FRESHMAN

CALCULUS

BOOK TWO
part one

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FRESHMAN CALCULUS

BOOK TWO

PART ONE

Sayın
Haluk Bingöl'e
saygı ve başarı dileklerimle

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P R E F A C E

Being well aware of the existence of excellent textbooks of similar content, before adding another one to the market, we humbly feel that the nature of the challenge, which motivated us to prepare this work, needs a justifying explanation. Summing up we may state briefly the following facts:

a. This work is designed primarily for students who were - like the ones at METU - to utmost one year intensive language training in English,

b. Its content is closely related to the syllabus traditionally followed at METU and similar institutions,

c. It is a practical answer to the ever increasing demand, caused by contemporary currency fluctuations which effectively curb the availability of the textbooks edited abroad.

We sincerely believe that the topics treated in the two volumes, each containing two distinct parts, are self contained and compact. Each part and each chapter is provided with numerous exercises, including the answers corresponding to the even numbered ones.

We express our gratitude to our colleagues for their constant encouragements and to Miss Zehra Öner for her careful typing.

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ANKARA

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CHAPTER I

SEQUENCES AND SERIES

I. I. SEQUENCES OF NUMBERS

A. DEFINITIONS

If $f: D \to R$ is a function whose domain D admits the set $\mathbb{Z}_p$ of consecutive integers $p, p+1, p+2, \dots, n, \dots$ as a subset, then the infinitely many numbers

$$f(p), f(p+1), \dots, f(n), \dots \tag{1}$$

written in this order, is called an <u>infinite sequence</u> or simply a <u>sequence</u>, where $f(p), f(p+1), \dots, f(n)$ are called the <u>first term</u>, the <u>second term</u>, ..., the <u>general term</u> respectively.

For brievity one denotes $f(n)$ usually by a letter with the subscript n, say a_n, and the sequence (1) by

$$(f(n))_p^\infty \quad \text{or} \quad (a_n)_p^\infty$$

or more simply by

$$(f(n))_p \quad \text{or} \quad (a_n)_p$$

<u>Examples</u>

$(n)_1$: $1, 2, 3, \dots, n, \dots$

$(\frac{n}{n-2})_3$: $3, \frac{4}{2}, \frac{5}{3}, \dots, \frac{n}{n-2}, \dots$

$(n!)_0$: $1, 1, 2, \ldots, n!, \ldots$

$((-1)^{n-1})_{-2}$: $-1, 1, -1, \ldots, (-1)^{n-1}, \ldots$

The notation $(a_n)_p^q$ is used to denote a <u>finite sequence</u> admitting also last term:

$$(\sqrt{n})_4^9 : 2, \sqrt{5}, \sqrt{6}, \sqrt{7}, 2\sqrt{2}, 3$$

In this Section we discuss briefly infinite sequences only.

A sequence is uniquely determined when the first and general term are given. Thus $a_3 = 4$, $a_n = 2^{n-1}$ define the sequence

$$(2^{n-1})_3 : 4, 8, \ldots, 2^{n-1}, \ldots$$

while some numbers written in succession followed by three dots, such as

$$5, 7, 9, \ldots$$

do not define uniquely a sequence, since the general term is not given, and as the 4th term any number can be assigned arbitrarily other than 11 (that one would expect). Indeed, the sequence $(a_n)_1$ with general term:

$$a_n = (n-1)(n-2)(n-3) + 2n + 3$$

gives $5, 7, 9$ as the first three terms and 17 as the 4th term.

Determination of Sequences By Recurrence Relations:

A sequence $(a_n)_p$ can be defined more generally by a recurrence relation

$$f(a_n, \ldots, a_{n+k}) = 0$$

and k consecutive terms $a_p, \ldots, a_{p+k-1}$.

The following are two examples for $k = 1$ and $k = 2$.

Example 1. Given the sequence defined by

$$a_1 = 3, \quad \text{and} \quad a_n = a_{n-1} + 2 \,.$$

a) obtain the first four terms,

b) find the general term.

