \iint_R f\ dA \leqslant M\ A$
7. $\left|\iint_R f\ dA\right| \leqslant \iint_R |f|\ dA$
8. $f(x,\ y) \leqslant g(x,\ y) \Rightarrow \iint_R f\ dA \leqslant \iint g\ dA$.

The first two are obvious from the definition of a double integral.

<u>Proof of (3)</u>:

$$\sum \sum \left[f(x_i^*,\ y_j^*) + g(x_i^*,\ y_j^*)\right] d\ A_{ij}$$

$$= \sum\sum f(x_i^*, y_j^*)\Delta A_{ij} + \sum\sum g(x_i^*, y_j^*)\Delta A_{ij} \qquad (3).$$

The other can be proved in a similar way. ∎

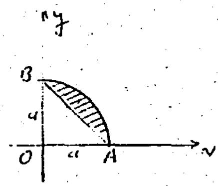
Example 1. Evaluate the double integral

$$I = \iint_R dA$$

where R is the normal region

$$R_{xy} = \left[0, a; a-x, \sqrt{a^2-x^2}\right]$$

Solution. Referring to the figure, we have



$$I = |OABO| - |OAB|$$

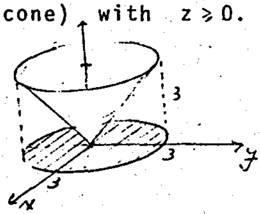
$$= \frac{1}{4}\pi a^2 - \frac{1}{2} a^2 = (\pi-2)a^2/4.$$

Example 2. Evaluate

$$\iint_R \sqrt{x^2 + y^2}\, dA$$

where R is the circular region with center at the origin and radius 3 units.

Solution. $z = \sqrt{x^2 + y^2} \Rightarrow x^2 + y^2 - z^2 = 0$ (cone) with $z \geq 0$. The given double integral is the volume under the cone over the circular region, i.e., the volume of the solid bounded by the cone and the circular cylinder with $z \geq 0$:

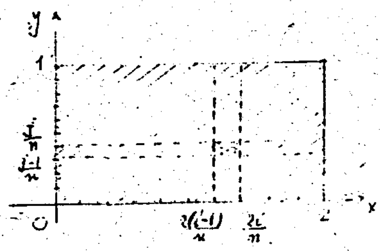


$$\iint_R \sqrt{x^2 + y^2}\, dA = V = \pi 3^2.3 - \frac{1}{3}\pi\, 3^2.3 = 18\pi \text{ unit}^3.$$

Example 3. Evaluate

$$I = \iint_R (x + y)dA,$$

where R is the rectangular region $[0, 2] \times [0, 1]$, by limit.



Solution. Using regular partitions on $(0, 2)$, $(0, 1)$ with $\Delta x_i = 2/n$, $\Delta y_j = 1/n$, and for simplicity taking (x^*, y^*) at the right upfor corner of R_{ij}, we have

$$I = \lim_{n \to \infty} \sum_{j=1}^{n} \sum_{i=1}^{n} \left(\frac{2i}{n} + \frac{j}{n}\right) \frac{2}{n} \frac{1}{n}$$

$$= \lim_{n \to \infty} \frac{2}{n^3} \sum_j \sum_i (2i + j)$$

$$= \lim_{n \to \infty} \frac{2}{n^3} \sum_j \left[2 \cdot \frac{n(n+1)}{2} + nj\right]$$

$$= \lim_{n \to \infty} \frac{2}{n^3} \left[n.n(n+1) + n \cdot \frac{n(n+1)}{2}\right]$$

$$= \lim_{n \to \infty} \frac{2}{n^3} \cdot \frac{3}{2} n^2(n+1) = 3.$$

B. Evaluation by Iterated Integrals

The previous examples were simple cases of evaluation. In the general case, the evaluation is performed by evaluating successive (iterated) integrals.

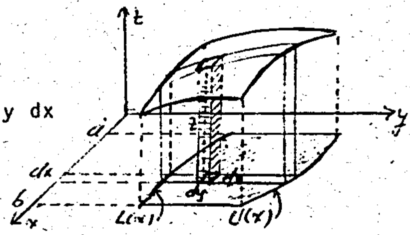
Let

$$I = \iint_R f(x, y)dA.$$

When $f(x, y) \geq 0$ over R, the double integral can be interpreted as the volume under the surface $z = f(x, y)$ over the region R. If $f(x, y) < 0$ in a subset R' of R, then "I" is the difference between the volume above $R-R'$ and the volume below R'. Hence we may suppose $f(x, y) \geq 0$.

Let R be a normal region, say

$$R_{xy} = \left[a, b; \ L(x), V(x)\right]$$



Consider an elementary rectangular prizm of height $z = f(x, y)$ and base dy dx (See Fig.), having the elementary volume

$$dV = z\,dy\,dx$$

When dV is integrated with respect to y partially (considering x constant) from $L(x)$ to $U(x)$, the elementary prizm sweeps out the slice under the surface from $L(x)$ to $U(x)$ over the strip of width dx. The volume of this slice is

$$(\text{Area of the slice}) \times (\text{thickness})$$
$$= \left[\int_{L(x)}^{U(x)} z\,dy\right] dx$$

The slice sweeps out the whole volume V, when the strip (the base of the slice) sweeps out the region R:

$$V = \int_a^b \left[\int_{L(x)}^{U(x)} z\,dy\right] dx.$$

Therefore

$$\iint_{R_{xy}} f(x, y)dA = \int_a^b \int_{L(x)}^{U(x)} f(x, y)dy\,dx$$

where the inner integral is performed first obtaining a function of x alone, then the outer one.

This process of evaluation is called *integration by iteration*. Then any double integral over a normal region can be written as iterated integrals:

$$\iint_{R_{xy}} f\,dA = \int_a^b \int_{L(x)}^{U(x)} f\,dy\,dx$$

or

$$\iint_{R_{yx}} f\,dA = \int_c^d \int_{l(y)}^{r(y)} f\,dx\,dy$$

if

$$R_{yx} = \left[c, \ d; \ \ell(y), r(y)\right]$$

The order of integrals in the second is the reverse of the order of the first.

If R is not a normal region, it can be split up into a finite number of normal regions $R_1, \ldots, R_k$. Then the double integral $\iint_R f\, dA$ is the sum of iterated integrals of f over normal regions $R_1, \ldots, R_k$ from Prop. 5.

Observe that in the notations of normal regions the first (second) ordered pair appear as limits in the outer (inner) integral.

REVERSING THE ORDER OF INTEGRATION:

Writing a given iterated integral defined a normal region, as iterated integral(s) over normal region(s) of other type is called the *reversing (changing) the order of integration.*

The reversing of order may be helpfull to evaluate the iterated integral, when the evaluation of the inner integral is impossible or too complicated.

If the given iterated integral is

$$\int_a^b \int_L^U f(x, y)dy\, dx$$

say, then the steps necessary in reversing the order are

(i) Sketching the normal region

$$R_{xy} = \left[a, b; \ L(x), U(x)\right]$$

(ii) Writing the sketched region as R_{yx} or as union of such region $R_1, \ldots, R_k$

(iii) Setting up the iterated integrals corresponding to $R_1, \ldots, R_k$ and adding them up.

<u>Corollary</u>. If a, b, c, d are constant, then

1. $\int_a^b \int_c^d f(x, y)dy\, dx = \int_c^d \int_a^b f(x, y)dy\, dx$

2. $\int_a^b \int_c^d f(x)\, g(y)dy\, dx = \int_a^b f(x)dx \int_c^d g(y)dy$

<u>Example 1</u>. Evaluate the iterated integral:

$$I = \int_0^{\pi/3} \int_0^{4\cos\theta} r \,\text{Sin}\theta\, dr\, d\theta$$

<u>Solution</u>.

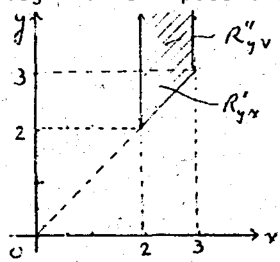
$$I = \int_0^{\pi/3} \left[\frac{r^2}{2}\,\text{Sin}\theta\right]_0^{4\cos\theta} d\theta = \int_0^{\pi/3} 8\cos^2\theta \sin\theta\, d\theta$$

$$= -\frac{8}{3}\cos^3\theta\Big|_0^{\pi/3} = -\frac{1}{3} + \frac{8}{3} = 7/3$$

<u>Example 2</u>. Evaluate

$$I = \int_2^3 \int_2^y \frac{1}{y^2}\,\ell n(\ell n\, x)dx\, dy + \int_3^\infty \int_2^3 \frac{1}{y^2}\,\ell n(\ell n\, x)dx\, dy.$$

<u>Solution</u>. Since the evaluation of inner integrals is impossible, we try changing the order of integration.



$$R = \underbrace{[2, 3;\ 2, y)}_{R'_{yx}} \cup \underbrace{[3, \infty;\ 2, 3)}_{R''_{yx}}$$

$$R = R_{xy} = [2, 3;\ x, \infty)$$

Then

$$I = \int_2^3 \int_x^\infty \frac{1}{y^2}\,\ell n\, \ell n\, x\, dy\, dx$$

$$= \int_2^3 -\frac{1}{y}\,\ell n\, \ell n\, x\Big|_x^\infty dx = \int_2^3 \frac{\ell n\, \ell n\, x}{x}\, dx$$

$$= \int_2^3 \ell n\, \ell n\, x\, d\,\ell n\, x = \int_{u_1}^{u_2} \ell n\, u\, du$$

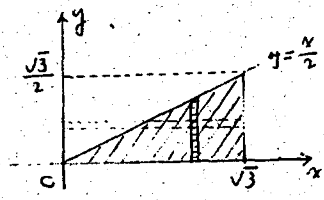
$$= u \ln u - u \Big|_{u_1}^{u_2} = \ln x \ln \ln x - \ln x \Big|_{2}^{3}$$

$$= \ln 3 \ln \ln 3 - \ln 3 - \ln 2 \ln \ln 2 + \ln 2.$$

Example 3. Evaluate by reversing the order of integration.

$$I = \int_0^{\sqrt{3}} \int_0^{x/2} \frac{x}{1+y^2}\, dy\, dx$$

Solution. $R_{xy} = [0, \ 3; \ 0, x/2]$



$$R_{yx} = \left[0, \sqrt{3}/2; \ 2y, \sqrt{3}\right]$$

$$I = \int_0^{\sqrt{3}/2} \int_{2y}^{\sqrt{3}} \frac{x}{1+y^2}\, dx\, dy$$

$$= \int_0^{\sqrt{3}/2} \frac{1}{2} \frac{x^2}{1+y^2} \Big|_{2y}^{\sqrt{3}} dy$$

$$= \frac{1}{2} \int_0^{\sqrt{3}/2} \left(\frac{3}{1+y^2} - \frac{4y^2}{1+y^2}\right) dy$$

$$= \frac{3}{2} \arctan y \Big|_0^{\sqrt{3}/2} - 2\,(y - \arctan y)_0^{\sqrt{3}/2}$$

$$= \frac{7}{2} \arctan \frac{\sqrt{3}}{2} - \sqrt{3}.$$

Evaluate the given integral without reversing the order!

C. Evaluation by Change of Variables

Let

$$\iint_R \varphi(x, y)\, dA \qquad (1)$$

be a double integral to be evaluated over R.

Suppose we make a change of variables:

$$x = F(u, v), \quad y = G(u, v) \qquad (2)$$

where F, G are continuous.

When (2) is substituted in (1), the latter reduces to

$$\iint_{R'} \phi(u, v)\,dA'$$

defined over a region R' in rectangular uv-system, where

$$\phi(u, v) = (F(u, v), G(u, v))$$

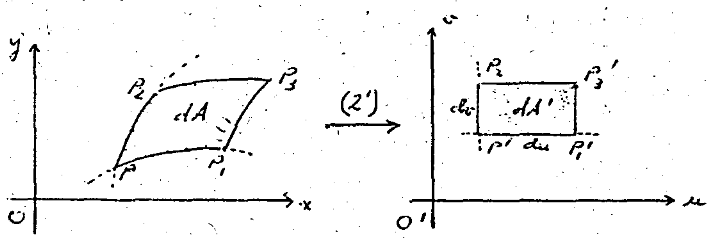
and R', dA', are to be determined.

If (2) is solved for u, v in terms of x, y we have

$$u = f(x, y), \quad v = g(x, y) \qquad (2')$$

This transformation defines a mapping from rectangular xy-system to rectangular uv-system, that is, the inverse transformation (2') of (2) maps a point P(x, y) of xy-system to a point P'(u, v) in uv-system, and maps the region R to a region R'. It also maps an elementary area dA into an elementary area dA' since F, G, f, g are continuous.

Taking dA' as area du dv of an elementary rectangle $P'\,P'_1\,P'_3\,P'_2$ with sides parallel to u- and v-axes., dA will be the area of an elementary curvilinear parallelogram $PP_1P_3P_2$ whose sides and vertices are mapped to the sides and vertices of $P'P'_1P'_3P'_2$, under (2'):



Now we evaluate dA interms of dudv.

Since the curves PP_1, PP_2 are mapped respectively to horizontal and vertical line $P'P'_1$, $P'P'_2$ along which v, u are constant, the equations of PP_1, PP_2 will be

$$g(x, y) = v, \quad f(x, y) = u,$$

so that along PP_1, v remains constant while u varies, and along PP_2, u remains constant while v varies. Therefore if P has coordinates $F(u, v)$, $G(u, v)$, those of the nearby points P_1 and P_2 will be

$$F(u + du, v), \quad G(u + du, v);$$
$$F(u, v + dv), \quad G(u, v + dv)$$

and by Mean Value Theorem, we have

$$P(F(u, v), \; G(u, v))$$
$$P_1(F(u, v) + F_u\, du, \; G(u, v) + G_u\, du)$$
$$P_2(F(u, v) + F_v\, dv, \; G(u, v) + G_v\, dv)$$

Now, since $PP_1P_3P_2$ is nearly a parallelogram for small du, dv, the area dA is twice the area of the triangle PP_1P_2:

$$dA = |PP_1P_2| = \pm \begin{vmatrix} F & G & 1 \\ F + F_u du & G + G_u\, du & 1 \\ F + F_v dv & G + G_v\, dv & 1 \end{vmatrix}$$

$$= \pm \begin{vmatrix} F & G & 1 \\ F_u du & G_u dv & 1 \\ F_v dv & G_v dv & 1 \end{vmatrix} = \pm \begin{vmatrix} F_u & G_u \\ F_v & G_v \end{vmatrix} du\, dv$$

$$\Rightarrow \quad dA = \pm \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial G}{\partial u} \\ \frac{\partial F}{\partial v} & \frac{\partial G}{\partial v} \end{vmatrix} du\, dv$$

where + or - is taken to make the right hand side positive.

The determinant

$$J = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial G}{\partial u} \\ \frac{\partial F}{\partial v} & \frac{\partial G}{\partial v} \end{vmatrix} \quad \text{or} \quad J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

is called the JACOBIAN of F, G (or of x, y) with respect to u, v, denoted by

$$\frac{\partial(F, G)}{\partial(u, v)} \quad \text{or} \quad \frac{\partial(x, y)}{\partial(u, v)}$$

Thus

$$\iint_R f(x, y)dA = \iint_{R'} \phi(u, v) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

where R' is the region bounded by the image $\partial R'$ of ∂R, in uv-system.

By a proper change of variables the region R or the integrand φ or both may be transformed into simpler ones.

When

$$\iint_{R'} \phi(u, v)du\, dv$$

is given, clearly

$$\iint_{R'} \phi(u, v)du\, dv = \iint_R f(x, y) \frac{\partial(u, v)}{\partial(x, y)} dx\, dy$$

under the same change of variables $u = f(x, y)$, $v = g(x, y)$ where the JACOBIANS satisfy the relation

$$\frac{\partial(x, y)}{\partial(u, v)} \cdot \frac{\partial(u, v)}{\partial(x, y)} = 1 \qquad (3)$$

so that each is the multiplicative inverse of the other.

(3) can be shown by direct multiplication of two determinants.

These two are the JACOBIANS of the transformation (2) and its inverse transformation (2').

Example 1. Evaluate

$$I = \iint_R x dy\, dx$$

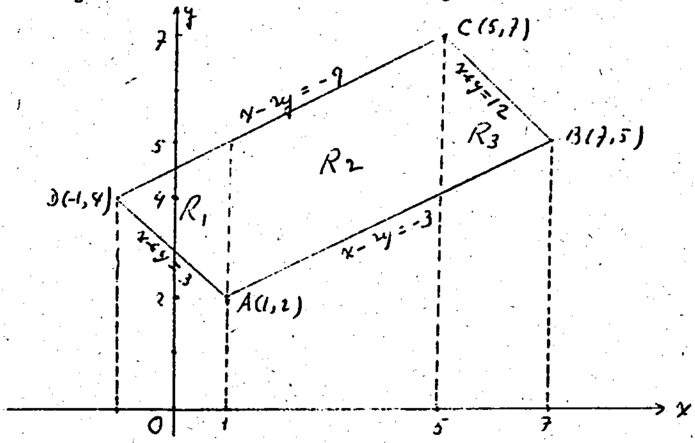
where R is the region bounded by pairs of parallel lines

$x-2y = -3$, $x-2y = -9$; $x+y = 12$, $x+y = 3$

a) directly, b) by a change of variables.

Solution.

a) Sketching of R is shown in the Figure:



R is the union of R_1, R_2, R_3:

$$\iint_{R_1} x\,dy\,dx = \int_{-1}^{1}\int_{3-x}^{\frac{x+9}{2}} x\,dy\,dx = 1$$

$$\iint_{R_2} x\,dy\,dx = \int_{1}^{5}\int_{\frac{x+3}{2}}^{\frac{x+9}{2}} x\,dy\,dx = 36$$

$$\iint_{R_3} x\,dy\,dx = \int_{5}^{7}\int_{\frac{x+3}{2}}^{12-x} x\,dy\,dx = 17$$

$$I = 1 + 36 + 17 = 54.$$

b) The equations of lines suggest the substitution:

$$x-2y = u \qquad x = \frac{1}{2}u + v$$
$$x+y = v \qquad y = -\frac{1}{3}u + \frac{1}{3}v$$

The parallel lines $x-2y = -3$, $x-2y = -9$ are transformed to vertical lines $u = -3$, $u = -9$ in rectangular uv-system, and the two other parallel lines co horizontal lines $v = 12$, $v = 3$, forming a rectangle A'B'C'D' with sides parallel to coordinate axes.

Since for $(x, y)\varepsilon R$ one has

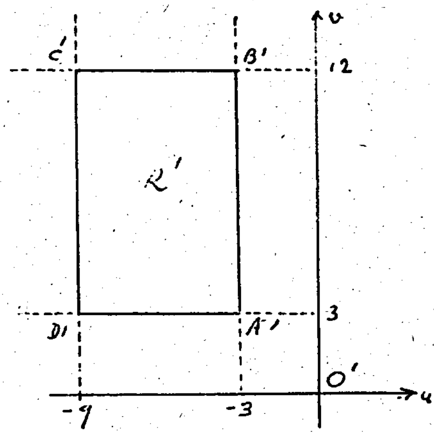
$$x-2y \leq -3, \quad x-2y \geq -9$$

$$x+y \leq 12, \quad x+y \geq 3,$$

it follows that for $(u, v)\varepsilon R'$

$$-9 \leq u \leq -3, \quad 3 \leq v \leq 12$$

hold, showing that the transformed region R' is the interior of the rectangle A'B'C'D'.



The JACOBIAN of the transformation being

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1/3 & 2/3 \\ -1/3 & 1/3 \end{vmatrix} = 1/3 > 0,$$

we have

$$I = \int_{-9}^{-3} \int_{3}^{12} \left(\frac{u}{3} + \frac{2v}{3}\right) . \frac{1}{3} \, . \, dv \, du = 54$$

<u>Example 2</u>. Evaluate

$$I = \int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{9 - x^2 - y^2} \, dy \, dx$$

by the use of polar substitution

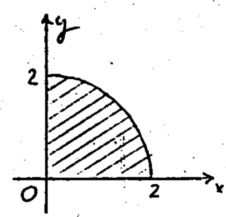
$$x = r\cos\theta, \quad y = r\sin\theta$$

<u>Solution</u>.

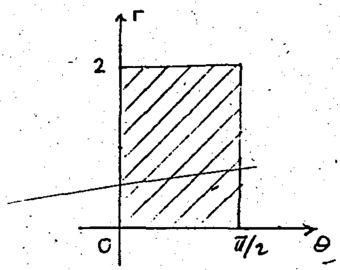
The region of integration is the normal region

$$R_{xy} = \left[0, 2; \ 0, \sqrt{4-x^2}\right]$$

where graph is the quarter of circular disc of radius 2.



In polar coordinates it is the normal region



$$R_{\theta r} = \left[0, \frac{\pi}{2} ; \ 0, 2\right]$$

Sketch is rectangular θr-system.

Then

$$I = \int_0^{\pi/2} \int_0^2 \sqrt{9-r^2} \cdot |r| \, dr \, d\theta$$

where $|r|$ is the absolute value of the JACOBIAN of the transformation:

$$I = \begin{vmatrix} x_\theta & x_r \\ y_\theta & y_r \end{vmatrix} = \begin{vmatrix} -r\sin\theta & \cos\theta \\ r\cos\theta & \sin\theta \end{vmatrix} = -r$$

$|I| = r$ (since $r > 0$ throughout the region)

$$I = \int_0^{\pi/2} \int_0^2 \sqrt{9-r^2} \, r \, dr \, d\theta$$

$$= \int_0^{\pi/2} d\theta \int_0^2 \sqrt{9-r^2} \, r dr$$

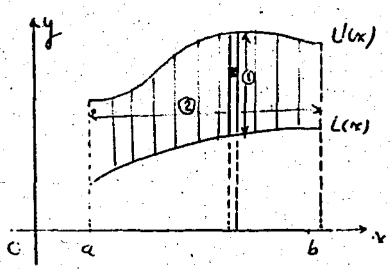
$$= \frac{\pi}{2} \left[- \frac{1}{3} (9-r^2)^{3/2}\right]_0^2 = \frac{\pi}{6} \left(-5\sqrt{5} + 27\right).$$

D. GEOMETRICAL APPLICATIONS

AREA

<u>Area of a cartesian normal region:</u>

Let



$$R_{xy} = \left(a, b \ ; \ L(x), \ U(x)\right)$$

be a normal region (Fig.)

Then from Prop. 1, we have

$$A = |R_{xy}| = \iint_{R_{xy}} dA$$

where

$$dA = dy\ dx \quad (= dx\ dy)$$

Integrating dx (independent of y) with respect to y from $L(x)$ to $U(x)$ one obtains the area of the vertical strip:

$$\int_{L(x)}^{U(x)} (dx)dy = \left[U(x) - L(x)\right]dx$$

This streep sweeps out the region when the last expression is integrated with respect to x from a to b:

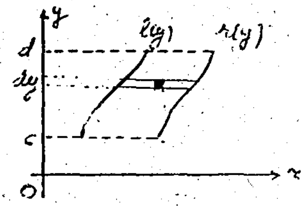
$$|R_{xy}| = \int_a^b \left[U(x) - L(x)\right]dx = \int_a^b \int_{L(x)}^{U(x)} dy\ dx \qquad (1)$$

When the region is the normal region

$$R_{yx} = (c,\ d;\ \ell(y),\ r(y))$$

We have similarly

$$|R_{yx}| = \int_c^d \int_{\ell(y)}^{r(y)} dx\ dy \qquad (2)$$



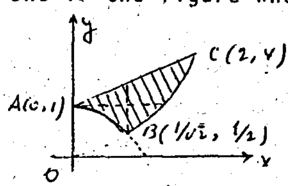
In the iterated integrals, for obtaining the limits of inner and outer integrals, drawing of vertical (or horizontal) strip is advisable.

If the region is not normal, it is to be partitioned into a finite number of normal regions by Prop. 5.

If R is unbounded or the integrand is discontinuous in a subset of R, then the double integral involves improper integral(s) which may or may not be convergent.

Example. Compute the area of the region in the I-quadrant bounded by $y = x^2$, $y = -x^2+1$ and $3x-2y+2 = 0$.

Solution. The region is the shaded one in the figure where the coordinates of A, B, C are obtained by simultaneous solutions of the equations taken two at a time.



R is not a normal region of either type. It can be split up into normal regions by drawing the horizontal line through A, or the vertical line through B.

By horizontal line through A:

$$R = R'_{yx} \cup R''_{yx}$$

$$= \left(\frac{1}{2}, 1;\ \sqrt{1-y}, \sqrt{y}\right) \cup \left(1, 4;\ \frac{2y-2}{3}, \sqrt{y}\right)$$

Then

$$|R| = |R'_{yx}| + |R''_{yx}|$$

$$= \int_{1/2}^{1} \int_{\sqrt{1-y}}^{\sqrt{y}} dx\, dy + \int_{1}^{4} \int_{\frac{2y-2}{3}}^{\sqrt{y}} dxdy$$

$$= \int_{1/2}^{1} (\sqrt{y}-\sqrt{y-1})dy + \int_{1}^{4} \left[\sqrt{y} - \frac{2}{3}(y-1)\right] dy$$

$$= \left(\frac{2}{3} y^{3/2} + \frac{2}{3}(1-y)^{3/2}\right)\Big|_{1/2}^{1} + \left(\frac{2}{3} y^{3/2} - \frac{2}{3}\left(\frac{y^2}{2} - y\right)\right)\Big|_{1}^{4}$$

$$= \frac{4-\sqrt{2}}{6} + \frac{10}{6} = \frac{7}{3} - \frac{\sqrt{2}}{6}$$

Compute the same area by splitting up the region into normal regions of type R_{xy}.