Solution.

a) $a_1=3$, $a_2=a_1+2 = 5$, $a_3 = a_2+2 = 7$, $a_4 = 7+2 = 9$

b) Writing the relation for $n = 2, 3, \ldots$ up to n, and adding these member to member, the intermediate terms $a_2, \ldots, a_{n-1}$ are canceled, and a_n is obtained:

$$\begin{aligned}
\cancel{a_2} &= a_1 + 2 \\
\cancel{a_3} &= \cancel{a_2} + 2 \\
&\vdots \\
a_n &= \cancel{a_{n-1}} + 2 \\
\hline
a_n &= a_1 + (n-1)2 \\
&= 3 + 2n - 2 = 2n+1.
\end{aligned}$$

Another definition of a sequence is obtained by giving the first two terms and a relation between a_n and a_{n-2} whose indices differ by 2:

Example 2. Given the sequence defined by

$$a_1 = 3, \quad a_2 = 2, \quad a_n = \frac{n-1}{n+1} a_{n-2},$$

a) obtain the first four terms,

b) find the general term.

Solution.

a) $a_1 = 3$, $a_2 = 2$, $a_3 = \frac{3-1}{3+1} a_1 = \frac{3}{2}$, $a_4 = \frac{3}{5} a_2 = \frac{6}{5}$

b) Since indices differ by 2, one evaluates a_{2n} and a_{2n+1} seperately. Replacing n by $2n$ in the given relation, one gets

$$a_{2n} = \frac{2n-1}{2n+1} a_{2n-2}$$

which, when written for $n = 1, 2, \ldots$ up to n, gives

$$\begin{aligned} a_4 &= \frac{3}{5} a_2 \\ a_6 &= \frac{5}{7} a_4 \\ &\vdots \\ a_{2n} &= \frac{2n-1}{2n+1} a_{2n-2} \end{aligned}$$

which in turn, when multiplied member to member yield

$$a_{2n} = \frac{3.\ 5.\ \ldots\ (2n-1)}{5.\ 7.\ \ldots\ (2n+1)}\, a_2 = \frac{3}{2n+1}\, a_2 = \frac{6}{2n+1}.$$

Now, replacing n by $2n+1$, one has

$$a_{2n+1} = \frac{n}{n+1}\, a_{2n-1} \Rightarrow a_{2n+1} = \frac{3}{n+1},$$

by similar process.

Algebra of Sequences:

Given two sequences $(a_n)_1$, $(b_n)_1$ having the common domain $\mathbb{N}_1$, we define new sequences on $\mathbb{N}_1$, namely

1. $(ca_n)_1$: $ca_1, ca_2, \ldots, ca_n, \ldots$ $c \in \mathbb{R}$

2. $(a_n+b_n)_1$: $a_1+b_1, a_2+b_2, \ldots, a_n+b_n, \ldots$

3. $(a_n-b_n)_1$: $a_1-b_1, a_2-b_2, \ldots, a_2-b_n, \ldots$

4. $(a_n \cdot b_n)_1$: $a_1b_1, a_2b_2, \ldots, a_nb_n, \ldots$

5. $(\frac{a_n}{b_n})_1$: $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}, \ldots$ ($b_n \neq 0$ for all $n > \mathbb{N}_1$)

Thus, if $a_n = \frac{1}{n}$, $b_n = \sqrt{n}$, then

$$(2a_n - b_n)_1 : 1, 1 - \sqrt{2}, \ldots, \frac{2}{n} - \sqrt{n}, \ldots$$

$$(a_n b_n^2)_1 : 1, 1, \ldots, 1, \ldots$$

<u>Subsequences</u>:

If every term of an (infinite) sequence $(b_n)_1$ is also a term of a sequence $(a_n)_1$, then $(b_n)_1$ is said to be a <u>subsequence</u> of $(a_n)_1$.

Clearly, every sequence is a subsequence of itself. Among other subsequences of $(a_n)_1$ we mention the following:

$$(a_{2n})_5 ,\quad (a_{n+3})_1 ,\quad (a_{n^2})_7 ,\ (a_{n!})_2 ,\ (a_n)_N$$

A notation for arbitrary subsequence of $(a_n)_1$ is $(a_{n_k})_{n=1}$ where (n_k) is a sequence of integers.

Some subsequences of $((-1)^n)_2$ are

$$1, 1, 1, \ldots , 1, \ldots$$

$$-1, -1, -1, \ldots , -1, \ldots$$

$$1, \underbrace{-1}_{1}, 1, \underbrace{-1, -1}_{2}, \ldots , 1, \underbrace{-1, \ldots , -1}_{n}, \ldots$$

B. <u>BEHAVIOR OF A SEQUENCE</u>

I. <u>Monotonocity</u>:

A sequence $(a_n)_1$ is called <u>monotone</u> if

$$a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n \leqslant \ldots$$

or else

$$a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n \geqslant \ldots$$

In the former (latter) case the sequence is said to be monotone increasing (decreasing).

To show that $(a_n)_1$ is monotone increasing, one proves either

$$\frac{a_{n+1}}{a_n} \geqslant 1 \quad \text{or} \quad a_{n+1} - a_n \geqslant 0 \quad \text{for all} \quad n \geqslant 1$$

In the second case, the inequality is to be reversed.

Example. Show that the sequence $(\frac{n-1}{n})_2$ is monotone increasing.

Solution.

$$a_{n+1} - a_n = \frac{n}{n+1} - \frac{n-1}{n} = \frac{1}{n(n+1)} \geqslant 0 \text{ for all } n \geqslant 2$$

2. Boundedness.

A sequence $(a_n)_1$ is bounded (from) above, if there is a number u such that

$$a_n \leqslant u \quad \text{for all} \quad n \geqslant 1,$$

and bounded (from) below, if there is a number ℓ such that

$$\ell \leqslant a_n \quad \text{for all} \quad n \geqslant 1.$$

A sequence which is bounded above and bounded below is said to be bounded, otherwise unbounded. In a bounded sequence all the terms lie in a closed interval $[\ell, u]$, where u is an upper bound, and ℓ is a <u>lower bound</u>. Any number larger then u is an upper bound, and any number smaller than ℓ is also a lower bound of the sequence.

If $(a_n)_1$ is bounded there exists, clearly, a positive number K such that

$$-K \leqslant a_n \leqslant K \quad \text{or} \quad |a_n| \leqslant K \quad \text{for all} \quad n \geqslant 1.$$

<u>Examples</u>.

1. $1, 2, \ldots, n, \ldots$ monotone and bounded below,
2. $1, 1/2, \ldots, 1/n, \ldots$ monotone and bounded,
3. $2, 2, \ldots, 2, \ldots$ monotone and bounded,
4. $-1, 1, \ldots, (-1)^n, \ldots$ non monotone, but bounded.

<u>Example</u>. Show boundedness of

a) $\left(\frac{\sin n}{\sqrt{n+8}}\right)_1$ b) $\left(\frac{n+8}{n^{3/2}}\right)_4$

<u>Solution</u>.

a) $\left|\frac{\sin n}{\sqrt{n+8}}\right| = \frac{|\sin n|}{\sqrt{n+8}} \leqslant \frac{1}{\sqrt{n+8}} \leqslant \frac{1}{\sqrt{1+8}} = \frac{1}{9} \quad (K = \frac{1}{9}),$

since $\max(\sin n) = 1$ and $\min n = 1$.

b) $\left|\frac{n+8}{n^{3/2}}\right| = \frac{n+8}{n\sqrt{n}} = \frac{1}{\sqrt{n}} + \frac{8}{n\sqrt{n}} \leqslant \frac{1}{\sqrt{4}} + \frac{8}{4\sqrt{4}} = \frac{1}{2} + 1 \quad (K = 3/2)$

since $\min n = 4$.

3. Convergence:

A sequence $(a_n)_1$ is said to be convergent if the general term a_n has a limit as $n \to \infty$, otherwise it is divergent.

If $\lim_{n \to \infty} a_n = a$, one says that $(a_n)_1$ converges to a, and one writes

$$(a_n)_1 \to a \quad \text{or} \quad a_n \to a$$

Clearly if $(a_n)_1 \to a$, then $(a_{n+p})_1 \to a$. (for, $n+p \to \infty$ as $n \to \infty$).