Area of polar normal regions:

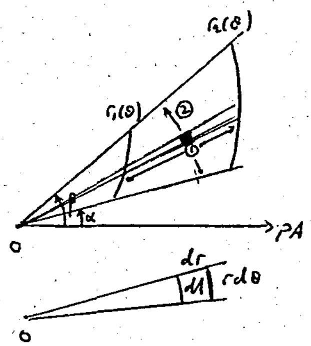
Let

$$R_{\theta r} = \left(\alpha, \beta;\ r_1(\theta), r_2(\theta)\right)$$

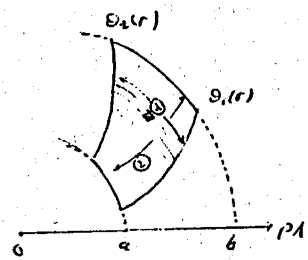
be a normal polar region. Having

$$A = |R_{\theta r}| = \iint_{R_{\theta r}} dA$$

where $dA = r\, dr\, d\theta$ (see Fig.) then



$$|R_{\theta r}| = \int_{\alpha}^{\beta} \int_{r_1(\theta)}^{r_2(\theta)} r\, dr\, d\theta,$$

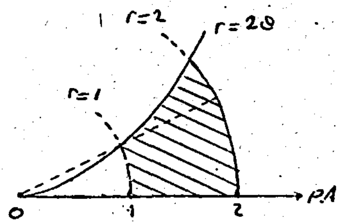


For the normal polar region

$$R_{r\theta} = \left[a,\ b;\ \ \theta_1(r),\ \ \theta_2(r)\right],$$

$$|R_{r\theta}| = \int_{a}^{b} \int_{\theta_1(r)}^{\theta_2(r)} r\, d\theta\, dr$$

Example. Compute the area of the region R bounded by the circles $r = 1$, $r = 2$, the polar axis and the spiral $r = 2\theta$.



Solution. The region R is of type $R_{r\theta}$. Then we have

$$R_{r\theta} = (1,\ 2;\ \ 0,\ r/2),$$

and

$$|R| = \int_1^2 \int_0^{r/2} r\, d\theta\, dr = \int_1^2 \frac{r^2}{2}\, dr = 7/6.$$

R is the union of two normal regions (separated by the line OA) of type $R_{\theta r}$:

$$R : \left[0,\ \frac{1}{2}\ ;\ \ 1,\ 2\right] \cup \left[\frac{1}{2}\ ,\ 1,\ \ 2\theta,\ 2\right]$$

$$|R| = \int_0^{1/2} \int_1^2 r\, dr\, d\theta + \int_{1/2}^{1} \int_{2\theta}^{2} r\, dr\, d\theta$$

$$= \int_0^{1/2} \frac{3}{2}\, d\theta + \int_{1/2}^{1} (2-2\theta^2)\, d\theta$$

$$= \frac{3}{4} + \frac{5}{12} = 7/6$$

VOLUME:

By the Prop. 2, we have the volume under the surface

$z = f(x, y)$ over R:

$$V = \iint_R f(x, y)dA \quad (f(x, y) > 0)$$

where

$dA = dy\,dx$ (or $dx\,dy$) in rectangular coordinates,

$dA = r\,dr\,d\theta$ (or $r\,d\theta\,dr$) in polar coordinates.

Then

$$V = \iint_R f(x, y)dy\,dx \quad \text{(in rectangular coord.)}$$

$$V = \iint_R f(r\cos\theta, r\sin\theta)r\,dr\,d\theta \quad \text{(in polar coord.)}$$

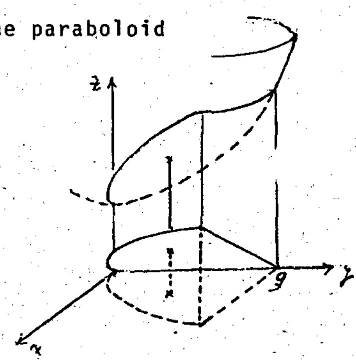
The above volume V is the volume of the solid bounded by the right cylinder where directrix is the boundary ∂R of R, the region R and the surface $z = f(x, y)$.

When the independent variables are y and z, say, and the region R is in yz-plane, then the volume is the volume of the solid bounded by the cylinder with directrix ∂R, the region R and the surface $x = f(y, z)$:

$$V = \iint_R f(y, z)dA$$

where $dA = dz\,dy$ (or $dy\,dz$).

Example 1. Compute the volume of the solid bounded by the vertical cylinder $y = x^2$ the planes $y = 9$, $x = 0$ and from below by the plane $x = z$ and above by the paraboloid $z = x^2 + y^2 + 4$. (In the I. octant)



Solution. The volume can be computed as the difference of two volumes under the surfaces $z = x^2+y^2+4$ and $z = x$ over the same region

$R_{yx} = [0, 9;\ 0, \sqrt{y}]$.

Then

$$V = \int_0^9 \int_0^{\sqrt{y}} \left[(x^2+y^2+4)-x\right]dx\, dy$$

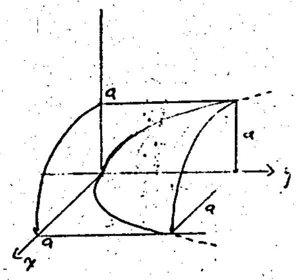
$$= \int_0^9 \left. \frac{x^3}{3} + y^2x + 4x - \frac{x^2}{2} \right|_0^{\sqrt{y}} dy$$

$$= \int_0^9 \left(\frac{1}{3}\, y^{3/2} + y^{5/2} + 4y^{1/2} - \frac{y}{2}\right)dy$$

$$= \left. \frac{1}{3} \cdot \frac{2}{5}\, y^{5/2} + \frac{2}{7}\, y^{7/2} + 4 \cdot \frac{2}{3}\, y^{3/2} - \frac{y^2}{4} \right|_0^9 = 709\ \frac{1}{140}$$

Example 2. Find the volume, in the I. octant, of the solid bounded by the cylinder $x^2 + y^2 = a^2$ and the paraboloid $y = x^2 + z^2$.

Solution. It is convenient to take the region on xz-plane as R_{xz}, say.



$$R_{xz} = \left[0,\ a;\ \ 0,\ \sqrt{a^2-x^2}\right]$$

$$V = \int_0^a \int_0^{\sqrt{a^2-x^2}} y\ dzdx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} (x^2+z^2)dz\ dx = \frac{\pi}{8}\, a^4.$$

E. Physical Applications

Mass (of a plate):

When in a double integral

$$I = \iint_R f(x, y)dA$$

the integrand $f(x, y)$ is taken as the density $\delta(x, y)$ per unit area of R, then $\delta(x, y)dA$ is the mass dm of the elementary area, and the integral will represent the mass of the plate in the shape of R:

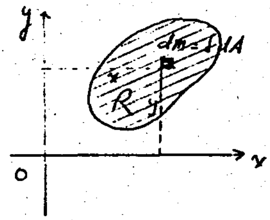
$$m = \iint_R \delta(x, y)\,dA$$

<u>Moments</u> (of a plate):

The moments of an element of area dA with density $\delta(x, y)$ with respect to x- and y-axes being

$$d\,M_{ox} = y\,dm = y\,\delta(x, y)\,dA$$

$$d\,M_{oy} = x\,dm = x\,\delta(x, y)\,dA$$



the corresponding moments of a plate with density function in the shape of R will be

$$M_{ox} = \iint_R y\,\delta\,dA$$

$$M_{oy} = \iint_R x\,\delta\,dA$$

<u>Center of mass</u> (center of gravity):

The center of mass of a plate with the density function $\delta(x, y)$ in the shape of R is that point $G(\bar{x}, \bar{y})$ charged with total mass m whose moments with respect to x- and y-axes are M_{ox} and M_{oy} respectively:

$$m\bar{y} = M_{ox}, \quad m\bar{x} = M_{oy}$$

$$\Rightarrow \quad G \left| \begin{array}{l} \bar{x} = \dfrac{M_{oy}}{m} = \dfrac{\iint_R x\,\delta\,dA}{\iint_R \delta\,dA} \\[2ex] \bar{y} = \dfrac{M_{ox}}{m} = \dfrac{\iint_R y\,\delta\,dA}{\iint_R \delta\,dA} \end{array} \right.$$

If δ is constant, the center of mass G is called the *centroid* of the region R and has coordinates

$$\overline{x} = \frac{\iint_R x\,dA}{|R|} \qquad \overline{y} = \frac{\iint_R y\,dA}{|R|}$$

Moments of inertia:

When the distances x and y in the formulas for moments are replaced by x^2 and y^2 respectively one obtains the *moments of inertia* of the plate:

$$I_{ox} = \iint_R y^2\,\delta\,dA$$

$$I_{oy} = \iint_R x^2\,\delta\,dA$$

The moment of inertia of the plate in the shape of R with density function $\delta(x, y)$ with respect to the origin is

$$I_o = \iint_R (x^2 + y^2)\delta\,dA = I_{ox} + I_{oy}$$

Example 1. Find the center of mass and average density of a semicircular plate of radius "a" if the density is proportional to the distance from the diameter.

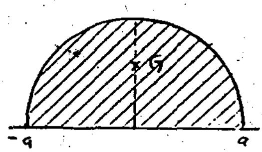
Solution. Taking the coordinate axes as shown in the Figure, the region R may be taken as

$$R_{xy} = \left[-a,\ a;\ \ 0,\ \sqrt{a^2-x^2}\right]$$

and the density will be

$\delta(x, y) = ky$ (k is the constant of proportionality)

Since the plate and the density function are symmetric with respect to y-axis, M_{oy} and hence $\overline{x}$ becomes zero. Then we need to compute the mass m and the moment M_{ox}:



$$m = \int_{-a}^{a} \int_{0}^{\sqrt{a^2-x^2}} ky.dy\ dx$$

$$= \frac{k}{2} \int_{-a}^{a} (a^2-x^2)dx = \frac{2k}{3} a^3,$$

$$M_{ox} = \int_{-a}^{a} \int_{0}^{\sqrt{a^2-x^2}} ky.y.dy\ dx$$

$$= \frac{k}{3} \int_{-a}^{a} (a^2-x^2)^{3/2} dx$$

$$(x = a \sin\theta, \quad dx = a \cos\theta\ d\theta)$$

$$= \frac{k}{3} \int_{-\pi/2}^{\pi/2} a^3 \cos^3\theta.a \cos\theta\ d\theta = \frac{k}{8} a^4$$

$$\bar{y} = \frac{M_{ox}}{m} = \frac{k}{8} \pi a^4/(\frac{2k}{3} a^3) = \frac{3}{16} \pi^a$$

$$G(0, \frac{3}{16} \pi^a)$$

$$\bar{\delta} = \frac{m}{|R|} = \frac{21}{3} a^3/(\frac{1}{2} \pi a^2) = \frac{4k}{3\pi} a.$$

Observe that $\bar{\delta}$ is between the extrema values 0 and ka of $\delta = ky$ $(0 \leq y \leq a)$.

Example 2. Find the moments of inertia of the plate given in Example 1 with respect to x-, y-axes, the origin 0, and the tangent line at $A(\pi/2, a\sqrt{3}/2)$.

Solution.

$$I_{ox} = \int_{-a}^{a} \int_{0}^{\sqrt{a^2-x^2}} y^2 . ky . dy\ dx = \frac{4k}{15} a^5$$

$$I_{oy} = \int_{-a}^{a} \int_{0}^{\sqrt{a^2-x^2}} x^2 . ky . dy\ dx = \frac{2k}{15} a^5$$

$$I_o = I_{ox} + I_{oy} = \frac{2k}{5} ka^5.$$

Since

$$\ell: \quad x + \sqrt{3}y - 2a = 0$$

the square of the distance of (x, y) from ℓ is

$$d^2 = (x + \sqrt{3}y - 2a)^2/4.$$

$$I_\ell = \int_{-a}^{a} \int_{0}^{\sqrt{a^2-x^2}} \frac{1}{4}\,(x+\sqrt{3}y-2a)^2 \,.\, ky \,.\, dy\, dx$$

$$= \frac{1}{4} \int_{-a}^{a} \int_{0}^{\sqrt{a^2-x^2}} (x^2+2\sqrt{3}\,xy+3y^2-4ax-4a\sqrt{3}y+4a^2)ky\, dy\, dx$$

$$= \frac{1}{4}\left(I_{oy} + 0 + 3I_{ox} - 4a\underbrace{M_{oy}}_{0} - 4a\sqrt{3}\, M_{ox} + 4a^2 m\right)$$

$$= \frac{k}{4}\left(\frac{2}{15}a^5 + 0 + 3 \,.\, \frac{4a^5}{15} - 0 - 4a\sqrt{3} \,.\, \frac{\pi}{8}a^4 + 4a^2 \,.\, \frac{2}{3}a^3\right)$$

$$= \frac{k}{4}a^5\left(\frac{2}{15} + \frac{12}{15} - \frac{\sqrt{3}}{2}\pi + \frac{5}{3}\right)$$

$$= \frac{k}{40}(36 - 5\sqrt{3}\,\pi)a^5.$$

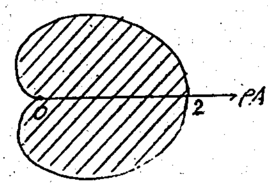
If R is a polar region, by usual transformations one can obtain m, M_{PA}, M_{CPA}, I_{PA}, I_{CPA}, I_0, and the coordinates $\overline{\theta}$, $\overline{r}$ of G can be computed by

$$\overline{\theta} = \arctan\frac{\overline{y}}{\overline{x}}, \qquad \overline{r} = \sqrt{\overline{x}^2 + \overline{y}^2}.$$

<u>Example</u>. Find the centroid, and moment of inertia I_0 of a homogeneous plate in the shape of the cardioid $r = a(1 + \cos\theta)$.

<u>Solution</u>.

$$m = \int_{-\pi}^{\pi} \int_{0}^{a(1+\cos\theta)} \delta \,.\, r\, dr\, d\theta$$



$$= \delta \int_{\pi}^{\pi} \frac{r^2}{2}\Bigg|_{0}^{a(1+\cos\theta)} d\theta$$

(since δ is const)

$$= \frac{1}{2}\,\delta\, a^2 \int_{-\pi}^{\pi} (1 + 2\cos\theta + \cos^2\theta)\,d\theta = \frac{3}{2}\,\delta\,\pi a^2.$$

$$I_{PA} = \int_{-\pi}^{\pi}\int_{0}^{a(1+\cos\theta)} r\sin\theta \,.\, \delta \,.\, r\,dr\,d\theta$$

$$= \delta \int_{-\pi}^{\pi} \sin\theta \,.\, \frac{r^3}{3}\Bigg|_{0}^{a(1+\cos\theta)} d\theta$$

$$= \frac{1}{3}\,\delta\, a^3 \int_{-\pi}^{\pi} -(1 + 3\cos\theta + 3\cos^2\theta + \cos^3\theta)\,d\cos\theta$$

$$= \frac{4\delta}{3}\, a^3$$

$$\bar{x} = \frac{I_{PA}}{m} = \frac{8}{9\pi}\,a \qquad\qquad \bar{\theta} = 0$$

$$\rightarrow \quad G \left| \begin{array}{l} \bar{\theta} = 0 \\ \bar{r} = \frac{8}{9\pi}\,a \end{array}\right.$$

$$\bar{y} = 0$$

$$I_o = \int_{-\pi}^{\pi}\int_{0}^{a(1+\cos\theta)} r^2 \,.\, \delta \,.\, r\,dr\,d\theta =$$

$$= \frac{\delta}{4}\, a^4 \int_{-\pi}^{\pi} (1 + 4\cos\theta + 6\cos^2\theta + 4\cos^3\theta + \cos^4\theta)\,d\theta$$

$$= \frac{\delta}{4}\, a^4 \int_{-\pi}^{\pi} (1 + 6\cos^2\theta + \cos^4\theta)\,d\theta = \frac{35}{16}\,\delta\pi a^4.$$

EXERCISES (5, 2)

26. Describe the region of integration, and evaluate:

a) $\int_1^2 \int_2^5 xy\,dxdy$ b) $\int_1^2 \int_2^5 xy\,dydx$

c) $\int_{-1}^2 \int_x^{x+2} dy\,dx$ d) $\int_{-1}^2 \int_{y+2}^{y} dx\,dy$

27. Evaluate $\iint_R \sqrt{9 - x^2 - y^2}\,dA$
where R is the region bounded by the circle $x^2 + y^2 = 9$ by the use of properties.

28. Sketch the region of integration, and evaluate:

a) $\int_{-1}^{2}\int_{x^2}^{x+2} dy\, dx$ b) $\int_{0}^{\pi}\int_{0}^{1-\cos\theta} r\, dr\, d\theta$

29. Same question for:

a) $\int_{0}^{a}\int_{a-x}^{\sqrt{a^2-x^2}} y\, dy\, dx$ b) $\int_{0}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\, dx$

30. Sketch the region of integration and compute

$$\int_{0}^{\pi/2}\int_{0}^{3\sec(\theta - \frac{\pi}{6})} r\, dr\, d\theta.$$

31. Without evaluating, find the largest and smallest possible value of

$$\iint_R \sqrt{1 + x^2 + y^2}\ dA$$

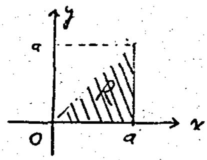
where R is the region bounded by the curves $y = 3x - x^2$ and $y = x^2 - 3x$ (Property 6).

32. Same question for

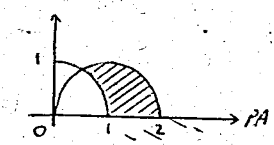
a) $\int_{-3}^{2}\int_{0}^{x+3} xy\, dy\, dx$ b) $\int_{-2}^{3}\int_{-2}^{x+2} (x^2+y^2)dy\, dx$

33. Determine $a > 0$ such that

$$\iint_R (x^2+y^2)dydx = \iint_R (x^2+y^2)dxdy$$



34. Evaluate $\iint_{R_{\theta r}} xy\ dA$ where $R_{\theta r}$ is the polar region bounded by two circles shown in the Fig.



35. Evaluate $\iint_R \frac{dA}{(x + y)^3}$ where $R = \{(x, y):\ x \geqslant 1,\ y \geqslant 1,\ x+y < 3\}$.

36. Given

$$\int_0^2 \int_0^{x^2/2} \frac{y}{\sqrt{1+x^2+y^2}}\, dy\, dx$$

a) Sketch the region of integration

b) evaluate the integral

37. Same question for

$$\int_0^{\arccos \frac{1}{5}} \int_1^{5\cos\theta} r^2\, dr\, d\theta$$

38. Without evaluating, show that the following integrals are zero:

a) $\displaystyle\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \frac{1}{x^2 y + y^3}\, dy\, dx$

b) $\displaystyle\int_0^{b} \int_{-\frac{a}{b}\sqrt{b^2-x^2}}^{\frac{a}{b}\sqrt{b^2-x^2}} \mathrm{Sin}\, xy\, dx\, dy$

39. Find the area of the region defined by the following inequalities

a) $r \leqslant 5 \sin\theta, \quad r \geqslant 2 + \sin\theta$

b) $r^2 \leqslant 2 \cos 2\theta, \quad r \geqslant 1$

40. Same question for

a) $y = x, \quad y = x^2$ b) $x + y = 2, \quad x^2 = 4 - 2y$

c) $y = 4 - x^2, \quad y = 4 - 2x$ d) $y = \sqrt{a^2-x^2}, \quad y = a - x$

41. Same question for:

a) $r^2 \leqslant 2a^2\cos 2\theta, \; r \geqslant a$ b) $r \leqslant 3a\cos\theta, \quad r \geqslant a\,(1+\cos\theta)$

42. Find the volume of the space region bounded by

a) cylinders $y = x^2, \quad x = y^2$ and the planes $z = 0, \quad z = 1$.

b) cylinders $x^2 + y^2 = a^2, \quad y^2 + z^2 = a^2$.

43. Find the volume under $z = x^2 + y^2$ above the region

$$R = \{0,\ 1;\quad 0,\ 1-x\}.$$

44. A cylindrical hole of radius b is drilled through a sphere of radius a $(b < a)$. What is the volume drilled out if the axis of the hole coincides with a diameter of the sphere.

45. Find the volume of the solid bounded by the surfaces $z = e^{-x^2-y^2}$, $x^2 + y^2 = 4$ and $z = 0$.

46. Find the centroid of the region enclosed by $y = 4-x^2$, $y=4-2x$.

47. Find $G(\overline{x}, \overline{y})$ of the plane region enclosed by $r = 2 \sin\theta$ and $r = 4 \sin\theta$.

48. Find the center of mass of the region R bounded by the curves $y = \sqrt{x}$, $y = x^3$ if the density function is $\delta = 3x$, and find the average density.

49. Given the transformation

$$x = u + v, \qquad y = u - v$$

a) find $\dfrac{\partial(x, y)}{\partial(u, v)}$ and $\dfrac{\partial(u, v)}{\partial(x, y)}$

b) find the image of the triangular region with vertices $(0, 0)$, $(4, 0)$, $(0, 3)$

50. Given the transformation

$$u = x^2 - y^2, \qquad v = 2xy$$

a) Sketch the curves $u = 4$, and $v = -2$ in xy-system,

b) find $\dfrac{\partial(x, y)}{\partial(u, v)}$ and $\dfrac{\partial(u, v)}{\partial(x, y)}$

c) find the image of the region defined by $x^2 + y^2 \leq 4$

d) find the image of the region defined by $y \leq 2x$

51. Reverse the order of integration of the double integrals given in Exercises 26 and 28.

52. Same question for those in Exercises 30 and 32.

53. Find the moments of intertia of the plate in the shape of the smaller regions enclosed by $y = a - x$ and $x^2 + y^2 = a^2$ about

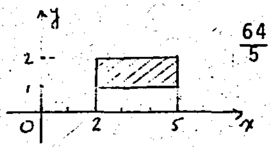
a) x-axis b) y-axis

54. Find I_{ox} of the plate in the shape of the region bounded by $x^2 + y^2 = a^2$ with density $\delta = \sqrt{x^2 + y^2}$.

55. Find I_o of the homogeneous triangular plate bounded by $y = 0$, $y = x$ and $x = 4$.

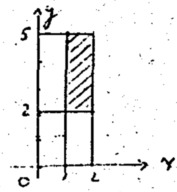
ANSWERS TO EVEN NUMBERED EXERCISES

26. a)



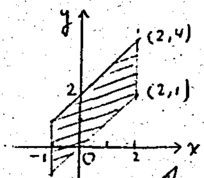
$\frac{64}{5}$

b)



$\frac{63}{4}$

c)



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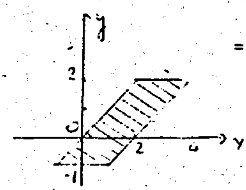
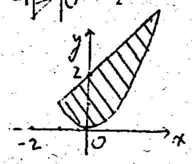
d)



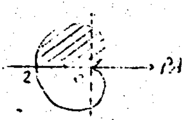
= 6

28. a)



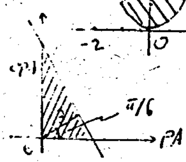
$\frac{9}{2}$

b)



$\frac{3\pi}{4}$

30.

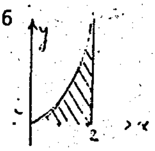


$6\sqrt{3}$

32. a) 125, -225/8 b) 765, 0

34. 9/16

36.



$\frac{1}{2}\left[\ln(2 + \sqrt{5}) + 2\sqrt{5}\right]$

38. a) the integrand is odd in y and the region is symmetric

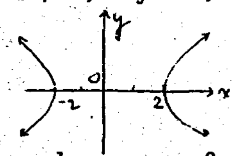
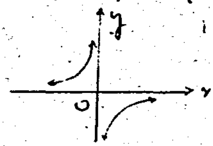
40. a) 1/6, b) 2/3, c) 4/3, d) $(\pi-2)a^2/4$.

42. a) 1/3, b) $2a^3/3$

44. $\frac{4}{3}\pi\left[a^3-(a^2-b^2)^{3/2}\right]$

46. $\overline{x}=1$, $\overline{y}=12/5$

48. $\overline{x}=25/42$, $\overline{y}=25/45$, $\overline{\delta}=36/25 \in (0,\ 3)$

50. a)  

b) $\dfrac{1}{4\sqrt{u^2+v^2}}$, $4(x^2+y^2)$, c) $u^2+v^2\leqslant 16$, d) $v\geqslant -\frac{4}{3}u$.

52. 30) $\displaystyle\int_{3}^{2\sqrt{3}}\int_{0}^{\frac{\pi}{6}+\operatorname{arcsec}\frac{r}{3}} r\,d\theta\,dr+\int_0^3\int_0^{\pi/2} r\,d\theta\,dr$

$\displaystyle+\int_3^6\int_{\frac{\pi}{6}+\operatorname{arcsec}\frac{r}{3}}^{\pi/2} r\,d\theta\,dr$

32a) $\displaystyle\int_0^5\int_{y-3}^{2} xy\,dx\,dy$ 32b) $\displaystyle\int_{-2}^{0}\int_{-2}^{3}(x^2+y^2)dx\,dy$

$\displaystyle+\int_0^5\int_{y-2}^{3}(x^2+y^2)dx\,dy$

54. $\frac{k\pi}{5}a^5$

5.3 SURFACE INTEGRAL

A. DEFINITIONS

Let

$$F = Pi + Qj + Rk$$

be a vector field defined over $D \subseteq \mathbb{R}^3$, and let S be a surface in D. Then the integral

$$\iint_S P\,\overline{dydz} + Q\,\overline{dzdx} + R\,\overline{dxdy} \qquad (1)$$

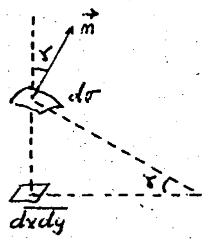
is called a surface integral if (x, y, z) in P, Q, R varies throughout S, where $\overline{dydz}$ for instance denotes free order of integration ($dydz$ or $dzdy$).