Since

$$(a_n)_1 \to a \Rightarrow a_n - a \to 0 \Rightarrow |a_n - a| \to 0,$$

it follows that in a convergent sequence $(a_n)_1$, for sufficiently large N, all the terms $a_{N+1}, a_{N+2}, \ldots$ (or almost every term of $(a_n)_1$) fall inside a neighborhood $(a-\varepsilon, a+\varepsilon)$ of a. In other words, the sequence $(a_n)_1$ converges to a, if given $\varepsilon > 0$ there exists a positive integer N such that

$$a_n \in (a-\varepsilon, a+\varepsilon) \quad \text{or} \quad |a_n - a| < \varepsilon \text{ for all } n > N.$$

We note that omission of finite number of terms from a sequence does not alter convergence or divergence of the sequence, and as far as convergence is concerned the index p in $(a_n)_p$ may be dropped.

The following statements hold true:

1. If a sequence is convergent, then every subsequence of it is convergent,
2. If a subsequence is divergent, the original sequence is divergent,
3. If two subsequences converge to distinct limits, the original sequence is divergent.

Example.

1. $3, \sqrt{10}, \ldots, \sqrt{n}, \ldots$ diverges to ∞,
2. $((1+\frac{1}{n})^n)_2$ converges to e,
3. $1, -1, 1, -1, \ldots, (-1)^{n-1}, \ldots$ diverges since it has the subsequences (1) and (-1) having distinct limits 1 and -1.

Theorem 1. If $(a_n) \to a$, $(b_n) \to b$, and $c \in \mathbb{R}$, then

a) $(c\, a_n) \to c\, a$

b) $(a_n + b_n) \to a+b$

c) $(a_n b_n) \to ab$

d) $(\frac{a_n}{b_n}) \to \frac{a}{b}$ (if $b_n \neq 0$, $b \neq 0$)

d) $(|a_n|) \to |a|$

Proof. We prove c) only. Those of the others are similar. The proof runs in the same way as that for functions with continuous variable.

Let $a_n \to a$, $b_n \to b$. Then given $\varepsilon > 0$ there exists $N > 0$ such that

$$|a_n - a| < \varepsilon, \quad |b_n - b| < \varepsilon \quad \text{for all} \quad n>N.$$

To show $a_nb_n \to ab$ we from $a_nb_n - ab$ and get

$$|a_nb_n - ab| = |a_nb_n - ab_n - ab + ab_n|$$

$$= |(a_n - a)b_n - a(b - b_n)$$

$$\leq |a_n - a|\,|b_n| + |a|\,|b - b_n|$$

$$\leq |b_n|\,\varepsilon + |a|\,\varepsilon.$$

Since for all $n>N$, b_n lies in the interval $(b-\varepsilon, b-\varepsilon)$ it follows that $b_n < K$ for some positive K, and one has

$$|a_nb_n - ab| < K\varepsilon + |a|\varepsilon = (K+|a|)\varepsilon$$

showing that $a_nb_n \to ab$.

Theorem 2.

a) A monotone bounded sequence is convergent,

b) A convergent sequence is bounded,

c) $(a_n) \to a$, $(b_n) \to b$ and $a_n \leqslant c_n \leqslant b_n$ for all $n>N$

$\Rightarrow$ $(c_n) \to c$ and $a \leqslant c \leqslant b$.

Proof. Omitted.

C. <u>Some Important Convergent Sequences</u>

1. a) $(\sqrt[n]{a})_1$: $a, \sqrt{a}, \sqrt[3]{a}, \ldots, \sqrt[n]{a}, \ldots$ converges to 1 (for $a > 0$)

   b) $(\sqrt[n]{n})_1$: $1, \sqrt{2}, \sqrt[3]{3}, \ldots, \sqrt[n]{n}, \ldots$ converges to 1.

2. $((1 + \frac{\lambda}{n})^n) \to e^{\lambda}$ or $(\frac{n+\lambda}{n})^n \to e^{\lambda}$

3. a) $(\frac{\ell n^p n}{n}) \to 0$ for any constant p.

   b) $(\frac{n^p}{e^n}) \to 0$ for any constant p.