Let $d\sigma$ be an elementary area of the surface S. The relations between $d\sigma$ and $\overline{dydz}$, $\overline{dzdx}$, $\overline{dxdy}$ can be obtained by projecting $d\sigma$ onto corresponding coordinate planes.

If

$$\vec{n} = (\cos\alpha, \cos\beta, \cos\gamma)$$

is a unit normal vector to S at P, then α, β, γ will be the angles of the tangent plane to S at P with coordinate planes, and one has



$$\overline{dydz} = d\sigma\cos\alpha, \quad \overline{dzdx} = d\sigma\cos\beta, \quad \overline{dxdy} = d\sigma\cos\gamma$$

so that (1) assumes the forms

$$\iint_S (P\cos\alpha + Q\cos\beta + R\cos\gamma)\,d\sigma \qquad (1')$$

and

$$\iint \vec{F}\cdot\vec{n}\,d\sigma = \iint_S \vec{F}_n\,d\sigma \qquad (1'')$$

B. Evaluation

When S has the equation $\phi(x, y, z) = 0$ defining $x = x(x, z)$, $y = y(z, x)$, $z = z(x, y)$, the surface integral (1) is the sum in

$$\iint_{S_{yz}} P(x(y, z), y, z)\overline{dydz} \qquad (2)$$

$$+ \iint_{S_{zx}} P(x, y(z, x), z)\overline{dzdx}$$

$$+ \iint_{S_{xy}} P(x, y, z(x, y)\overline{dxdy}$$

of three double integrals, where S_{yz} for instance is the projection of S onto yz-plane.

When $\phi(x, y, z) = 0$ defines z, for instance, as a function of x, y not uniquely, say z_1 and z_2, one evaluates surface integral for both surfaces (lower and upper surface).

When the equation of S is given parametrically as

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v),$$

then by the usual transformations (change of variables) from yz-, zx-, xy-planes to uv-plane, (1) becomes

$$\iint_{S_1} P(x(u, v), y(u, v), z(u, v))\left|\frac{\partial(y, z)}{\partial(u, v)}\right| \overline{dudv}$$

$$+ \iint_{S_2} Q(x(u, v), y(u, v), z(u, v))\left|\frac{\partial(z, x)}{\partial(u, v)}\right| \overline{dudv}$$

$$+ \iint_{S_3} R(x(u, v), y(u, v), z(u, v))\left|\frac{\partial(x, y)}{\partial(u, v)}\right| \overline{dudv}$$

where S_1, S_2, S_3 are the images of S_{yz}, S_{zx}, S_{xy} under the transformation.

When S is given by (1') (or by 1"), $d\sigma$ can be projected to one of the coordinate planes in which the transformed double integral is simpler. If S is projected onto xy-plane, say, (1') becomes

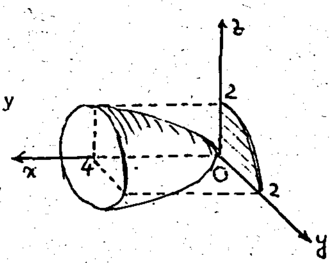
$$\iint_{S_{xy}} (P\cos\alpha + Q\cos\beta + R\cos\gamma)\sec\gamma \ \overline{dxdy}$$

Example 1. Evaluate

$$I = \iint_S (x - z)\ \overline{dydz}$$

where S is the surface $x = y^2 + z^2$, $0 \leq x \leq 4$.

Solution. R_{yz}: $y^2 + z^2 \leq 4$



$$I = \int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (y^2 + z^2 - z)dzdy$$

$$= \int_{-2}^{2} y^2 z + \frac{z^3}{3} - \frac{z^2}{2} \Bigg|_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dy$$

$$= 2 \int_{-2}^{2} y^2\sqrt{4-y^2} + \frac{1}{3}(4-y^2)\sqrt{4-y^2}\ dy$$

$$= 2 \int_{-2}^{2} \left(\frac{2}{3} y^2\sqrt{4-y^2} + \frac{4}{3}\sqrt{4-y^2}\right) dy$$

$$= \frac{8}{3}\pi + \frac{11}{3}\pi = 8\pi.$$

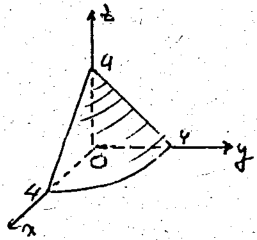
Example 2. Evaluate

$$I = \iint_S x^2 z\ \overline{dydz} + y^2 x\ \overline{dzdx} + xy\ \overline{dxdy}$$

where S is the cone $x^2 + y^2 = (z - 4)^2$ with $0 \leq z \leq 4$.

Solution.

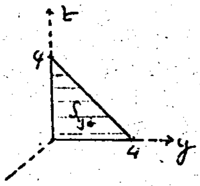
$$I_1 = \iint_{S_{yz}} x^2 z\, \overline{dydz} = \int_0^4 \int_0^{4-y} x^2 z\, dzdy$$



$$= \int_0^4 \int_0^{4-y} \left[(z-4)^2 - y^2\right] z\, dzdy$$

$$= \int_0^4 \int_0^{4-y} (z^3 - 8z^2 + 16z - y^2 z)\, dzdy$$

$$= \int_0^4 \underbrace{\frac{z^4}{4} - \frac{8}{3} z^3 + \frac{16-y^2}{z} z^2 \Big|_0^{4-y}}_{A} dy$$

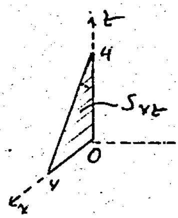


$$A = \left[(4-y)^2 \frac{(4-y)^2}{4} - \frac{8}{3}(4-y) + \frac{16-y^2}{2}\right]$$

$$= (y-4)^2 \frac{-3y^2 + 8y + 16}{12}$$

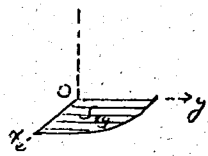
$$= -\frac{1}{2}(3y^4 - 32y^3 + 96y^2 - 256)$$

$$I_1 = -\frac{1}{12}\left(\frac{3}{5} y^5 - 8y^4 + 32y^3 - 256y\right)_0^4 = \frac{512}{15}$$



$$I_2 = \iint_{S_{xz}} y^2 x\, \overline{dzdx} = \int_0^4 \int_0^{4-z} y^2 x\, dxdz = I_1$$

$$I_3 = \iint_{S_{xy}} z^2 y\, \overline{dxdy} = \int_0^4 \int_0^{\sqrt{16-x^2}} z^2 y\, dydx$$



$$= \int_0^4 \int_0^{\sqrt{16-x^2}} (4 + \sqrt{x^2 + y^2})\, dydx$$

$$= 4 \int_0^4 \int_0^{\sqrt{16-x^2}} dydx + \int_0^{\pi/2} \int_0^4 r.r\, drd\Theta$$

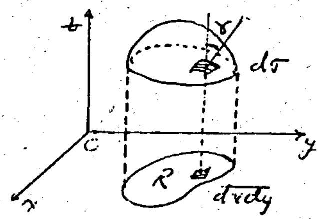
$$= 16\,\pi + \frac{32}{3}\,\pi = \frac{80}{3}\,\pi$$

$$I = \frac{2.512}{15} + \frac{80}{3}\,\pi.$$

C. Geometric and Physical Applications

GEOMETRIC APPLICATION (Area of a surface):

Let $S: F(x, y, z) = 0$ be a surface over the region R in xy-plane (i.e. R is the projection of S on xy-planes). Then the area of S is



$$|S| = \iint_S d\sigma$$

and we have

$$\iint_S d\sigma = \iint_R \sec\gamma \,\overline{dxdy}$$

where $d\sigma$ is the elementary area of S and γ is the acute angle between the normal line to S at a point in $d\sigma$ and z-axis.

Then

$$\cos\gamma = |\vec{n}.k|$$

$$\Longrightarrow \quad \sec\gamma = \frac{1}{|\vec{n}.\vec{k}|} = \begin{cases} \sqrt{1 + z_x^2 + z_y^2} & \text{if } z = f(x, y) \\ \dfrac{\sqrt{F_x^2 + F_y^2 + F_z^2}}{|F_z|} & \text{if } F(x, y, z) = 0 \end{cases}$$

and

$$|S| = \iint_R \sqrt{1 + z_x^2 + z_y^2} \;\overline{dxdy} \quad \left(\text{or } \iint_R \frac{\sqrt{F_x^2 + F_y^2 + F_z^2}}{|F_z|} \;\overline{dxdy}\right.$$

Similarly

$$|S| = \iint_R \sqrt{1 + x_x^2 + x_z^2} \;\overline{dxdz} \quad \left(\text{or } \iint_R \frac{\sqrt{F_x^2 + F_y^2 + F_z^2}}{|F_x|} \;\overline{dydz}\right.$$

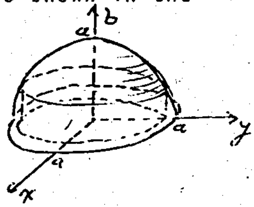
$$|S| = \iint_R \sqrt{1 + y_x^2 + y_z^2} \;\overline{dxdz} \quad \left(\text{or } \iint_R \frac{\sqrt{F_x^2 + F_y^2 + F_z^2}}{|F_y|} \;\overline{dxdz}\right.$$

To obtain a simpler region R or a simpler integrand, one projects S onto a convenient coordinate plane.

Note. If $S: F(x, y, z) = 0$ defines two functions, two parts of S must be considered separately.

Example 1. Find the area of the smallest part of a sphere of radius a, cut off by a plane π having distance $a/3$ from the center.

Solution. Choosing the coordinate system as shown in the Fig., we have



$$S: x^2 + y^2 + z^2 = a^2, \qquad \pi: z = a/3$$

$$R: x^2 + y^2 \leq 8a^2/9$$

$$|S| = \iint_R \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}}\, \overline{dxdy}$$

$$= \iint_R \frac{a}{z}\, \overline{dxdy} = a \iint_R \frac{1}{\sqrt{a^2 - x^2 - y^2}}\, \overline{dxdy}$$

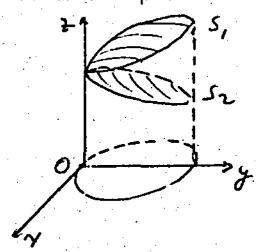
$$= 4a \int_0^{\pi/2} \int_0^{\frac{2\sqrt{2}}{3}a} \frac{1}{\sqrt{a^2 - r^2}}\, r\, dr d\Theta$$

$$= -4a \int_0^{\pi/2} \sqrt{a^2 - r^2}\,\Big|_0^{\frac{2\sqrt{2}}{3}a}\, d\Theta$$

$$= -4a \int_0^{\pi/2} \left(\frac{a}{3} - a\right) d\Theta = -\frac{8}{3}\,\frac{\pi}{2} = \frac{4\pi}{3} a^2$$

Example 2. Set up the integral for the surface area of the paraboloid $y = x^2 + (z - 4)^2$ cut off by the cylinder $x^2 + (y-2)^2 = 4$.

Solution. The vertical cylinder separates from the paraboloid two bounded surfaces S_1, S_2 with equal areas, since they are symmetrical with respect to the plane $z = 4$, and have the same projection



$$R: \quad x^2 + (y - 2)^2 \leq 4$$

on xy-plane. Then

$$|S| = 2|S_1| = 2\iint_R \sqrt{1 + z_x^2 + z_y^2}\ \ dxdy$$

$$y = x^2 + (z-4)^2 \Longrightarrow z_x = -\frac{x}{z-4}\ ,\quad z_y = \frac{1}{2(z-4)}$$

$$\Longrightarrow 1 + z_x^2 + z_y^2 = \frac{1+4y}{4(z-4)^2} = \frac{1+4y}{4(y-x^2)}$$

$$\Longrightarrow |S| = 4\int_0^4 \int_0^{4y-y^2} \sqrt{\frac{1+4y}{4(y-x^2)}}\ dxdy$$

Physical Applications

1. Mass, moments, center of mass, moments of inertia of shells

By the usual notations, we have

$$m = \iint_S \delta\, d\sigma \quad (\delta(x\ \ y,\ z) \text{ is the mass per unit area})$$

$$M_{yz} = \iint_S x\ \delta\, d\sigma,\quad M_{2x} = \iint_S y\ \delta\ d\sigma,\quad M_{xy} = \iint_S z\ \delta\ d\sigma$$

$$G(M_{yz}/m,\quad M_{zx}/m,\quad M_{xy}/m)$$

$$I_{yz} = \iint_S x^2\,\delta\, d\sigma,\quad I_{zx} = \iint_S y^2\ \delta\ d\sigma,\quad I_{xy} = \iint z^2\,\delta\ \ d\sigma$$

In general, for any plane π, line ℓ and point A we have

$$M_\pi = \iint_S d(P,\ \pi)\,\delta\, d\sigma,\qquad I_\pi = \iint_S d^2(P,\ \pi)\,\delta\, d\sigma$$

$$M_\ell = \iint_S d(P,\ \ell)\,\delta\, d\sigma,\qquad I_\ell = \iint_S d^2(P,\ \ell)\,\delta\, d\sigma$$

$$M_A = \iint_S d(P,\ A)\,\delta\, d\sigma,\qquad I_A = \iint_S d^2(P,\ A)\,\delta\, d\sigma$$

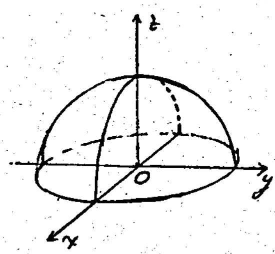
where P is the variable point on the shall.

$\rho_\ell^2 = I_\ell/m$ (ρ_ℓ is the radius of gyration)

Example. Given a semi-spherical shell of radius a with density proportional to the square of the distance from the axis of symmetry, find

a) G, b) ρ_ℓ

Solution. Placing the shell as shown in the Figure, we have



a) $m = \iint_S k(x^2 + y^2)d\sigma$

$= k\iint_R (x^2+y^2)\sqrt{1+z_x^2+z_y^2}\;\overline{dxdy}$

$= k\iint_R (x^2+y^2)\frac{a}{z}\;\overline{dxdy}$

$= ka\int_0^{2\pi}\int_0^a \frac{r^2}{\sqrt{a^2-r^2}}\, r\,drd\Theta$

$= \frac{4}{3}\pi ka^4$

From symmetry of the shell with respect to z-axis, and of δ in the variables x, y, the center of mass G lies on z-axis: $\bar{x} = 0$, $\bar{y} = 0$.

To find $\bar{z}$, we evaluate M_{xy}:

$$M_{xy} = \iint z\, k(x^2+y^2)\,.\,\frac{a}{z}\;\overline{dxdy}$$

$$= ka\int_0^{2\pi}\int_0^a r^2\,.\,r\,drd\Theta = \frac{1}{2}k\pi a^5$$

$$\Rightarrow \bar{z} = M_{xy}/m = \frac{3}{8}a$$

$$\Rightarrow G(0,\ 0,\ 3a/8)$$

$$b)\ I_{oz} = \iint_S (x^2+y^2)\cdot k(x^2+y^2)\,d\sigma$$

$$= k\iint_R (x^2+y^2)\,\frac{a}{z}\,\overline{dxdy}$$

$$= ka\int_0^{2\pi}\int_0^{a}\frac{r^4}{\sqrt{a^2-r^2}}\,r\,drd\Theta$$

$$= \frac{16}{15}k\pi a^5$$

$$\varsigma_{oz}^2 = I_{oz}/m = \frac{16}{15}k\pi a^5/(\tfrac{4}{3}\pi ka^4) = \frac{4}{5}a^2$$

$$\varsigma_{oz} = \frac{2}{\sqrt{5}}a$$

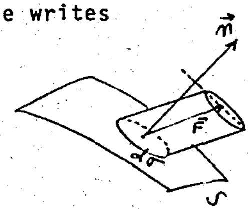
2. Fluid flow through a surface

Let

$$F = Pi + Qj + Rk$$

be the velocity vector field of a fluid flowing through a surface S. If one likes to determine the volume (or the mass) of the find passing through S in a unit time, one writes

$$q = \iint_S \vec{F}\cdot\vec{n}\,d\sigma \quad (\text{or}\quad q = \iint \vec{F}\cdot\vec{n}\,\delta\,d\sigma)$$



considering

$$dq = \vec{F}\cdot\vec{n}\,d\sigma \qquad (\text{or}\quad dq = \vec{F}\cdot\vec{n}\,\delta\,d\sigma)$$

as the volume (or as the mass) of the elementary cylindrical fluid with base $d\sigma$ and height $\vec{F}\cdot\vec{n}$.

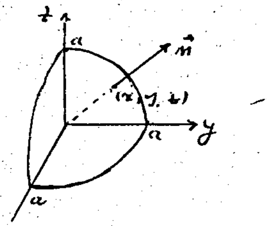
Example. Let a fluid be in motion with the velocity

$$F = 2x\,i + (y+z)j + zk$$

where distances are in meters and magnitude of the velocity is in m/sec. Find the volume of the fluid per second following through the first octant portion of the sphere $x^2+y^2+z^2 = a^2$.

Solution.

$$q = \iint_S \vec{F}\cdot\vec{n}\; d\sigma$$



$$\vec{n} = \frac{(x,\ y,\ z)}{\sqrt{x^2+y^2+z^2}} = \left(\frac{x}{a},\ \frac{y}{a},\ \frac{z}{a}\right)$$

$$\vec{F}\cdot\vec{n} = \frac{2x^2}{a} + \frac{(y+z)y}{a} + \frac{z^2}{a}$$

$$q = \frac{1}{a}\iint_{S_{xy}} (2x^2+y^2+z^2+yz)\,\frac{a}{z}\;\overline{dxdy}$$

$$= \iint \left(\frac{a^2+x^2}{z} + y\right)\overline{dxdy}$$

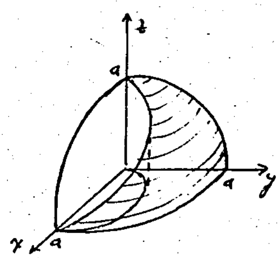
$$= \iint \frac{a^2+x^2}{\sqrt{a^2-x^2-y^2}}\,\overline{dxdy} + \iint y\;\overline{dxdy}$$

$$= \int_0^{\pi/2}\int_0^{a} \frac{a^2+r^2\cos^2\Theta}{\sqrt{a^2-r^2}}\, r\,drd\Theta + \int_0^{a}\int_0^{\sqrt{a^2-x^2}} y\,dydx$$

$$= \left[a^2\int_0^{\pi/2}\int_0^{a}\frac{r}{\sqrt{a^2-r^2}}\,drd\Theta + \iint\frac{r^2\cos^2\Theta}{\sqrt{a^2-r^2}}\,r\,drd\Theta\right] + \frac{a^3}{3}$$

$$= \frac{\pi}{2}a^3 - \frac{\pi}{6}a^3 + \frac{a^3}{3} = \frac{1}{3}(\pi+1)a^3\ m^3/sec$$

EXERCISES (5. 3)



56. Evaluate the area of the shaded portion of the sphere of radius a cut off by a cylinder (VIVIANI's window).

57. Find the area of the surface $z = \frac{2}{3}(x^{3/2} + y^{3/2})$ over the square region $0 \leqslant x \leqslant 3$, $0 \leqslant y \leqslant 3$.

58. Find the area of the portion of the cylinder $y^2 + z^2 = 4a^2$ that lies in the first octant and above the triangle bounded by the lines $x = 0$, $y = 0$, $x + y = a$ in xy-plane.

59. Find the area of the first octant portion of the surface $x = yz$ that lies inside the cylinder $y^2 + z^2 = 1$.

60. Find the area of the portion of the paraboloid $x^2 + y^2 = 2z$ that lies in the first octant and cut off by the parabolic cylinder $x^2 = z$ and the planes $x = 0$, $z = 4$.

61. Find the area of the portion of the cone $y^2 + z^2 = x^2$ that lies above the square bounded by $z = 0$, $y = 0$, $z = 1$, $y = 1$ in yz-plane.

62. For the solid bounded by xy-plane, the cylinder $x^2 + y^2 = a^2$ and the paraboloid $z = b(x^2 + y^2)$ with $b > 0$, find the area of the upper surface.

63. Find the area and centroid of the plate in the shape of first octant portion of the surface $x^2 + y^2 = z^2$ that lies between the planes $z = 0$ and $z = 1$.

64. Find the area and centroid of the plate in the shape of the first octant portion of the surface of the paraboloid $z = x^2 + y^2$ that lies between the xy-plane and the plane $z=2$.

65. Find the coordinates of the centroid of the homogeneous plate in the shape of the first octant portion of the surface $x^2 + y^2 + z^2 = a^2$.

66. Find the second moment, the radius of gyration with respect to z-axis of the plate in the shape of the first octant portion of homogeneous surface $x^2 + y^2 + z^2 = 1$.

67. If $F = yz\,i + zx\,j + xy\,k$, show that

$$\iint_S F \cdot n \, d\sigma = 0$$

for a closed surface S.

68. If $F = xi + yj + zk$, show that

$$\iint_S F \cdot n \, d\sigma = 3V$$

where V is the volume of the region enclosed by the closed surface.

69. Evaluate $\iint_S F \cdot n \, d\sigma$ where $\vec{F} = xi + yj + zk$ and S is the entire surface of the cube bounded by the coordinate planes and the planes $x = a$, $y = a$, $z = a$.

70. Same question if S is the sphere $x^2 + y^2 + z^2 = a^2$.

ANSWERS TO EVEN NUMBERED EXERCISES

56. a^2

58. $2a^2(\frac{\pi}{6} + \sqrt{3} - 2)$

60. $\frac{13}{6}\pi$

62. $\frac{\pi}{6}\left[(1 + 4a^2b^2)^{3/2} - 1\right]/b^2$

64. $13\pi/12$, $\bar{x} = \bar{y} = \frac{1}{208}\left[306\sqrt{2} - 3\ln(3 + 2\sqrt{2})\right]$, $\bar{z} = 149/130$.

66. $\pi/3$, $\sqrt{2/3}$

70. $4\pi a^3$.

5. 4 Triple Integrals

A. Definitions and Properties

Let R be a solid region, and $f(x, y, z)$ be a function defined on R Partioning R into elementary prisms $(dx \times dy \times dz)$ with faces parallel to coordinate planes, and considering the triple sum $\sum$ (as double sum in double integrals) one may show that $\lim \sum$ exists for continuous f and the limit is denoted by

$$\iiint_R f(x, y, z)dV$$

Properties. Let f, g be two continuous functions of there variables. Then

1. $\iiint_R dV = |R|$ V (Volume)
2. $\iiint_R (f+g)dV = \iiint_R f\ dV + \iiint_R g\ dV$
3. $\iiint_R c f\ dV = c \iiint_R f\ dV$ (c is a constant)
4. $\iiint_{R_1} f\ dV + \iint_{R_2} f\ dV = \iint_{R_1 \cup R_2} f\ dV$ (if $R_1 \cap R_2 = \phi$)
5. $m|R| < \iiint_R f\ dV < M|R|$
6. $\left|\iiint_R f\ dV\right| < \iiint_R |f| dV$
7. $f(x, y, z) \leqslant g(x, y, z) \Longrightarrow \iiint_R f\ dV \leqslant \iiint_R g\ dV$

The proofs are similar to ones given in double integral.

Example. Evaluate

$$I = \iiint_R dV$$

where R is the solid bounded by the coordinates planes and the plane $x/2 + y/4 + z/4 = 1$.

Solution. From Prop.1, we have

$$I = |R| = \frac{1}{3}\left(\frac{1}{2} \cdot 2 \cdot 4\right) \cdot 3 = 4.$$

B. Evaluation by Iterated Integrals

Let

$$I = \iiint_R f(x, y, z)dV,$$

and V be a normal solid region.

The six normal regions in $\mathbb{R}^3$ are denoted by

$$R_{xyz},\ R_{yxz};\quad R_{yzx},\ R_{zyx};\quad R_{zxy},\ P_{xzy}$$

obtained by permuting the letters x, y, z, in R_{xyz} where R_{xyz} is defined as the solid region

$$R_{xyz} = \{(x, y, z): (x, y) \varepsilon R_{xy},\ \phi(x, y) \leqslant z \leqslant \psi(x, y)\}$$

over normal plane region R_{xy}, bounded below and above by the surfaces $z = \varphi(x, y)$, $z = \psi(x, y)$, A shorter notation is

$$R_{xyz} = \left[R_{xy};\ \phi(x, y),\ \psi(x, y)\right]$$

where R_{xy} is a normal plane region $\left[a, b;\ L(x), U(x)\right]$

The six normal solid regions are:

$$R_{xyz} = \left[R_{xy};\ \phi(x, y),\ \psi(x, y)\right]$$
$$R_{yxz} = \left[R_{yx};\ \phi(x, y),\ \psi(x, y)\right]$$

$$R_{yzx} = \left[R_{yz};\ \phi(y, z),\ \psi(y, z)\right]$$
$$R_{zyx} = \left[R_{zy};\ \phi(y, z),\ \psi(y, z)\right]$$

$$R_{zxy} = \left[R_{zx};\ \phi(x, z),\ \psi(x, z)\right]$$
$$R_{xzy} = \left[R_{xz};\ \phi(x, z),\ \psi(x, z)\right]$$

If R is a normal region, say R_{xyz}, then the triple integral "I" above becomes

$$\iint_{R_{xy}} \int_{\phi(x,y)}^{\psi(x,y)} f(x, y, z)dz\, dydx$$

or

$$\int_a^b \int_{L(x)}^{U(x)} \int_{(x, y)}^{\psi(x, y)} f(x, y, z)dz\, dy\, dx$$

which is an iterated triple integral, and the integration is carried out in the order z, y, x.

When V is the union of some number of disjoint normal regio the integral is evaluated by the Prop. 4.

Example. Given

$$I = \int_0^4 \int_0^{3-\frac{3}{4}x} \int_{x^2+y^2}^{20} x\,dz\,dydx,$$

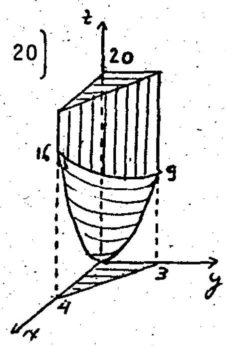
a) Sketch the solid region of integration,

b) evaluate the integral

Solution.

a) Since

$$V = R_{xyz} = \left[0,\ 4;\ \ 0,\ 3 - \frac{3}{4}x;\ \ x^2 + y^2,\ 20\right]$$



the region is over the triangular plane region and bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 20$.

b) $$I = \int_a^4 \int_0^{3-\frac{3}{4}x} xz \Big|_{x^2+y^2}^{20} dydx$$

$$= \int_0^4 \int_0^{3-\frac{3}{4}x} x(20 - x^2 - y^2)dydx = 26088/5.$$

REVERSING THE ORDER OF INTEGRATION:

Any given triple integral in an order of integration can be written as a triple integral (or as a sum of integrals) in any other order of the variables. To do this the necessary steps are:

(i) Sketching the normal region of integration of the given integral,

(ii) Writing this region as a normal region or as union of such regions in desired order.

(iii) Setting up the desired iterated integral(s)

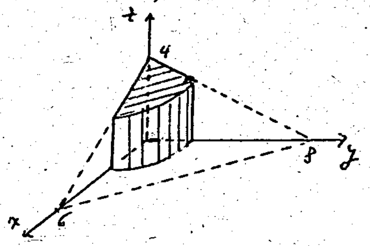
<u>Example</u>. Given the iterated triple integral

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_1^{4 - \frac{2}{3}x - \frac{y}{2}} f(x,\ y,\ z)\,dz\ dy\ dx,$$

reverse the order of integration in the order dx dy dz.

<u>Solution</u>. The region of the given integral is

$$R_{xyz} = \Big(\underbrace{0,\ 2;\ \ 0,\ \sqrt{4-x^2}}_{R_{xy}};\ \ 1,\ 4 - \frac{2}{3}x - \frac{y}{2}\Big)$$



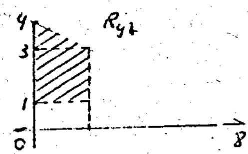
It is the solid bounded laterally by the vertical cylinders

$$x^2 + y^2 = 4, \quad x = 0, \quad y = 0$$

and below by the plane $z = 1$, above by the plane

$$z = 4 - \frac{2}{3}x - \frac{y}{2} \qquad \left(\frac{x}{6} + \frac{y}{8} + \frac{z}{4} = 1\right)$$

To write the integral over the solid region R_{zyx}, it is necessary to determine R_{zy} plane region which shown as the shaded one in the Fig:



$$R_{zy} = \underbrace{\Big[1,\ 3;\ 0,\ 2\Big]}_{R'_{zy}} \cup \underbrace{\Big[3,\ 4;\ 0,\ 8-2z\Big]}_{R''_{zy}}$$

Then

$$R_{zyx} = \Big[R'_{zy};\ 0,\ \sqrt{4-y^2}\Big] \cup \Big[R''_{zy};\ 0,\ 6 - \frac{3}{4}y - \frac{3}{2}z\Big],$$

and the integral becomes

$$\int_1^3 \int_0^2 \int_0^{\sqrt{4-y^2}} f\ dz\ dy\ dx + \int_3^4 \int_0^{8-2z} \int_0^{6 - \frac{3}{4}y - \frac{3}{2}z} f\ dz\ dy\ dx$$

C. Evaluation by Change of Variables

Let

$$\iiint_R \gamma(x, y, z)dV$$

be a triple integral, and let the following transformation (change of variables)

$$x = f(u, v, w), \quad y = g(u, v, w), \quad z = h(u, v, w)$$

be given.

As we have in double integral, dV becomes

$$dV = |J| \; \overline{du\,dv\,dw}$$

where J is the JACOBIAN

$$\frac{\partial(f, g, h)}{\partial(u, v, w)} = \begin{vmatrix} f_u & f_v & f_w \\ g_u & g_v & g_w \\ h_u & h_v & h_w \end{vmatrix}$$

If R' is the image of R under the given transformation the triple integral becomes

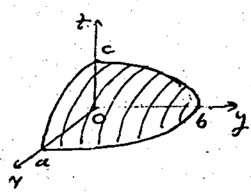
$$\iiint_{R'} \gamma(f, g, h)\; |J| \; \overline{du\,dv\,dw}.$$

Example 1. Find the volume of the ellipsoidal region

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leqslant 1.$$

Solution. The Fig. represents 1/8 of the solid.

$$|R| = 8 \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} \int_0^{\frac{c}{ab}\sqrt{a^2b^2-b^2x^2-a^2y^2}} dz\,dy\,dx$$



By the use of transformation

$$x = au, \quad y = bv, \quad z = cw,$$

V is mapped to the unit spherical regions

$$u^2 + v^2 + w^2 \leqslant 1.$$

with

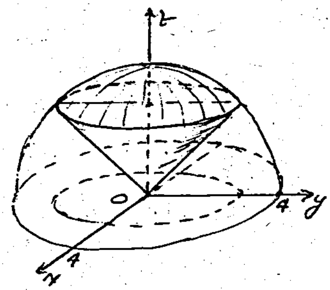
$$J = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

$$\Rightarrow \quad |V| = \iiint_{R'} abc\ dV' = abc|R'| = abc \cdot \frac{4}{3}\pi = \frac{4}{3}\pi\, abc.$$

Example 2. Find the volume of the solid defined by

$$x^2 + y^2 + z^2 \leqslant 16, \qquad x^2 + y^2 \leqslant z^2, \qquad z \geqslant 0$$

Solution.



$$|R| = 4 \int_0^{2\sqrt{2}} \int_0^{\sqrt{8-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} dz\ dy\ dx$$

Transforming it into spherical coordinates, we have the transformation

$$x = \varrho \sin\varphi \cos\Theta$$
$$y = \varrho \sin\varphi \sin\Theta$$
$$z = \varrho \cos\varphi$$

with Jacobian

$$J = \frac{\partial(x,\ y,\ z)}{\partial(\Theta,\ \varphi,\ \varrho)}$$

$$= \begin{vmatrix} -\varrho \sin\varphi \sin\Theta & \varrho \cos\varphi \cos\Theta & \sin\varphi \cos\Theta \\ \varrho \sin\varphi \cos\Theta & \varrho \cos\varphi \sin\Theta & \sin\varphi \sin\Theta \\ 0 & -\varrho \sin\varphi & \cos\varphi \end{vmatrix} = -\varrho^2 \sin\varphi$$

$$|R| = 4 \int_0^{\pi/2} \int_0^{\pi/4} \int_0^{4} \varrho^2 \sin\varphi .\ d\varrho\ d\varphi\ d\Theta = \frac{16}{3}\sqrt{2}\ \pi$$

D. GEOMETRIC AND PHYSICAL APPLICATIONS

1. Volume of a solid:

From Prop. 1, we have

$$|R| = \iiint_R dV = V \quad \text{(volume)}.$$

Then

$$|R| = \iiint_R \overline{dx\,dy\,dz} \quad \text{(in cart. coord.)}$$

$$|R| = \iiint_R r\,.\,dz\,\overline{dr\,d\Theta} \quad \text{(in cyl. coord.)}$$

$$|R| = \iiint_R \rho^2 \sin\varphi .\,\overline{d\rho\,d\varphi\,d\Theta} \quad \text{(in spher. coord.)}$$

Having examples on cartesian and spherical coordinates previously, we give one on cylindrical coordinates.

<u>Example</u>. Find the volume of the solid in the I. Octant bounded by the cylinders $x^2 + y^2 = a^2$, $x^2 + z = b^2$ $(b \geq a)$.

<u>Solution</u>.

$$|R| = \iiint_R dz\,dr\,d\Theta$$

$$= \int_0^{\pi/2} \int_0^a \int_0^{b^2 - r^2\cos^2\Theta} r\,.\,dz\,dr\,d\Theta$$

$$= \int_0^{\pi/2} \int_0^a r(b^2 - r^2\cos^2\Theta)dr\,d\Theta$$

$$= b^2 \int_0^{\pi/2} \int_0^a r\,dr\,d\Theta - \int_0^{\pi/2} \int_0^a r^3 \cos^2\Theta\,dr\,d\Theta$$

$$= \frac{1}{4}\pi a^2 b^2 - \frac{\pi}{4}\frac{a^4}{4}$$

$$= \frac{\pi}{16} a^2 (4b^2 - a^2)$$

2. Mass, moments, center of grevity, moments of inertia

By the usual notations

$m = \iiint_R \delta(x, y, z)dV$

$M_{xy} = \iiint_R z\ \delta\ dV, \quad M_{xz} = \iiint_R y\ \delta\ dV, \quad M_{yz} = \iiint_R x\ \delta\ dV$

$G(M_{yz}/m, \quad M_{xz}/m; \quad M_{xy}/m)$

$I_{ox} = \iiint_R (y^2+z^2)\ \delta\ dV, \quad I_{oy} = \iiint_R (x^2+z^2)\ \delta\ dV, \quad I_{oz} = \iiint_R (x^2+y^2)\delta dV$

and in general

$M_\pi = \iiint_R d(P,\pi)\delta dV, \quad M_\ell = \iiint_R d(P,\ell)\delta dV, \quad M_A = \iiint_R d(P,A)\ \delta\ dV$

$I_\pi = \iiint_R d^2(P,\pi)\delta dV, \quad I_\ell = \iiint_R d^2(P,\ell)\delta dV, \quad I_A = \iiint_R d^2(P,A)\ \delta\ dV$

Example. Find the centroid (δ is constant) of the solid bounded by $x^2 + y^2 + z^2 = a^2$, in the first octant.

Solution. From symmetry we have $\overline{x} = \overline{y} = \overline{z}$.

$$m = \delta V = \frac{\delta}{6}\pi a^3$$

$$m\overline{z} = M_{xy} = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{a} \rho \cdot \cos\varphi \cdot \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \frac{\delta}{2}\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{a} \rho^3 d\rho \sin 2\varphi\, d\varphi\, d\theta$$

$$= \frac{\pi}{8} a^4 \int_0^{\pi/2}\int_0^{\pi/2} \sin 2\varphi\, d\varphi\, d\theta$$

$$= \frac{\delta}{8} a^4 \cdot \frac{\pi}{2} = \frac{\delta}{16}\pi a^4$$

$$\overline{z} = \overline{x} = \overline{y} = \frac{\delta}{16}\pi a^4 / \left(\frac{\delta}{6}\pi a^3\right) = \frac{3}{8} a.$$

EXERCISES (5. 4)

71. Evaluate

a) $\int_0^2 \int_1^3 \int_1^2 xy^2 dzdy\, dx$ b) $\int_0^a \int_0^{\sqrt{a^2-y^2}} \int_0^{\sqrt{a^2-x^2}} dzdxdy$

72. Use cylindrical and spherical coordinates to evaluate:

$$\int_0^k \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\ dydxdz \quad (k > 0, \quad a > 0)$$

73. Use cylindrical or spherical coordinates to evaluate

$$\int_0^{1/\sqrt{2}} \int_0^x \int_0^{\sqrt{1-x^2-y^2}} z(x^2+y^2)^{-1/2}\ dzdydx$$

74. Find the volume of the solid bounded by the cylinder $y^2 = 4-4z$ and the planes $x = 0$, $x = z$.

75. Find by triple integral the volume

a) of one of the vedges cut off from the cylinder $x^2+y^2 = a^2$ by the planes $z = 0$, $z = x$.

b) enclosed by the cylinders $y^2 = z$, $x^2+y^2 = a^2$ and the plane $z = 0$.

c) enclosed by $y^2 + z^2 = 4x - 8$, $y^2+z^2 = 4$, $z = 0$.

76. Evaluate the volume of the solid bounded by the paraboloid $x = x^2 + z^2$ and the plane $x = 4$.

77. Same question for the region bounded by $x + 2y + 3z = 6$, $x = 0$, $y = 0$, $z = 0$.

78. Use spherical coordinates to find the volume of inside the cylinder $x^2 + y^2 = 2x$ bounded below by the cone $x^2+y^2 = 3z^2$ and above by xy-plane.

79. Use cylindrical coordinates to find the volume of the solid above the paraboloid $x^2+y^2 = z$ and below the plane $z = 4y$.

80. Use spherical coordinates to find the volume enclosed by the sphere $x^2 + y^2 + z^2 = 16$.

81. Find the moment of inertia with respect to the x-axis for the mass of the solid in the first octant bounded by the cylinders $z^2 = y$, $x^2 = y$ and the plane $y = 1$ if the density is proportional to z. ($\delta = kz$).

82. Find the centroid of a solid bounded by a sphere of radius a lying within a right circular cone with vertex at the center of the sphere and vertex angle α.

83. Find z coordinate of the center of gravity of the body of uniform density δ situated above the xy-plane and bounded by the surface $r = z(1-\cos\Theta)$, the cylinder $r = 4(1-\cos\Theta)$ and the plane.

84. Find the mass in the first octant of the solid bounded by the cylinders $z^2 = y$, $x^2 = y$ and the plane $y = 1$ if the density is $\delta = kz$.

85. Find the center of mass of the solid bounded by the planes $y + z = 2$, $x + z = 2$, $x = 0$, $y = 0$, $z = 0$ if $\delta = kx$.

ANSWERS TO EVEN NUMBERED EXERCISES

72. $\frac{\pi}{6} k a^3$

74. 16/15

76. 8π

78. $\pi/2$

80. $256 \pi/3$

82. $\bar{z} = \frac{3}{8} a (1 + \cos\alpha)$

84. k/5.

5.5 THEOREMS ON CHANGE OF MULTIPLICITIES OF INTEGRALS (GREEN'S, STOKES' AND GAUSS' THEOREMS)

A. GREEN'S THEOREM IN $\mathbb{R}^2$:

Theorem. $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \overline{dxdy} = \oint_{\partial R} Pdx + Qdy$

if ∂R is a simple piecewise smooth closed curve(*) and $P(x, y)$, $Q(x, y)$ are continuously differentiable functions on a region containing $R \cup \partial R$ and if ∂R is positively oriented (counter-clockwise).

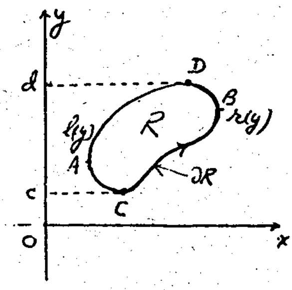
Proof. We prove that

$$\iint_R \frac{\partial Q}{\partial x} \overline{dxdy} = \oint_{\partial R} Qdy, \qquad \iint_R - \frac{\partial P}{\partial y} \overline{dxdy} = \oint_{\partial R} Pdx. \qquad (1)$$

The proof will be given for a region expressible in both types Rxy and Ryx:

$$R = \underset{Rxy}{\left[a, b; L(x), U(x)\right]} = \underset{Ryx}{\left[c, d; \ell(y), r(y)\right]}$$

The first equality in (1) will be shown for $R = R_{yx}$ and the second for $R = Rxy$:



$$\iint_R \frac{\partial Q}{\partial x} \overline{dxdy} = \iint_{Ryx} \frac{\partial Q}{\partial x} dxdy$$

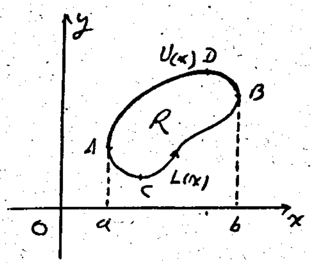
$$= \int_c^d \left[\int_{\ell(y)}^{r(y)} \frac{\partial Q}{\partial x} dx\right] dy$$

$$= \int_c^d \left[Q(r(y), y) - Q(\ell(y), y)\right] dy$$

$$= \int_{CBD} Q(x, y)dy - \int_{CAD} Q(x, y)dy$$

$$= \int_{CBD} Qdy + \int_{DAC} Qdy = \oint_{\partial R} Qdy.$$

(*) a curve $x = x(t)$, $y = y(t)$ is a simple curve if it has no double point, and piecewise smooth if it consists of pieces of curves, for each of which x, y, x', y' are continuous.



$$\iint_R \frac{\partial P}{\partial y}\,\overline{dxdy} = \iint_{R_{xy}} \frac{\partial R}{\partial y}\,dydx$$

$$= \int_a^b \left[\int_{L(x)}^{U(x)} \frac{\partial P}{\partial y}\,dy\right] dx$$

$$= \int_a^b \left[P(x,\ U(x)) - P(x,\ L(x)\right] dx$$

$$= \int_{ADB} P(x,\ y)dx - \int_{ACB} P(x,\ y)dx$$

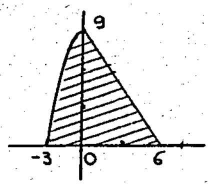
$$= \int_{ADB} P(x,\ y)dx + \int_{BCA} P(x,\ y)dx = -\int_{\partial R} Pdx . \blacksquare$$

<u>Example 1</u>. Evaluate

$$I = \oint_\Gamma y^2dx + x^2dy$$

by the use GREEN's Theorem, where Γ is the boundary of the region bounded by $x = -\sqrt{9-y}$, $x/6 + y/9 = 1$ and $y = 0$.

<u>Solution</u>. The region R is as shown in the Fig. Then



$$I = \oint_\Gamma y^2dx + x^2dy = \iint_R \left(\frac{\partial x^2}{\partial x} - \frac{\partial y^2}{\partial y}\right)\overline{dxdy}$$

$$= \iint_{R_{yx}} (2x-2y)dxdy$$

$$= 2\int_0^9 \int_{-\sqrt{9-y}}^{6(1-\frac{y}{9})} (x-y)dxdy$$

$$= \int_0^9 x^2-2yx \Big|_{-\sqrt{9-y}}^{6-\frac{2}{3}y} dy$$

$$= \int_0^9 \left[(6-\tfrac{2}{3}y)^2 - 2y(6-\tfrac{2}{3}y) - (9-y) - 2y\sqrt{9-y}\right] dy$$

$$= \int_0^9 \left[36 - 8y + \tfrac{4}{9}y^2 - 12y + \tfrac{4}{3}y^2 - 9 + y - 2y\sqrt{9-y}\right] dy$$

$$= \int_0^9 \left(\frac{16}{9}\, y^2 - 19y + 27 - 2y\sqrt{9-y}\right) dy$$

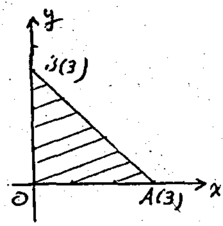
$$= \frac{16}{27}\, y^3 - \frac{19}{2} y^2 + 27y \Bigg|_0^9 - 2 \int_0^9 y\sqrt{9-y}\; dy$$

$$= \frac{16}{27} \cdot 9^3 - \frac{19}{2}\, 9^2 + 27.9 + \frac{12}{5}\, 3^3 = -191,7$$

Example 2. Evaluate

$$I = \iint_R x^3\, \overline{dxdy}$$

by the use of GREEN's Theorem, where R is the region bounded by $x = 0$, $y = 0$, $x+y = 3$.



Solution. Considering x^3 as $\partial Q/\partial x$ we have $Q = \frac{x^4}{4}$ (since integral is definite) and $I = \oint_\Gamma \frac{x^4}{4}\, dy$

$$= \int_{OA} \frac{x^4}{4}\, dy + \int_{AB} \frac{x^4}{4}\, dy + \int_{BO} \frac{x^4}{4}\, dy$$

$$= 0 + 3^5/20 + 0 = 3^5/20$$

Compute the same integral considering x^3 as $-\partial P/\partial y$.

Example 3. Evaluate the line integral

$$I = \int_\Gamma y^2 dx + x dy$$

along Γ: $y = x^2$ from $A(-2, 4)$ to $B(2, 4)$ by closing the curve with line segment (AB).

Solution. From GREEN's Theorem we have

$$\int_\Gamma \ldots + \int_{BA} \ldots = \iint_R \ldots .$$

Then

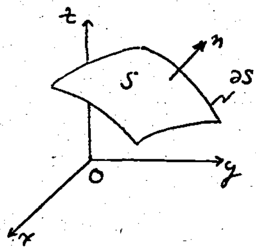
$$I = \iint_R \left(\frac{\partial x}{\partial x} - \frac{\partial y^2}{\partial y}\right)\overline{dxdy} - \int_{BA} y^2 dx + x dy$$

$$= \int_{-2}^{2}\int_{x^2}^{4} (1 - 2y)dydx - \int_{2}^{-2} 16dx$$

$$= \int_{-2}^{2} y - y^2 \Big|_{x^2}^{4} dx + 64 = 352/15.$$

B. STOKES' AND GAUSS' THEOREMS

Theorem. (STOKES')



$$\iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) \overline{dydz}$$

$$+ \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) \overline{dzdx}$$

$$+ \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \overline{dxdy} = \oint_{\partial S} Pdx + Qdy + Rdz$$

where S is an surface oriented by its unit vector $\vec{n}$ varying continuously over S, ∂S is a piecewise smooth closed curve which is the boundary of S, and $F = Pi + Qj + Rk$ is a vector field (smooth) defined over a region containing $S \cup \partial S$. The line integral is taken in the sense such that S is always on the observer's left as he moves on ∂S, and the sense of the observer (from foot to head) and $\vec{n}$ are the same.

Using vector notation the same theorem becomes

$$\iint_S \operatorname{curl} F \cdot \vec{n}\, d\sigma = \oint_{\partial S} F \cdot dr$$

Note that STOKES' Theorem is the GREEN's Theorem in $\mathbb{R}^2$ when S is flat.

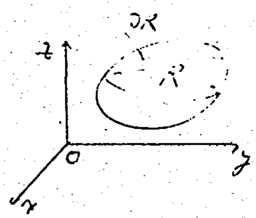
<u>Corollary</u>.

$$\oiint_S \text{curl } F \cdot \vec{n}\, d\sigma = 0$$

<u>Theorem</u>. (GAUSS)

$$\iiint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\overline{dxdydz} = \iint_{\partial R} P\,\overline{dydz} + Q\,\overline{dzdx} + R\,\overline{dxdy}$$

where R is a space region bounded by a piecewise smooth orientable closed surface $\partial R(= S)$, $F = Pi + Qj + Rk$ is a smooth vector field defined on region containing $R \cup \partial R$, and the boundary ∂R is oriented by taking $\vec{n}$ as the exterior normal.



This is also referred to as *GREEN's Theorem* in space.

Since its vector expression is

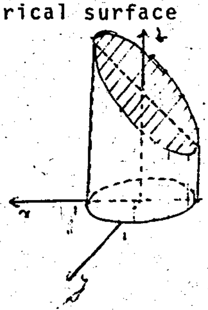
$$\iiint_R \text{Div } F \cdot dV = \oiint_{\partial R} F \cdot \vec{n}\, d\sigma$$

the same theorem is also called the *Divergence Theorem*.

<u>Example 1</u>. Evaluate by the use of STOKES' Theorem

$$I = \iint_S \text{curl } F \cdot \vec{n}\, d\sigma$$

where $F = yi + zj + xk$ and S is the cylindrical surface $x^2 + y^2 = 1$ cut by the planes $z = 0$ and $z = x + 2$ oriented with $\vec{n}$ pointing outward.



<u>Solution</u>. We have

$$I = \int_{\partial S} F.dr = \oint_{\Gamma_1} F.dr - \oint_{\Gamma_2} F.dr$$

since the given surface may be considered as the difference $(S \cup S')-S'$ (S': $z = x + 2$ bdd. by Γ_2).

$$\Gamma_1: \ x = \cos\Theta, \ y = \sin\Theta, \ z = 0 \Rightarrow F = \sin\Theta i + \cos\Theta k$$

$$dr = (-\sin\Theta i + \cos\Theta j)d\Theta$$

$$\oint_{\Gamma} -\sin^2\Theta = -\int_0^{2\pi} \sin^2\Theta \, d\Theta = -\pi.$$

$$\Gamma_2: \ x = \cos\Theta, \quad y = \sin\Theta, \quad z = 2 + \cos\Theta$$

$$\Rightarrow \quad F = \sin\Theta i + (2 + \cos\Theta)j + \cos\Theta \, k,$$

$$dr = (-\sin\Theta i + \cos\Theta j - \sin\Theta k)d\Theta$$

$$\oint_{\Gamma_2} (-\sin^2\Theta + 2\cos\Theta + \cos^2\Theta - \sin\Theta\cos\Theta)d\Theta = \int_0^{2\pi} \cdots = 0.$$

Example 2. Evaluate, by the use of Divergence Theorem,

$$I = \iint_S F \cdot \vec{n} \, d\sigma,$$

where $F = (x^2 + ye^z)i + (y^2 + ze^x)j + (z^2 + xe^y)k$ and S is the boundary of the solid R bounded by $z = 0$, $z = x+2$, $x^2+y^2 = 1$.

Solution. We have

$$I = \iiint_R \mathrm{Div}\, F \, dV = \iiint_R 2(x+y+z)dV$$

$$= 2\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1+x^2}} \int_0^{x+2} (x+y+z)dzdydx = \frac{19}{3}\pi .$$

EXERCISES (5.5)

86. By the use of GREEN's Theorem, evaluate

$$\oint_{\Gamma} 4xy^3dx + 6x^2y^2dy$$

where Γ is the circle $x^2 + y^2 = 1$.