4. If $(a_n)_1$: $a_1, a_2, \ldots, a_n, \ldots$ is a sequence with positive terms, converging to the limit a, then

   a) $A_n = \dfrac{a_1 + \ldots + a_n}{n} \to a$

   b) $G_n = \sqrt[n]{a_1 \ldots a_n} \to a$

where A_n, G_n are called the <u>arithmetic mean</u> and <u>geometric mean</u> of the positive numbers $a_1, a_2, \ldots, a_n$.

5. a) $(\frac{n}{\sqrt[n]{n!}}) \to e$, b) $(\frac{e^n}{n!}) \to 0$

The proofs of 1 to 3 are obtained by limit process considering the functions $a^{1/x}$, $x^{1/x}$, $(1 + \frac{\lambda}{x})^x$, $(\ell n^p x)/x$, x^p/e^x of continuous variable x.

<u>Proof</u>(4) Since $(a_n) \to a$, then given $\varepsilon > 0$ there is $N > 0$ such that

$$|a_n - a| < \varepsilon \quad \text{for all} \quad n>N.$$

a) considering the difference $A_n - a$, we have

$$A_n - a = \frac{a_1+\ldots+a_n}{n} - a = \frac{(a_1-a)+ \ldots + (a_n-a)}{n}$$

$$= \frac{(a_1-a)- \ldots -(a_N-a)}{n} + \frac{(a_{N+1}-a)- \ldots -(a_n-a)}{n}$$

$$|A_n - a| \leq \frac{|a_1-a| + \ldots + |a_N-a|}{n} + \frac{|a_{N+1}-a| + \ldots + |a_n-a|}{n}$$

$$< \frac{|a_1 - a| + \ldots + |a_N - a|}{n} + \frac{(n - N)\varepsilon}{n}$$

For sufficiently large n, say for $N_1>N$ the first term on the right hand side is less than ε, and

$$|A_n - a| < \varepsilon + \frac{N_1 - N}{N_1}\varepsilon = k\varepsilon. \blacksquare$$

b) $G_n = \sqrt[n]{a_1 \ldots a_n} \Rightarrow \ln G_n = \dfrac{\ln a_1 + \ldots + \ln a_n}{n} \to \ln a$

by (a) $\Rightarrow$ $G_n \to a$. $\blacksquare$

Proof (5)

a) Applying 4b to $(a_n) = (\frac{n+1}{n})^n \to e$, we have

$$G_n = \sqrt[n]{\left(\frac{2}{1}\right)^1 \left(\frac{3}{2}\right)^2 \cdots \left(\frac{n}{n-1}\right)^{n-1} \left(\frac{n+1}{n}\right)^n}$$

$$= \sqrt[n]{\frac{1}{1} \cdot \frac{1}{2} \cdots \frac{1}{n-1} \; \frac{(n+1)^n}{n}} = \frac{n+1}{\sqrt[n]{n!}} \to e.$$

$$\frac{n}{\sqrt[n]{n!}} = \frac{n+1}{\sqrt[n]{n!}} \; \frac{n}{n+1} \to e.1 = e$$

b) We use the lemma:

<u>Lemma</u>. $a_n \to 0 \Rightarrow a_n^n \to 0.$

Indeed, since $a_n \to 0$, given $\varepsilon > 0$ there is $N > 0$ such that $|a_n| = |a_n - 0| < \varepsilon$ for all $n > N$. Taking $\varepsilon < 1$,

$$|a_n^n| < \varepsilon^n < \varepsilon \Rightarrow a_n^n \to 0.$$

Now,

$$\frac{e}{\sqrt[n]{n!}} = \frac{e/n}{\sqrt[n]{n!}/n} \to \frac{0}{1/e} = 0 \Rightarrow \frac{e^n}{n!} \to 0. \quad \blacksquare$$

EXERCISES (I. I)

1. Which ones of the following are subsequences of $(1-(-1)^n)_1$?

   a) (1) b) (-1) c) (0) d) (2)

2. Write two subsequences of

a) $(\cos n\pi)_0$ b) $(\sin n\frac{\pi}{2})_0$

3. For each of the following cases find divergent sequences (d_n) and (d'_n) such that

a) $(d_n+d'_n)$ b) $(d_n - d'_n)$ c) $(d_n d'_n)$ d) (d_n/d'_n)

is convergent.

4. Examine the following sequences for monotonocity, boundedness and convergence:

a) $(\frac{n-1}{n})_2$ b) $(\frac{3^{n+1}-1}{3^n})_0$

5. Same question for:

a) $(\ell n\ n)_1$ b) $(1/\ell n\ n)_2$

6. Same question for:

a) $(\ell n\ \frac{1}{n})_1$ b) $(a_n)_0$ when $a_n = \begin{cases} 0, & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even} \end{cases}$

7. Same question for:

a) $([\![\frac{n-1}{n}]\!])_5$ b) $([\![\frac{n}{3}]\!])_0$

8. Same question for:

a) $(1/\sqrt[2]{n})_1$ b) $((-1)^n n)_1$ c) $(|(-1)^{n-1}|)_{-3}$ d) $(0,\underbrace{22\ldots2}_{n})_1$

9. Find the general term of the sequence defined by

a) $a_1 = 1$, $a_n = 3a_{n-1}$ b) $b_1 = 2$, $b_n = 3 + b_{n-1}$

c) $c_1 = 2$, $c_2 = 1$, $c_n = c_{n-2}$ d) $d_1 = 1$, $d_2 = 2$, $d_n = -d_{n-2}$

10. Given the sequence $a_1 = 1$, $a_n = \sqrt{1 + a_{n-1}}$ prove that (a_n)

a) is monotone increasing b) is bounded

c) has a limit, and find this limit.

ANSWERS TO EVEN NUMBERED EXERCISES

2. a) (1), (-1), b) (0), (1)

4. a) M, B, C, b) M, B, C

6. a) M, $\cancel{B}$, $\cancel{C}$, b) $\cancel{M}$, $\cancel{B}$, $\cancel{C}$

8. a) M, B, C, b) $\cancel{M}$, $\cancel{B}$, $\cancel{C}$, c) M, B, C, d) M, B, C.

10. c) $\frac{1}{2}(1+\sqrt{5})$

I. 2. SERIES OF NUMBERS

A. DEFINITIONS

A sum

$$a_1 + a_2 + \ldots + a_n + \ldots = \sum_{n=1}^{\infty} a_n \qquad (1)$$

of infinitely many numbers in the given order is called an infinite series or simply a series, where $a_n \in \mathbb{R}$ is the general term of the series.

In the sum (1) the numbers are to be added in succession, that is, a_2 is to be added to a_1, next a_3 is to be added to $a_1 + a_2$, then a_4 is to be added to $a_1 + a_2 + a_3$, and so on. This definition of (1) is equivalent to

$$\lim_{n \to \infty} s_n$$

where

$$s_n = a_1 + \ldots + a_n .$$

is called the general partial sum of the series.

If this limit exists which is equal to Σa_n, we call (1) convergent, otherwise divergent.

For instance the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} = 1 - 1 + 1 - 1 + \ldots + (-1)^{n+1} + \ldots$$

is divergent, since

$$s_n = 1 - 1 + \ldots + (-1)^{n+1} = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

has no limit.

Writing

$$\sum_{n=1}^{\infty} a_n = s_n + R_{n+1}$$

where

$$R_{n+1} = a_{n+1} + a_{n+2} + \dots$$

is the <u>remainder</u> after the general term, we have

$$R_{n+1} = \left(\sum_{1}^{\infty} a_n\right) - s_n = (\lim s_n) - s_n = s - s_n$$

showing that

$$R_{n+1} \to 0$$

for a convergent series.