87. Same question for

$$\oint_{\Gamma} e^x \sin y \, dx + e^x \cos y \, dy$$

where Γ is the rectangle with vertices $(0, 0)$, $(1, 0)$, $(1, \pi/2)$, $(0, \pi/2)$.

88. Use STOKES' Theorem to evaluate

$$\oint_{\Gamma} 2xy^2z \, dx + 2x^2yz \, dy + (x^2y^2 - 2z)dz$$

where Γ is $(\cos t, \sin t, \sin t)$, $t \in (0, 2\pi)$ and directed in increasing values of t.

89. By the Divergence Theorem, evaluate

$$\iint_S x \, \overline{dydz} + y \, \overline{dzdx} + z \, \overline{dxdy}$$

where S is the surface $x^2 + y^2 + z^2 = 1$ and $\vec{n}$ is the outer normal.

90. Same question for

$$\iint_S F \cdot n \, d\sigma$$

where S is the surface $x^2 + y^2 + z^2 = a^2$, and

a) $F = xi + yj + zk$ b) $F = xyi + yzj + zxk$ c) $F = xy^2i + yz^2j + zx^2k$

ANSWERS TO EVEN NUMBERED EXERCISES

86. 0

88. 0

90. a) $4\pi a^3$, b) 0, c) $4\pi a^5/5$.

A SUMMARY

(CHAPTER 5)

5.1 LINE INTEGRALS: $\int_{A\,\Gamma}^{B} Pdx + Qdy + Rdz = \int_{A\,\Gamma}^{B} F.dr$ is the line integral along a curve Γ from A to B, meaning that the integral is taken when the point (x, y, z) is moving on Γ.

Independence of path: A line integral which depends only on the limiting points A and B is said to be independent of path. The necessary and sufficient condition for independence of path is $\text{curl } F \equiv 0$.

5.2 DOUBLE INTEGRALS: $\iint_R f(x, y)\, \overline{dxdy}$ is the double integral of f over the plane region R. When R is a normal region R_{xy} or R_{yx}, we have

$$\int_a^b \int_{L(x)}^{U(x)} f\, dydx \quad \text{or} \quad \int_c^d \int_{\ell(y)}^{r(y)} f\, dxdy$$

as *iterated integrals.*

$|R| = \iint_R dA$ (area of R)

$V = \iint_R f\, dA$ (Volume under the surface of $f>0$ over R)

Change of variable: $x = F(u, v)$, $y = G(u, v)$ $\Rightarrow$

$$\iint_R \varphi(x, y) dA = \iint_{R'} \phi(u, v) dA' = \iint_{R'} \phi(u, v) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \overline{dudv}$$

where R' is the image of R under the transformation, and $\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$ is the *Jacobian* of the transformation.

5.3 SURFACE INTEGRALS: $I = \iint_S P\,\overline{dydz} + Q\,\overline{dzdx} + R\,\overline{dxdy} = \iint_S F \cdot \vec{n}\, d\sigma$ is the surface integral of $F \cdot n$ over S. "I" is equal to

$$\iint_{S_{yz}} (P\cos\alpha + \cos\beta + R\cos\gamma)\sec\alpha\, \overline{dydz}, \quad \text{or}$$
$$\iint_{S_{zx}} (P\cos\alpha + \cos\beta + R\cos\gamma)\sec\beta\, \overline{dzdx}, \quad \text{or}$$
$$\iint_{S_{xy}} (P\cos\alpha + \cos\beta + R\cos\gamma)\sec\gamma\, \overline{dxdy},$$

where S_{yz} say, is the projection of S on yz-plane, and $\vec{n} = (\cos\alpha, \cos\beta, \cos\gamma)$.

Area of a surface: $|S| = \iint_S d\sigma$.

5.4 TRIPLE INTEGRALS: $\iiint_R f(x, y, z)dV$ is a triple integral of f over the space region R.

$$V = |R| = \iiint_R dV \quad \text{(volume of the solid R)}$$

when R is one of the six normal regions, say R_{zyx}, we have the *iterated integral*

$$\int_a^b \int_{L(x)}^{U(x)} \int_{\phi(y,z)}^{\psi(y,z)} f\, dzdydx$$

Change of variable: $x = f(u, v, w)$, $y = g(u, v, w)$, $z = h(u, v, w)$

$$\iiint_R f\, dV = \iiint_{R'} (u, v, w)\, |J|\, \overline{dudvdw}$$

where R' is the image of R under the transformation and

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

is the Jacobian of the transformation.

Physical applications of integrals: If the symbol $\int$ is used to represent a line integral ($\int_\Gamma ds$) or double, surface or triple integral, we have

$$m = \int_R \delta\, dR, \qquad (R \text{ is } \Gamma,\ dR = ds \text{ for line integrals})$$

$$M_\ell = \int_R \delta . d(P, \ell) dR, \qquad M_\pi = \int_R \delta . d(P, \pi) dR$$

$$I_\ell = \int \delta . d^2(P, \ell) dR, \qquad I_\pi = \int_R \delta\, d^2(P, \pi) dR$$

$$G: \quad m\bar{x}_i = \int_R x_i\, \delta\, dR, \qquad (i = 1, \ldots)$$

$$\rho_\ell^2 = I_\ell / m .$$

5.5 GREEN'S, STOKES' AND GAUSS' THEOREM

$$\iint_R (Q_y - P_x)\overline{dxdy} = \oint_{\partial R} Pdx + Qdy \quad (\text{GREEN's Theorem in } \mathbb{R}^2)$$

$$\iint_S (R_y - Q_z)\overline{dydz} + (P_z - R_x)\overline{dzdx} + (Q_x - P_y)\overline{dxdy}$$

$$= \int_{\partial S} Pdx + Qdy + Rdz \quad (\text{STOKES' Theorem})$$

$$\iiint_R (P_x + Q_y + R_z)\overline{dxdydz}$$

$$= \iint_{\partial R} P\ \overline{dydz} + Q\ \overline{dzdx} + R\ \overline{dxdy} \quad (\text{GAUSS' Theorem})$$

MISCELLANEOUS EXERCISES

(CHAPTER 5)

91. Prove $\left| \int_{A\,\Gamma}^{B} Pdx + Qdy + Rdz \right| \leqslant s.M$ where s is the length of Γ and M is the maximum value of

$$\left| P \frac{dx}{ds} + Q \frac{dy}{ds} + R \frac{dz}{ds} \right| \quad \text{on } \Gamma.$$

92. Evaluate

$$\int_{A\,\Gamma}^{B} (e^{yz} + 1)dx + (xz\ e^{yz} + 1)dy + (xy\ e^{yz} + 2)dz$$

where
$$\Gamma: \; x^3yz + 2xy^3z = 3, \quad 2x^2y^2z - xyz = 1$$

$A(1, 1, 1)$, $B(1, 2, 1/6)$.

93. Evaluate
$$\int_A^B (e^y + ze^x)dx + (e^z + xe^y)dy + (e^x + ye^z)dx$$
along any curve from $A(0, 0, 0)$ to $B(\ln 2, \ln 3, \ln 4)$.

94. Same question for
$$\int_A^B y\,dx + x\,dy + z^2dz$$
if $A(0, 1, 3)$, $B(2, 3, 6)$

95. Evaluate
$$\int (x^2 + z^2)dx + (y^2 + z^2)dy + (2yz + 2xz)dz$$
for
$$\Gamma: \begin{cases} x^2 + 2xy + z^2 - 5x + y - 4 = 0, \\ x^2 - xy + 2y + 2z^2 + 6x - 8 = 0 \end{cases}$$
from $A(0, 0, 2)$ to $B(0, 3, 1)$ in two ways.

96. Find the work done by a particle moving along the helix Γ: $(a \cos t, a \sin t, bt)$ from $t = 0$ to $t = 2\pi$ if it is subject to the force field
$$F = (4x^3 + yz/x, \; z \ln x, \; y \ln x + 2z)$$

97. Find the mass of the wire in the shape of the curve Γ: $(2e^t, 2e^{-t}, 2\sqrt{2}t)$, $t\varepsilon[0, \ln 3]$, if the density per unit length is $\delta = xyz$.

98. Evaluate $\iint_R yx\,dA$ where
$$R = \left[[0, 1]; \; 1-x, \sqrt{4-x^2}\right]$$

99. Given $\int_0^2 \int_{x^2}^{2x} yxdydx$,

a) Sketch the region of integration, b) evaluate the integral

100. Evaluate $\iint_R e^{-x^2-y^2}\,dydx$ where $R = \{(x, y): x^2+y^2 \leq a^2\}$.

101. Find the area of the region outside $r = a(1 + \cos\Theta)$ and inside $r = 3a\cos\Theta$.

102. Find the area of the region bounded by the parabolas $y = x^2$, $y^2 = 2x$.

103. Evaluate $\int_0^{\sqrt{\ln 5}} \int_y^{\sqrt{\ln 5}} e^{x^2}\,dxdy$

104. Reverse the order of integration of the following double integrals

a) $\int_{-1}^{2} \int_{x^2}^{x+2} f(x, y)dydx$, b) $\int_0^2 \int_0^x f\,dydx + \int_2^{2\sqrt{2}} \int_0^{\sqrt{8-x^2}} f\,dydx$

105. Find $G(\overline{x}, \overline{y})$ of the plane region bounded by the arc of parabola $\sqrt{x} + \sqrt{y} = \sqrt{a}$ $(a > 0)$ and $x = 0$, $y = 0$.

106. Prove

a) $\dfrac{\partial(f, g)}{\partial(u, v)} \cdot \dfrac{\partial(u, v)}{\partial(f, g)} = 1$ b) $\dfrac{\partial(f, g)}{\partial(u, v)} \cdot \dfrac{\partial(u, v)}{\partial(x, y)} = \dfrac{\partial(f, g)}{\partial(x, y)}$

107. Find $G(\overline{x}, \overline{y})$ of the region bounded by one loop of the lemniscate $r^2 = 2a^2\cos 2\Theta$ and which is exterior to the circle $r = a$.

108. Find the center of mass of the plate described as:

a) bounded by $y = x^2$ and $y = x+2$ with $\delta = ky$

b) bounded by $y^2 = x$ and $x^2 = y$ with $\delta = ky$.

109. Same question for

a) bounded by $r = 2(1 + \cos\Theta)$ with δ constant

b) bounded by $4x^2 + 3y^2 = 48$ and $(y-2)^2 + x^2 = 1$ with δ constant.

110. Find $G(\overline{x}, \overline{y}, \overline{z})$ of the portion of the surface of a sphere radius a, cut off by a right circular cone whose vertex is at the center of the sphere and vertex angle is 2α.

111. Show that if R is symmetric with respect to the x-axis (or y-axis), and $\delta(x, -y) = \delta(x, y)$ (or $\delta(-x, y) = \delta(x,y)$ for all x, y on R, then $\bar{y} = 0$ (or $\bar{x} = 0$).

112. Find the center of mass of the solid in the first octant bounded by the parabolic cylinder $y^2 = x$ and the planes $x = 0$, $y = 1$, $z = 0$, $z = y$ if $\delta = k(x + 2z)$.

113. Use cylindrical coordinates to find the volume of the solid sphere 3 units of radius.

114. Find the volume of the solid bounded by the elliptic paraboloids $z = 4x^2 + y^2$, $z = 8 - 4x^2 - y^2$.

115. Same question for one bounded by $x = 0$, $y = 0$, $z = 0$ and $x/a + y/b + z/c = 1$ $(a, b, c > 0)$.

116. Find the volume of the region lying above the xy-plane and bounded by the surfaces $x^2 + y^2 = a^2$, $z = y$ and $z = 0$.

117. Compute the volume of the solid bounded by the cylinder $x^2 = y$ and the planes $z = y$, $y = z$, $x = 0$, $z = 0$.

118. Same question for one bounded by $z = 9 - y^2$, $z = 9 - x^2$ and coordinate planes.

119. Use cylindrical coordinates to find the centroid of the solid bounded by the cone $x^2 + y^2 = z^2$ and the plane $z = 4$.

120. Use cylindrical coordinates to find the moment of inertia with respect to a diameter of a sphere of radius a filled with a homogeneous material of density $\delta = k$.

121. Use spherical coordinates to find the centroid of the solid enclosed by the hemisphere $x^2 + y^2 + z^2 = 9$, $(z \geqslant 0)$ and the xy-plane.

122. Use spherical or cylindrical coordinates to evaluate

$$\int_0^b \int_0^{\sqrt{b^2-x^2}} \int_0^{\sqrt{b^2-x^2-y^2}} z\sqrt{b^2 - x^2 - y^2}\, dzdydx \quad (b > 0)$$

123. Find the volume of the solid bounded by the surfaces $y = e^x$, $y = z$, $z = 0$, $x = 0$ and $x = 2$.

124. Find the moment of inertia of the space region bounded by the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ about z-axis.

125. Find the centroid of the plane region bounded by the curve $y = 2(1 + x^3)$ and the coordinate axes.

126. Evaluate

$$\int_0^1 \int_y^{\sqrt{y}} \frac{\sin x}{x}\, dxdy$$

127. Evaluate

$$\int_{(1,\,0,\,1)}^{(2,\,3,\,1)} \vec{F} \cdot d\vec{r}$$

along a straight line joining the two given points if

$$F = x^2 i - xzj + y^2 k$$

128. Evaluate

$$\int_0^{\pi/2} \int_0^{\operatorname{Sec}\Theta} \frac{r\, dr d\Theta}{1 + r^2 \operatorname{Sin}^2\Theta}$$

by transforming to rectangular coordinates

129. Find the area of the plane region outside the circle $r = 3a$ and inside the cardioid $r = 2a(1 + \cos\Theta)$.

130. Given

$$\int_{-\infty}^{\infty} \int_{-\sqrt{4+y^2}}^{\sqrt{4+y^2}} f(x,\, y)\, dxdy,$$

write the integral in uv-system if $x = u+v$, $y = 2\sqrt{uv}$.

131. Evaluate $\int_{(0,\ 1)}^{(2,\ 8)} \nabla\phi . dr$ along the curve $y = x^3$, where $\phi(x,\ y) = x^2 - y^2$.

132. Show that the line integral is independent of path, and evaluate:

$$\int_{(1,\ 1,\ 1)}^{(2,\ 1,\ 3)} 2xy^3zdx + 3x^2y^2zdy + x^2y^3dz$$

133. Find the area of the region enclosed by the curves $r = \tan\Theta$, $\Theta = 0$, $\Theta = \pi/4$.

134. Find I_{ox} for the semicircular region $0 \leq r \leq a$, $0 \leq \Theta \leq \pi$ if the density is $\delta(\Theta,\ r) = \sin\Theta$.

135. Find the centroid, using polar coordinates, of the region bounded by the curves $y = 2x$, $y^2 = 4ax$ $(a > 0)$.

136. Evaluate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\ dydx$$

by transforming to polar coordinates.

137. Integrate $\int_\Gamma (x + y^2)dx + x^2ydy$ along the boundary of the region defined by $y^2 \leq x$, $|y| \geq 2x - 1$

a) by the use of GREEN's Theorem, b) without using it.

138. Find the mass of the wire in the shape of the arc Γ: $x = \cos t$, $y = \sin t$, $t \varepsilon [0,\ \pi/4]$ if the density is $\delta = x^2 + y^2 + \frac{y}{x}$.

139. Evaluate by the use STOKES' Theorem:

$$\iint_S \text{curl } F\ .\ \vec{n}\ dr$$

where $F = zyi + xj + xyk$ and S is the $4-z = x^2+y^2$ cut by $z = 0$.

140. If $F = p(y, z)i + q(x, z)j + r(x, y)k$, show that

$$\oiint_S F \cdot n \, d\sigma = 0$$

(if S is closed)

141. Evaluate by the use of Divergence Theorem:

$$\iint_S F \cdot n \, d\sigma$$

where $F = x^2 i + y^2 j + z^2 k$, S is the surface of the unit-cube: $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

142. Evaluate if possible by the use of GREEN's Theorem:

$$\int_\Gamma \frac{y}{x^2+y^2} dx - \frac{x}{x^2+y^2} dy$$

where Γ is the ellipse $4x^2 + y^2 = 4$.

143. Show that for an allowable region R of $\mathbb{R}^2$

$$\int_{\partial R} (x^2+y^2)^{-1} (xdy - ydx) = 0.$$

Why doesn't this result hold if R is bounded by Γ: $(\cos t, \sin t)$, $0 \leq t \leq 2\pi$?

144. Evaluate $\int_\Gamma \frac{1}{2} x^2 dy$ if Γ is

a) the square with vertices $(1, 1)$, $(-1, 1)$, $(-1, -1)$, $(1, -1)$

b) the square with vertices $(0, 0)$, $(a, 0)$, (a, a), $(0, a)$, $a > 0$.

c) the circle $(x-h)^2 + (y-\kappa)^2 = r^2$

by the use of GREEN's Theorem.

145. Suppose that $f(x, y)$ satisfies the LAPLACE equation $f_{xx} + f_{yy} = 0$ in a region R. Show that

$$\int_\Gamma f_y dx - f_x dy = 0$$

where $\Gamma \subset R - \partial R$.

ANSWERS TO EVEN NUMBERED EXERCISES

92. $e^{1/3} - e - 2/3$

94. 69

96. 4π

98. 5/6

100. $\pi(1 - e^{-a^2})$

102. 2/3

104. a) $\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f \, dxdy + \int_1^4 \int_{y-2}^{\sqrt{y}} f \, dxdy$, b) $\int_0^2 \int_y^{\sqrt{8-y^2}} f \, dydx$

108. a) $\overline{x} = 25/16$, $\overline{y} = 235/112$, b) $\overline{x} = 4/7$, $\overline{y} = 5/9$

110. $\overline{x} = 0$, $\overline{y} = 0$, $\overline{z} = a \cos^2 \frac{\alpha}{2}$

112. (95/238, 100/119, 185/357)

114. 8π

116. $2a^3/3$

118. 81/2

120. $8\pi a^5/15$

122. $\pi b^5/20$

124. $\frac{4}{15} \pi abc \, (a^2 + b^2)$

126. Inverting the order of integration one finds $1 - \sin 1$

128. $\pi/2$

130. $$\int_0^1 \int_0^{1+v} f(u+v, \ 2\sqrt{uv}) \ \frac{|u-v|}{\sqrt{uv}} \ dudv \ +$$
$$\int_1^{\infty} \int_{v-1}^{1+v} f(u+v, \ 2\sqrt{uv}) \ \frac{|u-v|}{\sqrt{uv}} \ dvdu \ .$$

132. 11

134. $a^4/3$ 136. $\frac{\pi}{4}(e - 1)$ 138. $\frac{\pi}{4} + \frac{1}{2} \ln 2$

142. not possible since P, Q are not continuous at the interior point (0, 0).

CHAPTER 6

DIFFERENTIAL EQUATIONS

6.1 DEFINITIONS

A relation involving a dependent variable, its variable(s) and derivatives is called a *differential equation (DE)*.

Thus

$$y'' - 2xy' + y = e^x, \qquad \frac{\partial z^2}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} = 0$$

are differential equations.

A differential equation involving an unknown function of a single variable and some of its derivatives is called an *ordinary differential equation* (ODE), and one involving an unknown function of several variables and some of its (partial) derivatives, a *partial differential equation* (PDE).

The equations

$$F(x, y \ \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}), \tag{1}$$

and

$$F(u(x, y, \ldots), \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \ldots, \frac{\partial^2 u}{\partial x^2}, \ldots) = 0 \tag{2}$$

represent ODE and PDE in their general form.

In this Book we will discuss only ODE's.

Order and degree of a DE.

The *order* of a DE is the order of the heighest-ordered derivative in the DE.

The *degree* of a DE is the exponent of the heighest-ordered derivative when the DE is expressed as a polynomial of derivatives.

In the following examples the order and degree are indicated:

	Order	Degree
$x^5y'' + xy'^3 = 4$	2	1
$(\frac{dr}{d\Theta})^2 - re^{\Theta} = 0$	1	2
$y'' = \sqrt{y} + \sqrt{y}$	2	2
$y'' = \sqrt{y + \sqrt{y'}}$	2	4
$\cos y' = xy'$	1	(no degree)
$\cos y' = xy$	1	1

Solution, general and particular solutions of a DE

Any relation $f(x, y) = 0$ satisfying a DE is called *a solution* of the DE.

A solution (of a DE of order n) involving n *arbitrary* constants is called the *general solution* (GS), while one involving less than n arbitrary constants a *particular solution* (PS).

Example. Show that $y = c_1e^x + c_2e^{-x}$ is a solution of $y'' - y = 0$.

Solution. $y' = c_1e^x - c_2e^{-x}$, $y'' = c_1e^x + c_2e^{-x}$

$$\Longrightarrow (c_1e^x + c_2e^{-x}) - (c_1e^x + c_2e^{-x}) = 0.$$

This solution involving two arbitrary constant is the GS, while $y = c_1e^x + e^x$, $y = 2e^x + c_2e^{-x}$, e^x, e^{-x} are particular solutions.

Finding the DE from the GS

To find the DE from the GE, differentiate it w.r.to the independent variable as many times as there are arbitrary constants, and then eliminate the constants between the obtained relations and the GS.

Example. Find the DE from the GS's:

a) $y = x\sin(x+c)$ b) $y = c_1x + c_2$

Solution.

a) $y = x\sin(x+c) \Longrightarrow y' = \sin(x+c) + x\cos(x+c)$

$\Longrightarrow \sin(x+c) = \frac{y}{x}, \qquad y' = \frac{y}{x} + x\cos(x+c)$

$$\Rightarrow \left.\begin{array}{l} \cos(x+c) = y' - \frac{y}{x} \\ \\ \sin(x+c) = \frac{y}{x} \end{array}\right\} \Rightarrow \left(y' - \frac{y}{x}\right)^2 + \left(\frac{y}{x}\right)^2 = 1 .$$

b) $y = c_1x + c_2 \Longrightarrow y' = c_1 \Longrightarrow y'' = 0$

Since in $y'' = 0$ there are no arbitrary constant it is the required DE.

EXERCISES (6. I)

1. Examine each DE for the dependent variable and independent variable(s), and then tell which DE is ordinary and which is partial, and also determine the order and degree of the ODE:

a) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = \sin x$ b) $4\frac{\partial^2 y}{\partial x^2} = \frac{\partial y}{\partial z}$

2. Same question for

a) $y' + \cos xy = 1$ b) $y''^2 = \sqrt{xy'} + \sqrt{y}$

c) $x(t) + 2\sin t = 0$ d) $y'' = \sqrt{xy'} + \sqrt{y'}$

3. State the order and degree of the DE:

a) $(x^2+y^2)dx + xdy = 0$ b) $y'^3 = 3x^2 + 1$

4. Same question for:

   a) $(\frac{d^3y}{dx^3})^4 - (\frac{d^2y}{dx^2})^5 + x = 0$ b) $x^4y' - x^2y'' = y^4y'''$

5. Show that the given function is a solution of the DE:

   a) $y = cx^2 - x;\quad xy' = 2y + x$

   b) $y = c_1 \sin x + c_2 \cos x;\quad y'' + y = 0$

   c) $y = (2x+c)e^{-x};\quad y' + y = 2e^{-x}$

6. Same question for:

   a) $y = c_1e^{2x} + c_2e^{-3x} + xe^{2x}\ ;\quad y'' + y' = 6y + 5e^{2x}$

   b) $y = r^2\tan\Theta = c\cos\Theta\ ;$

$$2r\frac{dr}{d\Theta}\tan\Theta + r^2\sec^2\Theta = r^2\sin^2\Theta\sec^2\Theta.$$

7. Find the DE from the given GS:

   a) $y = c_1e^x + c_2e^{-x} + x,$ b) $y = c\ \ln x$

8. Same question for:

   a) $y = c_1x^3 + c_2x^2 + c_3x + c_4$ b) $y = \sin cx$

9. Verify by substitution that for each case, the function given in the bracket is a solution of the corresponding DE:

   a) $y = xy' + y'^2\quad \left[y = cx + c^2\right]$

   b) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 3-2x^2\quad \left[y = c_1e^x + c_2e^{-2x} + x^2 + x\right]$

10. Same question for:

   a) $\frac{\partial z}{\partial x} - 2\frac{\partial z}{\partial y} = y - 2x\quad \left[z = xy + \phi(2x + y)\right]$

   b) $2z_{xx} + 3z_{xy} - 2z_{yy} = 0\quad \left[z = \phi(2x-y) + \psi(x + 2y)\right]$

ANSWERS TO EVEN NUMBERED EXERCISES

1. a) y; x, ODE, 2, 1 b) y; x, t, PDE, 2, 1

2. a) y; x, ODE, 1, 1   b) y; x; ODE, 3, 4   c) x; t, ODE, 2, 1

3. a) 1; 1,   b) 1, 3

4. a) 3, 4,   b) 3, 1

7. a) $y-y'' = x$,   b) $y = xy'\ln x$

8. a) $\frac{d^4y}{dx^4} = 0$,   b) $y' = \frac{1}{x}(\arcsin y)\ \frac{1}{\sqrt{1-y^2}}$

6.2 First Order Differential Equations.

The general form of such an equation is

$$F(x, y, \frac{dy}{dx}) = 0 \tag{1}$$

which when solved for dy/dx given

$$\frac{dy}{dx} = f(x, y) \tag{1'}$$

which becomes

$$f(x, y)dx - dy = 0$$

and more generally

$$P(x, y)dx + Q(x, y)dy = 0 \tag{2}$$

We classify (2) into four main types as separable, homogeneous, exact and linear equations. A DE may belong to more than one type.

A. Separable Differential Equations (SDE)

A first order DE which can be written or having the form

$$P(x)dx + Q(y)dy = 0 \tag{1}$$

is called a *separable differential equation* (SDE)

Its GS is obtained by direct integration of its terms:

$$\int P(x)dx + \int Q(y)dy = c$$

$$f(x) + g(y) = c$$

where c is an arbitrary constant.

The following are SDE:

$$(x^2+1)dx + ydy = 0, \qquad (y^2+1)dx - e^x dy = 0$$

$$r d\theta - \theta dr = 0, \qquad \frac{dy}{dx} = \frac{y+1}{x-1}$$

$$y' = (1-x)(y-1) \qquad y' = (2+x-2y-xy) \quad \text{(why?)}$$

Example. Solve $(y^2+1)dx - (x^2+4)dy = 0$

Solution. Seperating the variables, we have

$$\frac{dx}{x^2+4} - \frac{dy}{y^2+1} = 0$$

$$\Longrightarrow \frac{1}{2}\arctan\frac{x}{2} - \arctan y = c$$

Equation reducible to SDE:

The DE

$$\frac{dy}{dx} = f(ax+by+c) \qquad (a,\ b,\ c \ \text{const})$$

where f is a function of a single variable is reducible to a SDE upon the substitution

$$t = ax + by + c.$$

Indeed, differentiation given

$$\frac{dt}{dx} = a + b\frac{dy}{dx} \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{1}{t}\frac{dt}{dx} - \frac{a}{t}$$

$$\Longrightarrow \frac{1}{t}\frac{dt}{dx} - \frac{a}{t} = f(t) \quad \Longrightarrow \quad \frac{dt}{dx} - a = f(t)t$$

$$\Longrightarrow \quad \frac{dt}{dx} = a + f(t)t \quad \Longrightarrow \quad dx = \frac{dt}{a+f(t)dt} \quad \text{(SDE)}$$

Example. Solve

$$\frac{dy}{dx} = e^{2x-y+1} + 2x-y+1$$

Solution. Setting $t = 2x-y+1$, we have

$$y = 2x-t+1 \implies \frac{dy}{dx} = 2 - \frac{dt}{dx}$$

$$\implies 2 - \frac{dt}{dx} = e^t + t \implies \frac{dt}{dx} = 2 - e^t - t$$

$$\implies \frac{dt}{2-e^t-t} = dx$$

$$\implies x = \int \frac{dt}{2-e^t-t} = \varphi(t) + c$$

$$\implies t = 2\varphi(t) - y + 1 \implies y = 2\varphi(t) - t + 1$$

$$\text{GS:} \quad \left| \begin{array}{l} x = \varphi(t) + c \\ y = 2\varphi(t) - t + c + 1 \end{array} \right.$$

B. Homogeneous Differential Equations (HDE)

The DE

$$P(x, y)dx + Q(x, y) = 0 \qquad (1)$$

is called a *Homogeneous Differential Equations* if $P(x, y)$ and $Q(x, y)$ are homogeneous* of the same degree.

Thus the following are HDE:

$$(x^2 - 3xy)dx + (2x^2 + y^2)dy = 0$$

$$\frac{dx}{\sqrt{x^2 + y^2}} - \frac{dy}{y} = 0$$

Remark. Note that a DE of the form

$$\frac{dy}{dx} = f(\frac{y}{x}) \qquad (2)$$

is a HDE, since replacement of x, y by tx, ty $(t > 0)$ does not alter the equation.

(*) $f(x, y)$ is called homogeneous of degree γ into variables x and y if
$f(tx, ty) = t^{\gamma} f(x, y)$ for all $t > 0$

Method of solution.

The substitution

$$y = x\,t(x) \quad (\text{or} \quad x = y\,t(x))$$

transforms (1) into a SDE in the variables x and t.

Indeed,

$$y = xt \Longrightarrow dy = xdt + tdx$$

$$\Longrightarrow \quad P(x,\ tx)dx + Q(x,\ tx).(xdt + tdx) = 0$$

$$\Longrightarrow \quad x^{\gamma}P(1,\ t)dx + x^{\gamma}Q(1,\ t).(xdt + tdx) = 0$$

$$\Longrightarrow \quad \left[p(t) + q(t)t\right]dx + q(t)x\,dt = 0 \qquad \text{(SDE)}$$

Hence all HDE and ones reducible to HDE are reducible to SDE.

Example. Solve the HDE:

a) $y^2dx + (x^2-xy-y^2)dy = 0$ b) $\frac{dy}{dx} = (\frac{y}{x})^2 + \frac{y}{x}$

Solution.

a) Since the coefficient of dx is simpler, we set

$$x = t(y)y \text{ instead of } y = t(x)x$$

$$x = tdy \Longrightarrow dx = tdy + ydt$$

$$\Longrightarrow \quad y^2(tdy + ydt) + (t^2y^2 - ty^2 - y^2)dy = 0$$

$$\Longrightarrow \quad (tdy + ydt) + (t^2 - t - 1)dy = 0$$

$$\Longrightarrow \quad ydt + (t^2 - 1)dy = 0 \qquad \text{(SDE)}.$$

Then

$$\frac{dy}{y} + \frac{dt}{t^2-1} = 0 \quad \Longrightarrow \ell n\ y + \frac{1}{2}\ \ell n\ \frac{t-1}{t+1} = \ell n\ c$$

$$\Longrightarrow y\sqrt{\frac{t-1}{t+1}} = c \quad \Longrightarrow y\sqrt{\frac{\frac{x}{y} - 1}{\frac{x}{y} + 1}} = c$$

$$\text{GS:} \quad y\sqrt{\frac{x-y}{x+y}} = c.$$

b) Setting $y = tx$, we have

$$\frac{dy}{dx} = \frac{dt}{dx}x + t = t^2 + t$$

$$\Longrightarrow \quad \frac{dt}{dx}x = t^2 \Longrightarrow \frac{dt}{t^2} = \frac{dx}{x}$$

$$\Longrightarrow \quad -\frac{1}{t} = x + c \Longrightarrow x = -\frac{1}{t} + c$$

and

$$y = tx = -1 + ct$$

$$\text{GS: } x = -\frac{1}{t} + c, \quad y = ct - 1 \quad \text{(parametric)}$$

Differential Equations reducible to HDE:

The DE

$$\frac{dy}{dx} = f\left(\frac{ax + by + c}{dx + ey + f}\right),$$

where f is a function of a single variable and $a, b, \ldots, f$ are constant, is reducible to HDE:

Consider the straight lines

$$\ell_1: \quad ax + by + c = 0$$

$$\ell_2: \quad dx + ey + f = 0$$

1) If $\ell_1 // \ell_2$, from the proportionality

$$\frac{a}{d} = \frac{b}{e} \; (= \lambda) \quad \text{or} \quad a = \lambda d, \quad b = \lambda e$$

we have

$$\frac{dy}{dx} = f\left(\frac{\lambda(dx + ey) + c}{(dx + ey) + f}\right) = g(dx + cy)$$

which was shown to be reducible to SDE.

2) If ℓ_1, ℓ_2 intersect at a point (h, k) so that

$$ah + bk + c = 0, \quad dk + ek + f = 0 \qquad \text{(a)}$$

holds, by the substitution

$$\begin{matrix} x = u + h \\ y = v + k \end{matrix} \implies \begin{matrix} dx = du \\ dy = dv \end{matrix}$$

one translates the origin to (h, k).

Then by the use of (a), the linear expressions become

$$au + bv, \quad du + ev,$$

and

$$\frac{dy}{dx} = \frac{dv}{du} = f\left(\frac{au + bv}{du + ev}\right) = f\left(\frac{a + b\frac{v}{u}}{d + e\frac{v}{u}}\right) = g\left(\frac{v}{u}\right)$$

which is a HDE.

Example 1 Solve

$$(2x - 3y + 2)dx + (4x - 6y + 1)dy = 0$$

Solution. Since the lines $2x - 3y + 2 = 0$, $4x - 6y + 1 = 0$ are parallel, substitution $t = 2x - 3y$ gives

$$(t + 2)dx + (2t + 1)\left(\frac{2dx - dt}{3}\right) = 0$$

$$(3t + 6)dx + (2t + 1)(2dx - dt) = 0$$

$$(7t + 8)dx - (2t + 1)dt = 0$$

$$dx = \frac{2t + 1}{7t + 8}\, dt$$

$$x = \int \frac{2t + 1}{7t + 8}\, dt = \int \left(\frac{2}{7} - \frac{9/7}{(7t + 8)}\right)dt$$

$$= \frac{2}{7}t - \frac{9}{49} \ln\left(t + \frac{8}{9}\right) + c$$

$$x = \frac{2}{7}(2x - 3y) - \frac{9}{49}\left(2x - 3y + \frac{8}{9}\right) + c$$

Example 2. Reduce the following to HDE:

$$(2x - 3y + 1)dx + (y - 3x + 2)dy = 0$$

Solution. Since

$$\left.\begin{aligned} 2x - 3y + 1 &= 0 \\ -3x + y + 2 &= 0 \end{aligned}\right\} \quad \text{has the solution } (h, k) = (1, 1),$$

setting

$$x = u + 1, \quad y = v + 1; \quad dx = du, \quad dy = dv$$

we have

$$\left[2u + 2 - 3(v+1) + 1\right]du + \left[v + 1 - 3(u+1) + 2\right]dv = 0$$

$$\Longrightarrow \quad (2u - 3v)du + (v - 3u)dv = 0$$

C. Exact Differential Equations

The equation

$$P(x, y)dx + Q(x, y)dy = 0 \tag{1}$$

is called an *exact differential equations* (EDE) if the left hand side is the total differential of a function $u(x, y)$ called a primitive.

It follows that the condition for (1) to be exact is

$$P_y \equiv Q_x \quad \text{or} \quad P_y - Q_x \equiv 0 \tag{2}$$

(See Line Integral in $\mathbb{R}^2$).

Method of solution: Since (2) holds, there is a primitive $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = P, \quad \frac{\partial u}{\partial y} = Q \tag{3}$$

The DE becoming $du = 0$, the GS will be

$$u(x, y) = c$$

where c is an arbitrary constant.

u is determined by solving one of in (3), say $\frac{\partial u}{\partial x} = P$:

$$u(x, y) = \int P(x, y)dx + \varphi(y)$$

where integration is performed keeping y constant. Hence the constant of integration involves the variable y and is determined from the second of (3):

$$Q = u_y = \int P_y \partial x + \varphi'(y).$$

<u>Example 1</u>. Solve the differential equation

$$(2x \ln y + e^y)dx + (\frac{x^2}{y} + xe^y + y^2)dy = 0$$

<u>Solution</u>. Since

$$P_y - Q_x = (2x \frac{1}{y} + e^y) - (\frac{2x}{y} + e^y) \equiv 0,$$

the equation is exact. Hence a primitive $u(x, y)$ exists which is obtained on follows:

$$u(x, y) = \int (2x \ln y + e^y)\partial x$$

$$= x^2 \ln y + e^y x + \varphi(y)$$

$$\Longrightarrow \frac{\partial u}{\partial y} = \frac{x^2}{y} + e^y x + \varphi'(y) = Q = \frac{x^2}{y} + xe^y + y^2$$

$$\Longrightarrow \varphi'(y) = y^2 \Longrightarrow \varphi(y) = y^3/3 \quad (c = 0 \text{ is taken})$$

$$\Longrightarrow u(x, y) = x^2 \ln y + xe^y + \frac{y^3}{3}$$

$$\text{GS:} \quad x^2 \ln y + xe^y + \frac{y^3}{3} = c.$$

DIFFERENTIAL EQUATIONS SOLVABLE BY INTEGRATING FACTOR:

<u>Integrating factor</u>: For a non exact DE

$$P(x, y)dx + Q(x, y)dy = 0, \qquad (1)$$

if a function $\mu(x, y)$ exists such that

$$\mu Pdx + \mu Qdy = 0 \qquad (1')$$

is exact, then the factor $\mu(x, y)$ is called an *integrating factor* of (1).

From the exactness of (1'), we have

$$\frac{\partial}{\partial y}(\mu P) - \frac{\partial}{\partial x}(\mu Q) = 0$$

$$\Longrightarrow (\mu_y P + \mu P_y) - (\mu_x Q + \mu Q_x) = 0$$

$$\Longrightarrow \mu_x Q - \mu_y P - \mu(P_y - Q_x) = 0 \qquad (1")$$

which is a PDE in the unknown function μ, whose solu n is more difficult than that of (1). However if we take μ as a function of x or y alone, the PDE (1") becomes more simpler:

1) <u>Case $\mu = \mu(x)$</u>:

$$\mu' Q - 0 - \mu.(P_y - Q_x) = 0$$

$$\Longrightarrow \frac{\mu'}{\mu} = \frac{P_y - Q_x}{Q}$$

which is solvable if $(P_y - Q_x)/Q$ is a function of x alone. Then

$$\ell n\ \mu = \int \frac{P_y - Q_x}{Q}\, dx$$

$$\mu(x) = \exp \int \frac{P_y - Q_x}{Q}\, dx \qquad (2)$$

2) <u>Case $\mu = \mu(y)$</u>:

For this case one finds

$$\mu(y) = \exp \int -\frac{P_y - Q_x}{P}\, dy \qquad (2')$$

where the integrand is a function of y alone.

<u>Example 2</u>. Solve

$$(2xy\ \ell ny + ye^y)dx + (x^2 + xy\ e^y + y^3)dy = 0$$

by finding an integrating factor of the form $\mu(x)$ or $\mu(y)$.

<u>Solution</u>.

From

$$P_y - Q_x = (2x \ \ln y + 2x + e^y + ye^y)$$
$$-(2x + y + e^y) = 2x \ \ln y + e^y$$

we observe that $(P_y - Q_x)/P = \frac{1}{y}$. Hence

$$\mu(y) = \exp \int -\frac{1}{y}\, dy = \exp(-\ln y) = e^{-\ln y} = \frac{1}{y}$$

Then multiplying the terms of the DE one gets the EDE

$$(2x \ \ln y + e^y)dx + (\frac{x^2}{y} + xe^y + y^2)dy = 0$$

where the GS was obtained as

$$x^2 \ln y + xe^y + \frac{y^3}{3} = c$$

in Example 1.

D. Linear Differential Equations

The equation of the form

$$A_0(x)\frac{dy}{dx} + A_1(x)y + A_2(x) = 0 \tag{1}$$

linear in dy/dx and y is called a *linear differential equation* (LDE).

Such a LDE (1) is usually expressed in its standard form*

$$\frac{dy}{dx} + p(x)y = f(x) \tag{1'}$$

obtained from (1) through division by $A_0(x)$.

(1') is solved by putting it in differential form

$$\left[p(x)y - f(x)\right]dx + dy = 0$$

and by finding an integrating factor of the form $\mu(x)$:

(*) Changing the roles of x and y, the equation

$$\frac{dx}{dy} + p(y)dx = f(y)$$

is also linear in dx/dy and x.

$$P_y - Q_x = p(x) \implies \frac{P_y - Q_x}{Q} = p(x) \implies$$

$$\mu(x) = e^{\int p(x)dx} \qquad (2)$$

Then

$$\mu(x)\frac{dy}{dx} + \mu p(x)y = \mu f(x)$$

is exact.

Since

$$\frac{d\mu}{dx} = \frac{d}{dx}e^{\int p\,dx} = e^{\int p\,dx}p = \mu p,$$

we have

$$\mu\frac{dy}{dx} + \frac{d\mu}{dx}y = \mu f(x)$$

$$\implies \quad \frac{d}{dx}\mu y = \mu f(x)$$

$$\implies \quad \mu y = \int \mu f(x)dx + c \implies$$

$$\implies \quad y = \frac{1}{\mu}\left[\int \mu f(x)dx + c\right] \qquad (3)$$

which is the GS of (1). Explicitly

$$y = e^{-\int p(x)dx}\left[\int e^{\int p(x)dx}f(x)dx + c\right]$$

Example 1. Solve

a) $\frac{dy}{dx} + \frac{1}{x}y = e^x$ b) $(y+1)dx + (2x-y-y^2)dy = 0$

Solution.

a) Since it is linear in dy/dx and y, from

$$\mu(x) = e^{\int \frac{dx}{x}} = e^{\ln x} = x$$

and (3), we have

$$y = \frac{1}{x}\left[\int x\, e^x dx + c\right]$$
$$= \frac{1}{x}(x\, e^x - e^x + c)$$
$$= -\frac{x-1}{x} e^x + \frac{c}{x}.$$

b) Arranging it in dy/dx and y as

$$\frac{dy}{dx} + \frac{y+1}{x-y-y^2} = 0,$$

we see that it is non linear. Arranging in dx/dy and x we have

$$\frac{dx}{dy} + \frac{2}{y+1} x = y$$

which is linear. Then from

$$\mu(y) = e^{\int \frac{2}{y+1} dy} = e^{2\, \ell n(y+1)} = (y+1)^2$$

and (3) we have

$$x = \frac{1}{(y+1)^2}\left[\int (y+1)^2 y\, dy + c_1\right]$$
$$= \frac{3y^4 + 8y^3 + 6y^2 + c}{12(y+1)^2}$$

EQUATIONS REDUCIBLE TO LDE:

1. BERNOULLI Differential Equation (BDE):

$$\frac{dy}{dx} + p(x)y = f(x)y^{\alpha}.$$

This DE is linear when $\alpha = 0$ or $\alpha = 1$. So we suppose $\alpha \neq 0$, $\alpha \neq 1$. It is reducible to linear upon the substitution

$$y = u^{\frac{1}{1-\alpha}} \quad \text{or} \quad u = y^{1-\alpha}$$

Indeed, setting

$$\frac{dy}{dx} = \frac{1}{1-\alpha}\, u^{\frac{\alpha}{1-\alpha}} \frac{du}{dx}$$

in the DE, we have

$$\frac{1}{1-\alpha}\, u^{\frac{\alpha}{1-\alpha}} \frac{du}{dx} + p(x).u^{\frac{1}{1-\alpha}} = f(x)\, u^{\frac{\alpha}{1-\alpha}}$$

$$\Longrightarrow \quad \frac{du}{dx} + (1-\alpha)p(x)u = (1-\alpha)f(x) \qquad \text{(LDE)}$$

It can be shown that the BERNOULLI DE becomes a SDE by the substitution

$$y = v(x)\, e^{-\int p\, dx}$$

Example. Solve the BDE

$$\frac{dy}{dx} + xy = e^{x^2} y^3 \qquad (\alpha = 3)$$

Solution. Substituting $y = u^{\frac{1}{1-3}} = u^{-\frac{1}{2}}$, we have

$$\frac{dy}{dx} = -\frac{1}{2} u^{-\frac{3}{2}} \frac{du}{dx} \quad \text{and}$$

$$-\frac{1}{2} u^{-\frac{3}{2}} \frac{du}{dx} + x\, u^{-\frac{1}{2}} = e^{x^2} u^{-\frac{3}{2}},$$

$$\Longrightarrow \quad \frac{du}{dx} - 2x\, u = -2e^{x^2}$$

$$\mu(x) = e^{-2\int x dx} = e^{-x^2}$$

$$u = e^{x^2}\left(\int e^{-x^2} e^{x^2} dx + c\right) = e^{x^2}(x+c)$$

$$y = u^{-\frac{1}{2}} = \frac{1}{e^{x^2/2}\sqrt{x + c}}$$

2. RICCATI Differential Equation (RDE).

$$\frac{dy}{dx} + p(x)y = f(x)y^2 + g(x) \qquad (1)$$

This DE is usually written as

$$\frac{dy}{dx} = A(x)y^2 + B(x)y + C(x) \qquad (1')$$

so that the derivative is a quadratic expression in y.

The RDE cannot be solved unless a particular solution is known. If y_1 is a PS of (1') the transformation

$$y = y_1 + u(x)$$

transforms it to a BDE with $\alpha = 2$ which in turn, is transformed into a LDE by $u = t^{-1}$. Hence the transformation

$$y = y_1 + \frac{1}{v}$$

reduces (1') into a LDE.

Indeed,

$$\frac{dy_1}{dx} - \frac{v'}{v^2} = A(y_1 + \frac{1}{v})^2 + B(y_1 + \frac{1}{v}) + C$$

$$\Longrightarrow \quad y_1' - \frac{v'}{v^2} = (Ay_1^2 + By_1 + C) + (\frac{2Ay_1 + B}{v}) + \frac{A}{v^2} +$$

$$\Longrightarrow \quad -v' = (2Ay_1 + B)v + A$$

$$\Longrightarrow \quad v' + (2Ay_1 + B)v = -A(x) \qquad \text{(LDE)}$$

Example. Given the RDE

$$y' + y^2 + 2xy + x^2 + 1 = 0.$$

a) find a PS of the form $y_1 = ax$ (a const)

b) Solve.

Solution.

a) $a + a^2x^2 + 2x.ax + x^2 + 1 = 0$

$\Longrightarrow (a^2 + 2a + 1)x^2 + (a+1) = 0$

$\Longrightarrow a^2 + 2a + 1 = 0,\ a+1 = 0 \quad \Longrightarrow a = -1$

$\Longrightarrow y_1 = -x.$

b) Setting

$$y = -x + \frac{1}{v}$$

we have

$$-1 - \frac{v'}{v^2} + (-x + \frac{1}{v})^2 + 2x(-x + \frac{1}{v}) + x^2 + 1 = 0$$

$$\Longrightarrow \quad -1 - \frac{v'}{v^2} + x^2 - \frac{2x}{v} + \frac{1}{v^2} - 2x^2 + \frac{2x}{v} + x^2 + 1 = 0$$

$$\Longrightarrow \quad -\frac{v'}{v^2} + \frac{1}{v^2} = 0 \Longrightarrow v' = 1 \Longrightarrow v = x + c \Longrightarrow$$

$$\text{GS:} \quad y = -x + \frac{1}{x + c}.$$

DIFFERENTIAL EQUATIONS LINEAR IN X AND Y:

Consider the DE

$$A(y')x + B(y')y + C(y') = 0$$

linear in x and y, which is usually written as solved for y:

$$y = x\,\phi(y') + \psi(y')$$

This DE is called a *CLAIRAUT Differential equation* (CDE) if $\phi(y') \equiv y'$, and LAGRANGE Differential equation otherwise.

1. <u>CLAIRAUT DE</u>: $y = xy' + \psi(y')$ (1)

One sets $y' = p$ in (1) obtaining

$$y = xp + \psi(p)$$

which when differentiated once gives

$$p = p + x\frac{dp}{dx} + \psi'(p)\frac{dp}{dx}$$

$$\Longrightarrow \left[x + \psi'(p)\right]\frac{dp}{dx} = 0$$

$$\Longrightarrow \quad x = -\psi'(p) \quad \text{or} \quad \frac{dp}{dx} = 0$$

The first of which gives a parametric solution

$$\begin{aligned} x &= -\psi'(p) \\ y &= xp + \psi(p) \end{aligned} \qquad (2)$$

This solution not involving an arbitrary constant cannot the GS. The second gives $p = c$ or the solution

$$y = cx + \psi(c) \qquad (3)$$

which the GS.

The solution (2) cannot be obtainable from the GS (3) by a choice of the arbitrary constant c. Hence (2) is not a solution. It is called a *singular solution* of (1). The graph of this singular solution can be seen to be the envelope of the family (3).

Summarizing, the GS of (1) is obtained by replacing y' by an arbitrary constant in the DE and the envelope of (3) is the singular solution.

Example. Find the singular and general solution of the CDE:

$$y = xy' + \frac{1 + y'}{1 - y'}$$

Solution. The GS is obtained by replacing y' by c:

$$\text{GS:} \quad y = cx + \frac{1 + c}{1 - c}$$

$$\text{SS:} \quad x = -2/(1-p)^2, \quad y = xp + (1+p)/(1-p)$$

2. LAGRANGE Differential Equation (Lg DE):

$$y = x\,\phi(y') + \psi(y'), \quad (\phi(y') \not\equiv y') \qquad (1)$$

Setting $y' = p$ in (1) and considering p as a new variable, differentiation of (1) gives

$$p = \phi(p) + x\,\phi'(p)\,\frac{dp}{dx} + \psi'(p)\,\frac{dp}{dx}$$

$$\Longrightarrow \quad \Big[\phi(p) - p\Big]dx + \Big[x\,\phi'(p) + \psi'(p)\Big]dp = 0 \qquad (1')$$

Integrating factor:

$$P_p - Q_x = \Big[\phi'(p) - 1\Big] - \phi'(p) = -1$$

$$\Longrightarrow \quad \mu(p) = e^{\int \frac{dp}{\phi - p}}$$

yielding $x = x(p)$.