<u>Arithmetic, geometric, harmonic series</u>:

These series are related to arithmetic, geometric and harmonic means of numbers. If $a, b>0$, then the numbers A, G, H defined by

$$A = \frac{a+b}{2}, \quad G = \sqrt{ab}, \quad \frac{2}{H} = \frac{1}{a} + \frac{1}{b}$$

are respectively called the <u>arithmetic</u>, <u>geometric</u> and <u>harmonic means</u> of a, b. Observe that H is the reciprocal of the arithmetic mean of the reciprocals of a and b.

A series whose all terms are positive is called an <u>arithmetic</u>, <u>geometric</u> or a <u>harmonic series</u> according as every term a_n of the series, except the first one, is the arithmetic, geometric or harmonic mean of its adjacent terms a_{n-1} and a_{n+1}. The first term is an arbitrary positive number.

<u>Arithmetic series</u>: $a + (a+d) + (a+2d) + \dots + (a+nd) + \dots$

<u>Geometric series</u>: $a + ar + ar^2 + \dots + ar^n + \dots$

<u>Harmonic series</u>: $\frac{1}{a} + \frac{1}{a+d} + \frac{1}{a+2d} + \dots + \frac{1}{a+nd} + \dots$

In an arithmetic series $a_{n+1} - a_n = d$ is the <u>common difference</u>, in a geometric series $a_{n+1}/a_n = r$ is the <u>common ratio</u>.

These three series defined for $a, d, r>0$ have obvious generalizations o negative a, d and r.

An arithmetic series is convergent only when $a=0$ and $d=0$ and divergent in other cases.

A geometric series (for $a\neq 0$) is obviously divergent for $r=1$. In other cases (for $a\neq 0$), having

$$S_n = a(1+r+ \dots + r^n) = a\,\frac{r^{n+1}-1}{r-1}$$

there is convergence when $|r|<1$ or when $-1<r<1$, and divergence when $|r|\geqslant 1$. Hence

$$\sum_{n=0}^{\infty} ar^n = \begin{cases} \frac{a}{1-r} & \text{if } |r|<1 \\ \infty & \text{if } |r|\geqslant 1 \end{cases}$$

The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \dots$$

corresponds to $a=1$ and $d=1$. We show that it is a divergent series. To prove divergence we assume its convergence and produce a contradiction by establishing a relation between h_{2n} and h_n where h_n is the general partial sum:

$$h_{2n} = (1 + \tfrac{1}{2})+(\tfrac{1}{3} + \tfrac{1}{4})+ \dots +(\tfrac{1}{2n-1} + \tfrac{1}{2n})$$

$$> (1 + \tfrac{1}{3})+(\tfrac{1}{4} + \tfrac{1}{4})+ \dots +(\tfrac{1}{2n} + \tfrac{1}{2n})$$

$$= \tfrac{1}{3} + 1 + \tfrac{1}{2} + \dots + \tfrac{1}{n} = \tfrac{1}{3} + h_n$$

$$\Longrightarrow h_{2n} - h_n > \tfrac{1}{3} \quad \text{contradicting } h_{2n}-h_n \to h-h = 0.$$

Theorem.

a) In a convergent series, the general term tends to zero,

b) If, in a series, the general term does not tend to zero, the series is divergent.

We call Part b test for divergence by general-term lest

Proof.

a) Let $\sum_1 a_n = s$. Then $a_n = s_{n+1} - s_n \to s - s = 0$

b) Let $a_n \to k \neq 0$, but suppose the series is convergent.

Then by a), $a_n \to 0$ contradicting $k \neq 0$.

Remark. As the harmonic series shows, the approach of a_n to zero does not imply the convergence of the series, that is, the series in which $a_n \to 0$ may converge or diverge.

Tests for convergence and divergence are generally given for series of positive terms.

B. SERIES OF POSITIVE TERMS

A series having almost every term (all terms except perhaps finitely many ones) positive is called a series of positive terms. Thus the series

$$\sum_1 1/n, \quad \sum_0 |a_n|, \quad \sum_1 \frac{n^2-2n-3}{n^2} = -4 - \frac{3}{4} - 0 + \frac{5}{6} + \dots + \frac{n^2-2n-3}{n^2} + .$$

are series of positive terms, while $\sum_0 (-1/2)^n$ is not.