So the GS of (1) is the parametric function

$$\left|\begin{array}{l} x = x(p) \\ y = x\,\phi(p) + \psi(p) \end{array}\right.$$

<u>Example</u>. Solve the Lg DE

$$y = xy'^2 + 1 - y'$$

<u>Solution</u>. Setting $y' = p$ in the DE and differentiating $y = xp^2 + 1 - p$ with respect to x we have

$$p = p^2 + 2xpp' - p'$$

$$(p^2 - p)dx + (2xp - 1)dp = 0$$

with an integrating factor

$$\mu(p) = \exp\int \frac{dp}{p^2 - p} = \exp \ln \frac{p-1}{p} = \frac{p-1}{p}$$

Then

$$\frac{p-1}{p}(p^2 - p)dx + \frac{p-1}{p}(2xp-1)dp = 0 \qquad \text{or}$$

$$(p-1)^2 dx + \frac{p-1}{p}(2xp-1)dp = 0$$

is exact and solved as follows:

$$(p-1)^2 dx + 2(p-1)x\,dp = \frac{p-1}{p}\,dp$$

$$\Longrightarrow \quad d\Big[(p-1)^2 x\Big] = d\int \frac{p-1}{p}\,dp$$

$$\Longrightarrow \quad (p-1)^2 x = (p - \ln p) + c$$

$$\Longrightarrow \begin{cases} x(p) = \dfrac{p - \ln p + c}{(p-1)^2} \\ y(p) = xp^2 + 1 - p \end{cases}$$

EXERCISES (6. 2)

11. Find the GS of the following SDE:

   a) $2y\,dy + 4x^2 \sqrt{4-y^2}\,dx = 0$, b) $\dfrac{\ln y}{\ln y}\,dy - \dfrac{x^4}{y^2}\,dx = 0$

12. Same question for:

   a) $x^3 dy - x^3 dx = dx$; b) $(1+y^2)dx + x(x+1)dy = 0$

13. Same question for:

   a) $3x^2 - 2y^3 y' = 0$, b) $\sin\Theta\, dr + r\cos\Theta\, d\Theta = 0$

14. Solve

   a) $\dfrac{dy}{dx} = \cos(x-y)$ b) $\dfrac{dx}{dt} = e^{x/t} + \dfrac{x}{t}$

15. Find the PS under the given condition:

   a) $dr = r\tan\Theta\, d\Theta$; $r = 1$ when $\Theta = 0$,

   b) $e^x \sec y\, dx + (1+e^x)\sec y \tan y\, dy = 0$, $y(3) = \pi/3$

16. Find the GS of the following HDE:

   a) $\left(\dfrac{1}{x} - \dfrac{y}{x^2} e^{y/x}\right)dx + \left(\dfrac{1}{x} e^{y/x} - \dfrac{1}{y}\right)dy = 0$

   b) $(x\sqrt{x^2 + y^2} - y^2)dx + xy\,dy = c$

17. Solve

   a) $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}$ b) $x(\ln x - \ln y)dy - y\,dx = 0$

18. Find the GS of the following HDE:

   a) $2\,y dx - (x^2-y^2)dy = 0$ b) $y dx = x dy - \sqrt{x^2+y^2}\,dx$

19. Same question for:

a) $dy = (\frac{y}{x} - \csc^2 \frac{y}{x})dx$, b) $(3x^2+2xy+4y^2)dx+(20x^2+6xy+y^2)dy=0$

20. Find the PS under the given condition:

a) $(x + y)dy - y\,dx = 0$; $y(0) = 1$

b) $(x^2 + 2xy - y^2)dx + (y^2 + 2xy - 2x^2)dy = 0$, $y(0) = 3$

21. Find the GS of the following EDE:

a) $(2x^2 + 5xy^2)dx + (5x^2y - 2y^4)dy = 0$

b) $\mathrm{Sec}^2 x \tan y\,dx + \mathrm{Sec}^2 y \tan x\,dy = 0$

22. Same question for:

a) $\arcsin y\,dx + \dfrac{x + 2\sqrt{1 - y^2}\cos y}{\sqrt{1 - y^2}}\,dy = 0$

b) $(x \ln y + y \ln x + y)dx + (\frac{x^2}{2y} + x \ln x)dy = 0$

23. Same question for:

a) $(y \,\mathrm{ch}\, xy + \mathrm{ch}\, y)dx + (x \,\mathrm{ch}\, xy + x \,\mathrm{Sh}\, y)dy = 0$

b) $(\sin x + \sin y)dx + (x \cos y + \cos y)dy = 0$

24. Find the PS passing through the given point:

a) $(2x \sin y + 2x + 3y \cos x)dx + (x^2 \cos y + 3 \sin x)dy = 0$, $(\pi/2, 0)$

b) $(ye^{2x} - 3xe^{2y})dx + (\frac{1}{2} e^{2x} - 3x^2e^{2y} - e^y)dy = 0$; $(1, 0)$

25. Find an integrating factor $\mu(xy)$ and solve

$$x\,dy + y\,dx = x^2y^2dx$$

26. Find the GS of the following DE by finding and integrating factor:

a) $dx + (x \tan y - 2 \sec y)dy = 0$ b) $(2-xy)y\,dx + (2+xy)x\,dy = 0$

27. Find the GS:

a) $x^3 dy - x^2 y\, dx = x^5 y\, dx$ b) $x dy - y dx = x^2 \sqrt{x^2 - y^2}\, dx$

28. Find the GS of the following LDE:

a) $\sin x \frac{dy}{dx} - y = 5$ b) $\sin x \frac{dy}{dx} - y = \cos x$ c) $x \frac{dy}{dx} - y = x^2$

29. Same question for:

a) $x^2 y' + xy = x^4 + x^2$ b) $y' + 4y = x^2$

30. Same question for:

a) $dx + 2x dy = y dy$ b) $dx - x dy = \ln y . dy$

31. Find the type and then solve:

a) $\frac{dy}{dx} + y = 2 + 2x$ b) $\frac{dy}{dx} + y = y^2 e^x$

c) $yy' + y^2 \tan x = \cos^2 x$ d) $y' = -2y^2 + \frac{6}{x^2}$ (PS: $y_1 = a/x$)

32. Find the GS:

a) $(2-x-y)dx + (3+x+y)dy = 0$ b) $(2+3x-5y)dx + 7dy = 0$

33. Solve

$$x \frac{dy}{dx} + y = x^2 y^2$$

34. Solve:

a) $xy(x dy + y dx) = 4x^3 dx$

b) $(1 + e^{y/x})dy + (1 - \frac{y}{x})e^{y/x}\, dx = 0$ (Hint: Set $y/x = z$)

35. Find the GS and the singular solution of the CDE:

a) $y = xy' + 2y'^2$ b) $xy'^2 - yy' + 1 = 0$

ANSWERS

11. a) $y^2 = 2 \sin(r - x^4)$

b) $\frac{1}{3} y^2 \ln|y| - \frac{1}{9} y^3 = \frac{x^5}{5} \ln|x| - \frac{x^5}{25} + c$

12. a) $y - x + \dfrac{1}{2x^2} = c,$ b) $\ln \dfrac{x}{x+1} \quad \arctan y = c$

13. a) $x^3 - y^4/2 = c,$ b) $r \sin\Theta = c$

14. a) $x + \cot \dfrac{x-y}{2} = c,$ b) $\ln t = c - e^{-x/t}$

15. a) $r = \sec\Theta,$ b) $1 + e^x = 2(1 + e^3)\cos y$

16. a) $y = cx,$ b) $x \ln|x| + \sqrt{x^2 + y^2} = cx$

17. a) $\sqrt{x^2 + y^2} = ce^{-\arctan \frac{y}{x}},$ b) $x = ye^{cy+1}$

18. a) $x^2 + y^2 = cy,$ b) $cx^2 = y + \sqrt{x^2 + y^2}$

19. a) $2y - x \sin \dfrac{2y}{x} + 4x \ln x = cx$ b) $x^2 + 7xy + y^2 = c(3x + y)$

20. a) $y \ln y = x,$ b) $x^2 - xy + y^2 - 3(x+y) = 0$

21. a) $20x^3 + 75x^2y - 12y^5 = c,$ b) $\tan x \tan y = c$

22. a) $x \arcsin y + 2 \sin y = c$ b) $x^2 \ln|y| + 2xy \ln|x| = c$

23. a) $\mathrm{Sh}\, xy + x\, \mathrm{Ch}\, y = c,$ b) $(x+2)\mathrm{Sin}\, y - \cos x = c$

24. a) $x^2 \sin y + x^2 + 3y \sin x = \pi^2/4$ b) $ye^{2x} - 3x^2e^{2y} - 2e^y + 5 = 0$

25. $\mu = x^{-2}y^{-2}, \quad x^2y + cxy + 1 = 0$

26. a) $x \sec y - 2 \tan y = c$ b) $\dfrac{2}{xy} + \ln|x| - \ln|y| = c$

27. a) $\ln \dfrac{y}{x} = \dfrac{1}{3} x^3 + c,$ b) $\arcsin \dfrac{y}{x} = \dfrac{1}{2} x^2 + c$

28. a) $y = c \tan \dfrac{x}{2} - 5,$ b) $y = -1 - x \tan \dfrac{x}{2} + c \tan \dfrac{x}{2}$ c) $y = x^2 + cx$

29. a) $4xy = x^4 + 2x^2 + c,$ b) $y = ce^{-4x} + \dfrac{x^4}{4} - \dfrac{x}{8} + \dfrac{1}{32}$

30. a) $4x = 2y - 1 + ce^{-2y}$ b) $x + \ln y = e^y\left(\int \dfrac{1}{y} e^{-4}\, dy + c\right)$

31. a) $y = 2x + ce^{-x},$ b) $x = -e^{-y}/(y + c)$

c) $y^2 = (2x + c)\cos^2 x$ d) $y = \dfrac{2}{x} + \dfrac{7}{cx^8 - 2x}$

32. a) $2y - x - 5\ \ln(2x + 2y + 1) = c$ b) $5x - 7\ \ln(3x - 5y + \frac{31}{5}) = c$

33. $x^2y + cxy + 1 = 0$

34. a) $x^2y^2 = 2x^4 + c$, b) $y + xe^{y/x} = c$

35. a) $y = cx + 2c^2$, b) $y = -x^2/8$ c) $y = cx + \frac{1}{c}$ d) $y^2 = 4x$

6.3 Some Problems Leading to First Order Differential Equations

GEOMETRIC PROBLEMS (Trajectories)

A. Determination of the family of curves under a given condition involving first order derivative

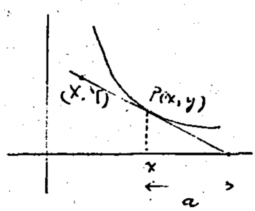
A condition involving the first order derivative is expressible by a relation

$$f(x, y, y') = 0$$

which is an ordinary differential equation where the GS is the required family.

Example. Find the family of curves having constant subtangent.

Solution. Let $y = f(x)$ be a member of the family and let $P(x, y)$ be a point on the curve.



Tangent-line: $Y - y = f'(x)(X - x)$

y-intercept : $0 - y = y' \cdot (X - x)$

$$\Longrightarrow \qquad X - x = \frac{y}{y'}$$

Setting $|X - x| = a$ (const), we have

$$a = \left|\frac{y}{y'}\right|$$

$$\Longrightarrow \quad \frac{y'}{y} = \pm \frac{1}{a} \quad \Longrightarrow \quad \ln y = \pm \frac{1}{a} x + \ln c$$

$$\Longrightarrow \quad y = ce^{\pm x/a}$$

B. Determination of the family which is in a relation to a given family (at all common points).

Let

$$f(x, y, c) = 0$$

be a given family of curves $\Gamma_1(c)$ where DE is

$$F(x, y, y') = 0.$$

The given condition and $F = 0$ yields a DE

$$G(x, y, y') = 0$$

where solution.

$$g(x, y, c) = 0$$

is the equation of the required family of curves $\Gamma_2(c)$.

One of $\Gamma_1(c)$, $\Gamma_2(c)$ is said to be the *trajectory* of the other under the given condition. If the condition between them is the orthogonality then the trajectories are called *orthogonal trajectories*.

Example 1. Find the orthogonal trajectories of the family of parabolas with common axis as the y-axis, and passing through the origin.

Solution. The equation of the given family is

$$\Gamma_1 : y = cx^2$$

Then the DE of the family is obtained by elementary c between $y = cx^2$ and $y' = 2cx$:

$$y' = 2\,\frac{y}{x}$$

Hence the differential equations of the orthogonal trajectories becomes

$$-\frac{1}{y'} = 2\,\frac{y}{x} \implies 2y\,dy + x\,dx = 0 \quad (SDE)$$

since $y'_{\Gamma_1}\, y'_{\Gamma_2} = -1$, and its solution is

$$\Gamma_2 : \frac{x^2}{2} + y^2 = c \quad (c > 0) \quad \text{(family of ellipses)}$$

Example 2. Find the orthogonal trajectories of the family of cardiods

$$\Gamma_1(c) : \quad r = c + c\cos\Theta$$

Solution. The DE of $\Gamma_1(c)$ is obtained by eliminating c between

$$r = c + c\cos\Theta \quad \text{and} \quad \frac{dr}{d\Theta} = -c\sin\Theta :$$

$$\text{DE: } \frac{dr}{d\Theta} = \frac{-r}{1 + \cos\Theta}\sin\Theta \quad \text{or} \quad \frac{r}{r'} = -\frac{1 + \cos\Theta}{\sin\Theta}$$

Then the DE of the orthogonal trajectories will be

$$\frac{r'}{r} = \frac{1 + \cos\Theta}{\sin\Theta}$$

Since $\mu_{\Gamma_1}\,\mu_{\Gamma_2} = -1$ (See Book 1/2 p.535). Separating the variables and integrating, we have

$$\frac{dr}{r} = \frac{1 + \cos\Theta}{\sin\Theta}\,d\Theta \implies \ln r = \int \frac{2\cos^2\frac{\Theta}{2}}{2\sin\frac{\Theta}{2}\cos\frac{\Theta}{2}}\,d\Theta$$

$$\ln r = \int \frac{2d\,\sin\frac{\Theta}{2}}{\sin\frac{\Theta}{2}}$$

$$\Longrightarrow \ell n\, r = \int \frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}d\theta = -\int \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\,d\theta$$

$$= -\int \frac{2d\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}} = 2\,\ell n \sin\frac{\theta}{2} + \ell n\, c$$

$$\Longrightarrow \quad r = \frac{c}{\sin^2\frac{\theta}{2}} = \frac{2c}{1-\cos\theta} \quad \text{(parabolas)}$$

Physical Problems

A. Population Problems

The law governing the problem is the following: *The time rate of change at time t is proportional to the amount present at time t.* When this law is formulated mathematically we arrive at a DE:

$x(t)$: The amount of population at time t

$\frac{dx}{dt}$: The time rate of change or the rate of change of $x(t)$ with respect to time :

DE : $\frac{dx}{dt} = kx$ where k is the constant of population. Population increases (decreases) when $k>0$ $(k<0)$.

Note that rate problems on interest and on decay of radioactive elements are the same as population problems.

Example. The population in Turkey is 41.000.000 in the year 1975, and increases 3 % per unit time. Estimate the population in the year 2000!

Solution. The DE of the problem is the SDE

$$\frac{dx}{dt} = kx \quad \text{or} \quad \frac{dx}{x} = kdt$$

where GS is

$$x(t) = ce^{kt}$$

where the constants are determined by the given data:

If in $dx/x = kdt$ dt is unit, k becomes 3/100. Then

$$x(t) = ce^{0,03t}$$

Considering 1975 as initial time $t = 0$, we have

$$x(0) = c = 41.000.000$$

Then

$$x(t) = 41.000.000\, e^{0,03\, t}$$

$$\Longrightarrow \quad x(25) = 41.10^6.e^{0,75}$$

$$= 41.10^6.2,0923$$

$$\cong 86.000.000$$

since 2000-1975 = 25.

B. COOLING OF A BODY

The phenomenon of cooling of a hot body in a given medium is governed by NEWTON's law of cooling: *The time rate of cooling of a body at time t is proportional to the difference between the temperature of the body and that of the medium at time t.*

Formulation:

$x(t)$: temperature of the body at time t,

a : temperature of the medium, usually taken as constant

$\frac{dx}{dt}$: The time rate of cooling at time t

DE : $\frac{dx}{dt} = -k(x-a) \qquad (k > 0)$

Example. A piece of metal of temperature of $80^{o}c$ is placed at time $t = 0$ in a medium of constant temperature of $20^{o}c$. At the end of 5 min the metal has cooled down to $70^{o}c$. What will the temperature be at the end of 10 min?

Solution. From the SDE

$$\frac{dx}{dt} = -k(x-20)$$

we have

$$\frac{dx}{x - 20} = -k\ dt$$

$$\Longrightarrow \quad \ln(x-20) = -kt + \ln c.$$

The constants c and k are determined from $x(0) = 80$ and $x(5) = 70$:

$$\ln(80-20) = \ln c \quad \Longrightarrow \quad c = 60$$

$$\ln(70-20) = -5k + \ln 60 \quad \Longrightarrow \quad -5k = \ln \frac{5}{6}$$

Then

$$\ln(x-20) = \frac{1}{5}\,(\ln \frac{5}{6})t + \ln 60$$

$$\ln(x-20) = \ln(\frac{5}{6})^{t/5} + \ln 60$$

$$x = 20 + 60\ (\frac{5}{6})^{t/5}$$

$$\Longrightarrow x(10) = 20 + 60 \cdot \frac{25}{36} \cong 61.5^{o}$$

C. STREAM LINES (VECTOR LINES)

Let

$$F = P(x, y)i + Q(x, y)j$$

be a vector field in $\mathbb{R}^2$. We define a *vector line* (stream line) of F a curve at every point of which the corresponding vector of the field is tangent to the curve at that point.

$$\text{DE: } \frac{dx}{P} = \frac{dy}{Q} \quad \text{or}$$

$$Q(x, y)dx - P(x, y)dy = 0$$

The orthogonal trajectories of the vector lines are called *equipotential curves* of the field with DE. $Pdx + Qdy = 0$.

Example. Find the family of vector lines (stream lines) of the vector field.

$$F = 2xy\, i - (x^2 - y^2)j$$

and the equipotential curves.

Solution. $\dfrac{dx}{2xy} = \dfrac{dy}{-(x^2 - y^2)}$

$$\Longrightarrow \quad (x^2 - y^2)dx + 2xy\, dy = 0$$

$$\text{GS: } x^2 + y^2 = cx \quad \text{(circles)}$$

Then the DE of the equipotential curves will be

$$(x^2 - y^2)dy - 2xy\, dx = 0$$

with solution

$$x^2 + y^2 = cy$$

EXERCISES (6. 3)

36. Find the DE of the family of parabolas having the origin as focus and $x = -p$ as directrix.

37. Find the curve having length of subnormal equal to 3 and passing through the point (1, 4).

38. Find the equation in polar coordinates of the curves such that the tangent of the angle ψ between the radius vector

and the tangent line is equal to the square of the radius vector.

39. Find the equation of the curves such that its slope at any point $P(x, y)$ is equal to the difference of the squares of the distances of $P(x, y)$ from the points $(1, 0)$ and $(4, 0)$.

40. Find the orthogonal trajectories of the following family:

 a) $y^2 = 4ax$, b) $xy = c$.

41. Same question for:

 a) $r = c \cos\Theta$ b) $(1 + 2 \cos\Theta)r = 2$

42. Same question for:

 a) $2x^2 + 3y^2 = c$ b) $x^2 + y^2 = cx$

43. Find the stream line through $A(3, 1)$ of the vector field:

$$F = (2xy, x^2-y^2)$$

44. The surface area A of a snowball increases proportional to the area at that moment. If $A = 4\pi$ dm^2 at $t = 0$ and 64π dm^2 at $t = 2$ min, find the area at $t = 3$ min.

45. A body falls from rest through air. If air resistance is proportional to the square of the velocity, determine its equation of motion $s = s(t)$.

ANSWERS

36. $yy'^2 + 2xy' - y = 0$
37. $y^2 = 6x + 10$
38. $r^2 = \pm 20 + c$

39. $y = 3x^2 - 15x + c$

40. a) $y^2 = ce^{-x/a}$, b) $y^2 - x^2 = c^2$

41. a) $r = c \sin\Theta$, b) $r^2(1-\cos\Theta)\sin\Theta = c$

42. $y^2 = cx^3$, b) $x^2 + y^2 = cy$

43. $x^3 - 3xy^2 - 18 = 0$

44. $192\pi \text{ dm}^2$

45. $s = \frac{1}{c} \ln \operatorname{ch}(act)$, $a = \sqrt{g/c}$

6. 4 Linear Differential Equations of Higher Order with Constant Coefficients

A. DEFINITIONS

A differential equation of the form

$$A_{o(x)} \frac{d^n y}{dx^n} + \ldots + A_{n-1}(x) \frac{dy}{dx} + A_{n(x)} y = f(x) \tag{1}$$

is called a *LDE of order* n (which is linear in the dependent variable y and all its derivatives).

(1) is *homogeneous (non homogeneous)* LDE if $f(x) \equiv 0$ $(f(x) \not\equiv 0)$.

The LDE (1) may be expressed as

$$\left(A_o D^n + \ldots + A_{n-1} D + A_n\right) y = f(x) \tag{2}$$

or

$$P(D) y = f(x) \tag{2'}$$

where the operator $P(D)$ is linear.

When all the coefficients $A_o(x)$, ... , $A_n(x)$ are constants, we have a LDE with constant coefficients.

In the following, only LDE with constant coefficients will be treated.

B. Solution of a HLDE

Let

$$P(D)y = (a_0 D^n + \ldots + a_{n-1}) + a_n)y = 0 \tag{3}$$

be a given HLDE with constant coefficients a_i.

This HLDE admits the *trivial solution* $y = 0$, and non trivial solutions are obtained by the substitution

$$y = e^{\lambda x} \qquad (\lambda \text{ constant})$$

yielding the polynomial equation

$$P(\lambda) = 0, \tag{4}$$

Since

$$y = e^{\lambda x},\ D_y = \lambda e^{\lambda x}, \ldots, D^n y = \lambda^n e^{\lambda x},$$

where $P(\lambda) = 0$ is the *auxiliary equation.*

The auxiliary equation (4) has n complex roots $\lambda_1, \ldots, \lambda_n \varepsilon \mathbb{C} \quad (\supset \mathbb{R})$, simple or multiple.

The GS of (3) can be obtained by the use of the following theorem:

Theorem. Let $P(\lambda) = 0$ be the auxiliary equation of a HLDE (3) of order n. Then

1. if $\lambda = a$ is simple real root, the corresponding solution is

$$c\, e^{ax}$$

2. if $\lambda = a + ib$ $(b \neq 0)$ is an imaginary simple root ($\Longrightarrow \bar{\lambda} = a - ib$ is a root too), then the corresponding

solution is

$$c^{ax}\ (c_1 \cos bx + c_2 \sin bx)$$

3. if $\lambda = a$ is a real r-fold root, then the corresponding solution is

$$(c_1 + c_2 x + \ldots + c_r\ x^{r-1})e^{ax}$$

4. if $\lambda = a + ib$ $(b \neq 0)$ is an r-fold imaginary root, then the corresponding solution is

$$e^{ax}\Big[(c'_1 + c'_2 x + \ldots + c'_r x^{r-1})\cos bx$$
$$+(c''_1\ \ c''_2 x + \ldots + c''_r x^{r-1})\sin bx\Big]$$

and the GS is the sum of the solutions corresponding to all roots.

For instance, if the equation is

$$(D-4)(D+2)^3(D^2-6D+13)^2 y = 0,$$

which is of order 8, the auxiliary equation being

$$P(\lambda) = (\lambda-4)(\lambda+2)^3(\lambda^2-6\lambda+13)^2 = 0$$

the roots are 4, -2, 3+2i, 3-2i of multiplicities 1, 3, 2, 2 respectively. Then by the theorem the solutions corresponding to 4, -2 and $3 \pm 2i$ are

$$c_1\ e^{4x},\ (c_2 + c_3 x + c_4 x^2)e^{-2x} \quad \text{and}$$
$$e^{3x}\Big[(c_5 + c_6 x)\cos 2x + (c_7 + c_8 x)\sin 2x\Big].$$

Hence the GS is

$$y = c_1\ e^{4x} + (c_2 + c_3 x + c_4 x^2)e^{-2x}$$
$$+\ e^{3x}\Big[(c_5 + c_6 x)\cos 2x + (c_7 + c_8 x)\sin 2x\Big].$$

C. SOLUTION OF NHLE:

Let

$$P(D)y = (a_0D^n + \ldots + a_{n-1}D + a_n)y = f(x) \qquad (1)$$

be a NHDE with constant coefficients.

We call the HDE

$$P(D)y = 0$$

the *reduced equation* of (1) and it GS is said to be the *complementary solution* of (1), written y_c.

We show that the GS of (1) is the sum

$$y = y_c + y_p$$

where y_p is any particular solution of (1). Indeed,

$$P(D)y = P(D)\left(y_c + y_p\right) = P(D)y_c + P(D)y_p = 0 + f(x) = f(x)$$

and since y_c involves n arbitrary constants it is the GS.

Let (1) has the form

$$P(D)y = f_1(x) + \ldots + f_k(x) \qquad (1')$$

where non of $f_i(x)$ is a constant multiple of any other.

We state the following:

If y_1 , ... , y_k are particular solution of the DE's

$$P(D)y = f_1(x) \ , \ \ldots \ , \ P(D)y = f_k(x)$$

respectively, then a particular solution of (1) is the sum

$$y_p = y_1 + \ldots + y_k$$

of particular solutions.

Indeed, since $P(D)$ is linear and $P(D)y_i = f_i(x)$, we have

$$\begin{aligned} P(D)y_p &= P(D)\sum y_i \\ &= \sum P(D)y_i \qquad \text{(linearity of } P(D)) \\ &= \sum f_i(x) \end{aligned}$$

METHODS FOR FINDING A PARTICULAR SOLUTION

Among various method for finding a PS y_p of a NHLDE we mention

I. The method of Undetermined Coefficients (UC method)

II. The method of Variation of Parameters (VP method)

I. UC Method

This method is applicable when in the DE

$$P(D)y = \sum f_i(x) \tag{2}$$

the function $f_i(x)$ are all UC function, a *UC function* being one whose the set of all its successive derivatives is finite.

The basic UC functions are

$$P(x), \quad e^{ax}, \quad \sin bx, \quad \cos cx,$$

the others being their sums and products, where $P(x)$ is a polynomial.

Note that $\tan x$ is not a UC function, since its successive derivatives

$$\sec^2 x, \quad 2\sec^2 x \tan x, \ldots$$

do not form a finite set.

To determine a particular solution y_p of (2) the steps are as follows:

1. Obtain the complementary solution

$$y_c = c_1\ u_1(x) + \ldots + c_n\ u_n(x) \qquad \text{(obtained by } P(\lambda)=0\text{)}$$

2. For

$$P(D)y = f_i(x) \tag{3}$$

write the set

$$S_i = \{\varphi_1(x)\ , \ldots\ , \varphi_m(x)\}$$

whose elements are $f_i(x) = \varphi_1(x)$ and functions appearing as terms in all derivatives (constants being omitted)

a) If none of φ_i are contained in $\{u_1(x), \ldots, u_n(x)\}$, then a particular solution y_i is a linear combination of $\varphi_1\ , \ldots\ , \varphi_m$:

$$y_i = A_1 \varphi_1(x) + \ldots + A_m \varphi_m(x)$$

and set this y_i in (3) to obtain an identity yielding m equations in the m unknowns $A_1\ , \ldots\ , A_m$.

b) If some of φ_i are contained in $\{u_1\ , \ldots\ , u_n\}$, multiply each element of S_i by the least power of x to obtain a set S'_i having no element in common with $\{u_1\ , \ldots\ , u_n\}$. Then a linear combination of the elements in S'_i is a particular solution whose coefficients are obtained as in a).

3. Obtain in this manner all particular solutions $y_1\ , \ldots\ , y_k$ corresponding to $f_1(x), \ldots ,f_k(x)$. Then the required PS will be

$$y_p = y_1 + \ldots + y_k.$$

Example 1. Find a PS of

$$y'' - 4y = x^2 + e^x$$

Solution.

1. $\lambda^2 - 4 = 0 \implies \lambda_1 = 2, \quad \lambda_2 = -2$

$$y_c = c_1 e^{2x} + c_2 e^{-2x}, \qquad \{e^{2x}, e^{-2x}\}.$$

2. $f_1(x) = x^2$:

$$f_1(x) = x^2, \quad f_1'(x) = 2x, \quad f_1''(x) = 2$$

$$\implies S_1 = \{x^2, x, 1\} \implies y_1 = Ax^2 + Bx + C$$

$f_2(x) = e^x$:

$$\implies S_2 = \{e^x\} \implies y_2 = D.e^x \quad \text{(D is a const.)}$$

Then

$$y_p = Ax^2 + Bx + C + D e^x$$

3.

$$\begin{array}{r|l} -4 & y_p = Ax^2 + Bx + C + D e^x \\ 0 & y_p' = 2Ax + B + D e^x \\ 1 & y_p'' = 2A + D e^x \end{array}$$

$$-4Ax^2 - 4Bx - 4C - 4De^x + 2A + De^x = x^2 + e^x$$

$$\implies -4Ax^2 - 4Bx + (2A-4C) - 3De^x = x^2 + e^x$$

$$\implies -4A = 1, \quad -4B = 0, \quad 2A-2C = 0, \quad -3D = 1.$$

$$\implies A = -1/4, \; B = 0, \quad C = -\frac{1}{8}, \; D = -\frac{1}{3}$$

$$y_p = -\frac{x^2}{4} - \frac{1}{8} - \frac{1}{3} e^x$$

Example 2. Find the GS of

$$y''' - 2y'' + y' = (x+2) + e^x$$

Solution.

1. $\lambda^3 - 2\lambda^2 + 2\lambda = 0 \implies \lambda(\lambda-1)^2 = 0 \Rightarrow \lambda_1 = 0,\ \lambda_2 = 1,\ \lambda_3 = 1$

$$\implies y_c = c_1 + c_2\, e^x + c_3\, x\, e^x, \qquad \{1,\ e^x,\ xe^x\}$$

2. $\underline{f_1(x) = x+1}$. $S_1 = \{x+2,\ 1\}$

where 1 is contained in y_c. Then

$$S_1' = \{x^2 + 2x,\ x\}$$

$\underline{f_2(x) = e^x}$: $S_2 = \{e^x\}$

e^x is contained in y_c. Then

$$S_2' = \{x^2\, e^x\}$$

So

$$y_p = A(x^2 + 2x) + Bx + Cx^2\, e^x$$

3. $y_p = Ax^2 + (A+B)x + Cx^2\, e^x$

$$\begin{array}{r|l} 1 & y_p' = 2Ax + (A+B) + 2Cxe^x + Cx^2e^x \\ -2 & y_p'' = 2A + 2Ce^x + 4Cx\, e^x + Cx^2e^x \\ 1 & y_p''' = 6C\, e^x + 6C\, xe^x + Cx^2\, e^x \end{array}$$

$$\begin{aligned} (x+2) + e^x = {} & 2Ax + (A+B) + 2Cx\, e^x + Cx^2\, e^x \\ & -4A - 4C\, e^x - 8Cx\, e^x - 2Cx^2\, e^x \\ & +6C\, e^x + 6Cx\, e^x + Cx^2e^x \end{aligned}$$

$$x + 2 + e^x = 2Ax + (B-3A) + 2C\, e^x,$$

$$\implies 2A = 1, \quad B-3A = 2, \quad 2C = 1$$

$$\Longrightarrow \quad A = \frac{1}{2}, \quad B = \frac{7}{2}, \quad C = \frac{1}{2}$$

$$\Longrightarrow \quad y_p = \frac{1}{2}x^2 + 4x + \frac{1}{2}x^2 e^x$$

II. VP Method (Variation of parameters):

This method is applicable also when f_i's are not UC functions.

For simplicity in explanation, we consider a DE of order 3. Let

$$P(D)y = f(x) \tag{1}$$

be a DE of order 3 with complementary solution

$$y_c = c_1 u_1(x) + c_2 u_2(x) + c_3 u_3(x)$$

This method is based on replacing the arbitrary constants (parameters) c_1, c_2, c_3 by functions $c_1(x), c_2(x), c_3(x)$ and on the determination of them to get a PS in the form

$$y_p = c_1(x)u_1 + c_2(x)u_2 + c_3(x)u_3$$

We have

$$y_p' = c_1u_1' + c_2u_2' + c_3u_3' + \left[c_1'u_1 + c_2'u_2 + c_3'u_3\right]$$

Setting the expression in the bracket equal to zero, i.e.

$$c_1'u_1 + c_2'u_2 + c_3'u_3 = 0 \tag{a}$$

and redifferentiating y_p', we get

$$y_p'' = c_1u_1'' + c_2u_2'' + c_3u_3'' + \left[c_1'u_1' + c_2'u_2' + c_3'u_3'\right]$$

Just as before, setting

$$c_1'u_1' + c_2'u_2' + c_3'u_3' = 0 \tag{b}$$

and differentiating y''_p, we have

$$y_p''' = c_1 u_1''' + c_2 u_2''' + c_3 u_3''' + \left[c_1' u_1'' + c_2' u_2'' + c_3' u_3''\right].$$

Now substituting y_p and its derivatives in (1) one gets

$$c_1' u_1'' + c_2' u_2'' + c_3' u_3'' = f(x), \qquad (c)$$

since $P(D)\, u_i(x) = 0, \quad i = 1, 2, 3.$

This may, we obtain the three linear equations

$$u_1 c_1' + u_2 c_2' + u_3 c_3' = 0 \qquad (a)$$

$$u_1' c_1' + u_2' c_2' + u_3' c_3' = 0 \qquad (b)$$

$$u_1'' c_1' + u_2' c_2' + u_3'' c_3' = f(x) \qquad (c)$$

in the unknowns $c_1'(x)$, $c_2'(x)$, $c_3'(x)$.

Since u_1, u_2, u_3 are linearly independent, we have

$$W = \begin{vmatrix} u_1 & u_2 & u_3 \\ u_1' & u_2' & u_3' \\ u_1'' & u_2'' & u_3'' \end{vmatrix} \neq 0$$

Then $c_1'(x)$, $c_2'(x)$, $c_3'(x)$ and $c_1(x)$, $c_2(x)$, $c_3(x)$ are determined uniquely, except the constants of integration which are unneaded.

Example 1. Solve $y'' - 4y = e^x$.

Solution. Since $y_c = c_1 e^{2x} + c_2 e^{-2x}$, we have

$$y_p = c_1(x) e^{2x} + c_2(x) e^{-2x}$$

Then

$$c_1' e^{2x} + c_2' e^{-2x} = 0$$

$$2c_1' e^{2x} - 2c_2' e^{-2x} = e^x$$

$$\Longrightarrow \quad c_1'(x) = \frac{1}{4} - e^{-x}, \qquad c_2' = -\frac{1}{4} e^{3x}$$

$$\Longrightarrow \quad c_1(x) = \frac{1}{4} \int e^{-x} dx = -\frac{1}{4} e^{-x}$$

$$\Longrightarrow \quad c_2(x) = -\frac{1}{4} \int e^{3x} dx = -\frac{1}{12} e^{3x}$$

$$\Longrightarrow \quad y_p = -\frac{1}{4} e^{-x} e^{2x} - \frac{1}{12} e^{3x} e^{-2x} = -\frac{1}{3} e^{x}$$

$$\Longrightarrow \quad y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{3} e^{x}.$$

Solve this DE by UC method.

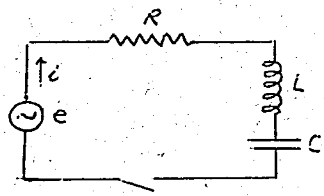
Example 2. (On electricity). In the given electrical circuit with the data.

$e = 220 \operatorname{Sin}(100\,\pi t)$ volts (emf),

$R = 3$ ohms (resistance),

$C = \frac{1}{2}$ farads (capacitance),

$L = 1$ henry (inductance),



compute the quantity q (coulomb) of electricity, and current i (ampere) if $q = 0$, $i = 0$ at $t = 0$.

Solution. From the KIRCHOFF's law(*)1, we have

$$Ri + L\frac{di}{dt} + \frac{1}{C} q = e$$

Since $i = dq/dt$ and $di/dt = d^2q/dt^2$, it becomes

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C} q = e \quad \text{(General equation of a circuit)}$$

Then

(*) 1. Around any closed circuit (path) the sum of instantaneous voltage drops in a specified direction is zero,

2. The sum of currents following into (or away from) any point on the circuit is zero.

$$(D^2 + 3D + 2)q = 220\ \sin(100\ \pi t)$$

$$\Longrightarrow \quad (D + 1)(D + 2)q = 200\ \sin(100\ \pi t)$$

$$q_c = c_1 e^{-t} + c_2 e^{-2t}$$

$$\Longrightarrow$$

$$q_p = c_1(t)e^{-t} + c_2(t)e^{-2t}$$

$$\Longrightarrow \quad \begin{cases} c_1' \ e^{-t} + c_2' \ e^{-2t} = 0 \\ -c_1' \ e^{-t} - 2c_2' e^{-2t} = 220\ \sin\ 100\ \pi t \end{cases}$$

$$\Rightarrow \quad \begin{cases} c_1(t) = 200 \ \int \sin(100\ \pi t)e^{t}\ dt \\ c_2(t) = -200 \int \sin(100\ \pi t)e^{2t}\ dt \end{cases}$$

Using

$$\int \sin\ at\ e^{bt}\ dt = \frac{b\ \sin\ at - a\ \cos\ at}{a^2 + b^2}\ e^{bt},$$

we have

$$c_1(t) = \frac{\sin\ at - a\ \cos\ at}{a^2 + 1}\ e^{t}$$

$$(a = 100\ \pi)$$

$$c_2(t) = \frac{2\sin\ at - a\ \cos\ at}{a^2 + 1}\ e^{2t}$$

Then

$$q_p = \frac{1}{a^2 + 1}\ (\sin\ at - a\ \cos\ at) + \frac{1}{a^2+1}\ (2\ \sin\ at - a\ \cos\ at)$$

$$\Longrightarrow q(t) = c_1 e^{-t} + c_2 e^{-2t} + \frac{1}{a^2+1}\ (3\ \sin\ at - 2a\ \cos\ at)$$

$$\Rightarrow \quad i(t) = -c_1 e^{-t} - 2c_2 e^{-2t} + \frac{1}{a^2+1}\ (3\ a\ \mathrm{cost} + 2a^2\ \sin\ at)$$

Using the conditions $q(0) = 0$ and $i(0) = 0$, we have

$$\left.\begin{array}{l} c_1 + c_2 - \dfrac{2a}{a^2+1} = 0 \\ -c_1 - 2c_2 + \dfrac{3a}{a^2+1} = 0 \end{array}\right. \Rightarrow \begin{cases} c_1 = \dfrac{a}{a^2+1} \\ c_2 = \dfrac{a}{a^2+1} \end{cases}$$

and

$$q(t) = \frac{a}{a^2+1}(e^{-t}+e^{-2t}) + \frac{1}{a^2+1}(3 \sin at - 2a \cos at)$$

$$i(t) = -\frac{a}{a^2+1}(e^{-t}+2e^{-2t}) + \frac{a}{a^2+1}(3 \cos at + 2a \sin at)$$

where $a = 100\pi$.

EXERCISES (6. 4)

46. Find the solution of the following HDE:

a) $y'' - y' - 6y = 0$ b) $2y''' - 5y'' - y' + 6y = 0$

47. Find the GS of the following NHLDE by two methods (UC, VP):

a) $y'' - y' = \sin 2x$ b) $y'' - y' = 6x^5 e^x$

48. Find the GS:

a) $y'' + 4y = 2 \tan 2x$ b) $y'' + 4y = \sec 2x$

49. Solve:

a) $(D^4 - 4D^3 + 5D^2 - 4D + 4)y = 0$ (check 2 as a double root)

b) $(D^4 - 8D^3 + 42D^2 - 104D + 169)y = 0$

(Find the constant b if $P(D) = (D^2 + bD + 13)^2$)

50. Find the DE in operator form admitting

$c_1 + c_2 x + c_3 e^{4x}$ as GS.

ANSWERS

46. a) $y = c_1 e^{3x} + c_2 e^{-2x}$, b) $y = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{3x/2}$

47. a) $y = c_1 + c_2 e^x - \frac{1}{5} \sin 2x + \frac{9}{10} \cos 2x$

b) $y = c_1 + c_2 e^x + (x^6 - 6x^5 + 30x^4 - 120x^3 + 360x^2 - 720x)e^x$

48. a) $y = c_1 \sin 2x + c_2 \cos 2x - \frac{1}{2}\cos 2x \ln(\tan 2x + \sec 2x)$

b) $y = c_1 \sin 2x + c_2 \cos 2x + \cos 2x \ln$:os $2x$

49. a) $y = (c_1 + c_2 x)e^{2x} + c_3 \sin x + c_4 \cos x$

b) $y = e^{2x}\left[(c_1 + c_2 x)\sin 3x) + (c_3 + c_3 x)\cos 3x\right]$

50. $D^2(D - 4)y = 0$

A SUMMARY
(CHAPTER 6)

6.1 DEFINITIONS

Order of a DE is the order of highest ordered derivative in the DE.

A solution of a DE is any relation satisfying the DE.

The *general solution* of a DE is the solution involving n arbitrary constants if the order of the DE is n.

6.2 FIRST ORDER ORDINARY DE:

General form: $f(x, y, \frac{dy}{dx}) = 0$, $\frac{dy}{dx} = g(x, y)$

Separable DE: $P(x)dx + Q(y)dy = 0$

Homogeneous DE: $P(x, y)dx + Q(x, y)dy = 0$, $\frac{dy}{dx} = f(\frac{y}{x})$
where P, Q are homogeneous functions of the same degree.

Exact DE: $P(x, y)dx + Q(x, y)dy = 0$ if $P_y \equiv Q_x$.

Linear DE: $\frac{dy}{dx} + p(x)y = f(x)$ (linear in $y, \frac{dy}{dx}$)

$\frac{dx}{dy} + p(y)x = f(y)$ (linear in $x, \frac{dx}{dy}$)

Bernoulli DE: $\frac{dy}{dx} + p(x)y = f(x)y^{\alpha}$

Riccati DE: $\frac{dy}{dx} = A(x)y^2 + B(x)y + C(x)$

Clairant DE: $y = xy' + \psi(y')$

Lagrange DE: $y = x\phi(y') + \psi(y') \quad (\phi(y') \not\equiv y')$

Integrating factor of a DE $Pdx + Qdy = 0$ is a function $\mu(x, y)$ such that $\mu Pdx + \mu Qdy = 0$ is exact.

6. 3 A *trejectory* of a given family of curves is a curve satisfying a given relation at all common points with the given family, if the given relation is orthogonality, the trajectories are *orthogonal trajectories*.

The DE of a *Population problem:* $\frac{dx}{dt} = k(x+a)$ (a, k const)

The DE of *stream lines* of a vector field $F = Pi + Qj$:

$$\frac{dx}{P} = \frac{dy}{Q}$$

The DE of a *equipotential curves* (orthogonal trajectories of the stream lines):

$$Pdx + Qdy = 0$$

6. 4 LDE OF ORDER n (constant coefficients)

$$P(D)y = (a_0D^n + \ldots + a_{n-1}D + a_n)y = f(x)$$

GS: $y = y_c + y_p$

where y_c is the complementary solution (GS of the reduced form $P(D)y = 0$) which is obtained by the use of auxiliary equation $P(\lambda) = 0$, and y_p is a particular solution of $P(D)y = f(x)$ which can be obtained by several methods, say by UC and VP methods.

MISCELLANEOUS EXERCISES

51. State the order and degree of:

a) $\sqrt[3]{1+y''^2} + x^5 y''' + 2xy = 0$ b) $\sin(y''^5) + x^2 + y^3 = 0$

52. Find the DE of the following family of curves

a) The circles centered on $y = 2x$ and radius 2 units,

b) The parabolas intersecting x-axis at -2 and 3, with axes parallel to y-axis.

53. Find the DE admitting the given family as GS:

a) $y = ce^x - c^2$ b) $(r+1)e^{2\Theta} = c\Theta$ c) $y = x + c_1 \cos x + c_2 e^x$

54. Same question for

a) $z = \frac{x}{y} + \phi(x^2 y)$ b) $z = \phi(x) + \psi(2x-3y)$

55. Find the DE of all circles in R^2.

56. Solve $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

57. Solve the SDE:

a) $\tan y - (\cot x)y' = 0$ b) $\tan\Theta \, dr + 2r \, d\Theta = 0$

58. Solve

a) $(1+x+y)dx + (3+2x+2y)dy = 0$ b) $\frac{dy}{dx} = \frac{x-y+1}{x+y+3}$

59. Solve the DE. If not exact find an integrating factor.

a) $2xy \, dx + (1+x^2)dy = 0$ b) $(x^2+y\, e^{2x})dx + (2xy+x)e^{2y}dy = 0$

60. Same question for:

a) $xdy - ydx = x^2y^2dy$ b) $(x^2+3y^2)dx - 2xy\, dx = 0$

c) $ydx = (2x^2y^3 - x)dy$, $y(1) = 1$.

61. Find an integrating factor $\mu = \mu(x+y)$ and solve:

$$(x + 3y + 5)dx + (2x + 4y + 5)dy = 0$$

62. Find a PS of the following DE, through the given point:

a) $x^2y' - xy = x^2 - y^2$; $(1, 0)$ b) $y\,dx = (2x^2y^3 - x)dy$; $(1, 1)$

63. Find an integrating factor of the form $x^m y^n$ and solve:

$$(2y + 3x^2y^3)dx + (3x + 5x^3y^2)dy = 0$$

64. Find the GS of the following LDE:

a) $\dfrac{dy}{dx} - \dfrac{xy}{x^2-1} = x$

b) $(y \cos 2x + 2 \sin^{3/2} 2x)dx + \sin 2x\, dy = 0$

65. Find the GS:

a) $(x-y-3)dx + (3x-3y+1)dy = 0$ b) $(2x-y-1)dx + (3x+2y-5)dy = 0$

66. Solve:

a) $xy' + y = y^2x^3 \sin x$

b) $(3x^2 + 2xy - 2y^2)dx + (2x^2 + 6xy + y^2)dy = 0$

67. Find the GS of the BERNOULLI DE:

a) $(1 + x^2)\dfrac{dy}{dx} + xy = x^3y^3$ b) $\dfrac{dr}{d\theta} = \dfrac{\theta^2 + r^2}{2\theta r}$

68. Find the singular and general solution of the CDE:

a) $2xy' - 2y - y'^2 = 0$ b) $y = xy' + \ln y'$

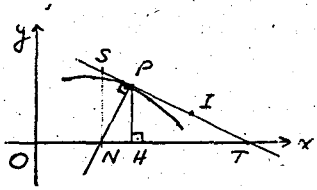
69. Solve $y = 2xy' + y'^2$

70. Same question as Exercise 66:

a) $y + y'^2 = xy' + 1$, b) $y = y'x + \ln y'$

71. Find curves such that

a) $|OT|^2 / |OH| = a$ const.



b) $|NS| = a$

72. Same question for:

a) I is fixed ($|PI| = |IT|$) b) $|ON| = |PN|$

73. Find curves having constant

a) subnormal p, b) subtangent q, c) normal R, d) tangent t

74. Find a family of curves such that the perpendicular distance from the origin to each tangent is equal to the value of x of the point of contact.

75. Find the orthogonal trajectories of the family:

a) $y^2 = 2cx + c^2$ b) $r = c\ (\mathrm{Sec}\Theta + \tan\Theta)$

76. Find the stream lines and equipotential curves of the vector field.

$$F = (x^2 + y^2,\quad -2xy)$$

77. An object of $W = 64$ kg is falling in the air with a resistance force proportional to the square of velocity. If the velocity $v = 0$ at $t = 0$, find v.

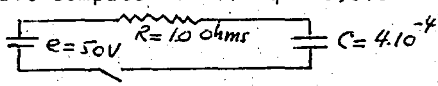
[The law of equation: $\frac{W}{g}\ \frac{dv}{dt} = W - kv^2$]

78. A certain radioactive substance has a half-life of 38 hours. Find how long it takes for 90 % of the radiractivity to be dissipated.

79. Find the orthogonal trajectories of the family:

a) $r = a(\mathrm{Sec}\Theta + \tan\Theta)$, b) $a = a\ \mathrm{Sin}\Theta\ \tan\Theta$

80. For the given electrical circuit compute i if $q = 0{,}015$ when $t = 0$.



ANSWERS

51. a) 3, 3. b) 2, 5

52. a) $(2x-y)^2(1+y'^2) = 4(1+2y')^2$, b) $(x^2-x-6)y'-(2x-1)y = 0$

53. a) $y'^2 - x(e^x-2)y' + (e^x-2)^2 y = 0$,

b) $\frac{dr}{d\Theta} + (r+1)(2 - \frac{1}{\Theta}) = 0$

c) $(\cos x + \sin x)y'' - 2(\cos x)y' + (x-y)(\sin x - \cos x) + 2\cos x = 0$

54. a) $xyz_x - 2y^2z_y + 3x = 0$, b) $3z_{xy} + 2z_{yy} = 0$

55. $(1+y'^2)y''' - 3y'y''^2 = 0$

56. $xy + c(x-y) + 1 = 0$

57. a) $\cos y = c \cos x$, b) $r = c^2 \csc^2\Theta$

58. a) $x + y + z - \ell n(x+y+z) = c$ b) $x^2 - 2x - y^2 + x + 6y = c$

59. a) $x^2y + y = c$, b) $x^3 + 3xy + 2y = c$

60. a) $xy^3-3y+cx = 0$, b) $x^2-3y^2 = cy$, c) $xy^3 - 2xy - 1 = 0$

61. $\mu = x+y$, $\frac{x^3}{3} + 2x^2y + 3xy^2 + \frac{4}{3}y^3 + \frac{5}{2}(x^2+y^2) + 5xy = c$

62. a) $x + y = x^2(x-y)$, $xy^3 - 2xy + 1 = 0$

63. $\mu = x^{-9}y^{-13}$; $2x^2y^2 - cx^8y^{12} + 1 = 0$

64. a) $y = x^2-1+c\sqrt{x^2-1}$ b) $y = \sin 2x.(\cos 2x + c)$

65. a) $5\,\ell n(2x - 2y - 1) = 2x + 6y + c$

b) $\ell n\sqrt{x^2 + xy + y^2 - 3x - 3y + 3} + \frac{2}{\sqrt{3}} \arctan \frac{x + 2y - 3}{\sqrt{3}(x - 1)} = c$

66. a) $xy(x \cos x - \sin x + c) = 1$, b) $x^2+xy+y^2 = c(3x+y)$

67. a) $y\sqrt{-(x^2+1)\ \ell n(x^2+1) + c(x^2+1)-1} = 1$, b) $r^2 = \Theta^2 + c\Theta$

68. a) $y = x^2/2$; $y = cx-c^2/2$, b) $y = -1 + \ell n \frac{-1}{x}$; $y = cx + \ell n\, c$

69. $x = -2p/3 + c/p^2, \quad y = -p^2/3 + 2c/p$

70. a) $4y = x^2+4; \quad y = cx - c^2 + 1$, b) $xe^{y+1} + 1 = 0; \; y = cx + \ln c$

71. a) Parabolas tangent to coordinate axes, b) cycloids

72. a) parabolas having fixed focus b) circles

73. a) $y^2 = 2p(x-c)$, b) $y = ce^{\pm x/q}$, c) $(x-c)^2 + y^2 = R^2$

d) $t\left(\pm \ln \frac{t + \sqrt{t^2 - y^2}}{y} \pm \frac{\sqrt{t^2 - y^2}}{y}\right) = x + c$

74. $x^2 + y^2 - cx = 0$

75. a) $y^2 = 2cx + c^2$, b) $r = c\, e^{-\sin\theta}$

76. $x^2y + \frac{y^3}{3} = c, \quad x - \frac{y^2}{x} = c$

77. $n\sqrt{\frac{8 + \sqrt{k}\, v}{8 - \sqrt{k}\, v}} = \frac{g\sqrt{k}}{8}\, t$

78. 123 h

79. a) $r = c\, e^{-\sin\Theta}$, b) $r^2 = c(1 + \cos^2\Theta)$

80. $i = 1{,}25\, e^{-250t}$

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