The tests that are given here are comparison tests by which a given series of positive terms is compared either with a positive improper integral or with another series of positive terms. There are also intrinsic tests which are applied directly to a given series itself.

1. Integral test (of McLAURIN):

Theorem. Let $\sum\limits_{p}^{\infty} a_n$ be a series of positive term with $a_n = f(n)$ and $D_f = [p, \infty)$. If $f(x)$ is positive, continuous and decreasing on $[N, \infty)$, $N>p$, then

$$\sum_{p}^{\infty} a_n \quad \text{and} \quad \int_N^{\infty} f(x)\,dx$$

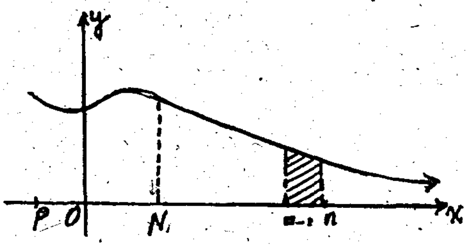
are both convergent or both divergent.

Proof. Consider the partial sum s_n. Separating from it the terms $a_p, \ldots, a_{N-1}$ of $[p, N-1]$ on which $f(x)$ may not satisfy the hypothesis, we set

$$s_n = s_{N-1} + \underbrace{a_N + \ldots + a_n}_{s'_n} \quad (s_{N-1} \text{ is finite}).$$

$\lim s_n$ exists if $\lim s'_n$ exists. Then the series is convergent, otherwise divergent.

Consider the graph of $f(x)$. $f(x)$ being decreasing, one has the double inequality



$$f(n) < \int_{n-1}^{n} f(x)dx < f(n-1)$$

which, when written for all subintervals on $[N, n]$ and added gives

$$s'_n - s_N < \int_N^n f(x)dx < s'_{n-1} - s_{N-1}.$$

If the integral has a limit when $n \to \infty$, this limit is an upper bound for the increasing sequence $(s'_n - s_N)$. Then (s'_n) and hence $\sum\limits_{p} a_n$ is convergent.

If the integral diverges to ∞ the same is true for $(s'_{n-1} - s_{N-1})$. Then (s'_{n-1}) and hence $\sum\limits_{p} a_n$ is divergent.

p-series:

The series

$$\sum_{n=1} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{n^p} + \dots$$

is called a p-series. Clearly the harmonic series is a p-series for $p=1$.

Theorem. The p-series is convergent when $p>1$, divergent when $p\leq 1$.

Proof. For $p=1$ the p-series is harmonic and therefore divergent.

The function $f(x) = 1/x^p$ is positive, continuous and decreasing on $[1, \infty)$. Then

$$\int_1^\infty \frac{dx}{x^p} = \int_1^\infty x^{-p}dx = \frac{x^{-p+1}}{-p+1} = \frac{1}{1-p} \left. x^{1-p} \right|_{x=1}^{\infty}$$

which is convergent if $1 - p < 0$ or when $p>1$, and divergent when $p<1$.

Example 1. Test the series

$$\sum_0^\infty \frac{1}{n^2+1} = 1 + \frac{1}{2} + \frac{1}{5} + \dots + \frac{1}{n^2+1} + \dots$$

for convergence.

Solution. The series is of positive terms. The function $f(x) = 1/(x^2+1)$ being positive, continuous and decreasing on $[1, \infty)$ the integral test is applicable:

$$\int_1^\infty \frac{dx}{x^2+1} = \arctan x \Big|_1^\infty = \frac{\pi}{2} \quad \text{(convergent)}.$$

Hence the given series is convergent.

<u>Example 2</u>. Test the series $\sum\limits_{2} \frac{1}{n \; \ell n^p n}$ for convergence.

<u>Solution</u>. The series is of positive terms. The function $f(x) = 1/(x \; \ell n^p x)$ fulfills the conditions of the integral test on $[2, \infty)$:

$$\int_2^\infty \frac{dx}{x \; \ell n^p x} = \int_2^\infty \frac{d \; \ell n \; x}{\ell n^p \; x} = \frac{1}{1-p} \left(\ell n \; x\right)^{1-p} \Big|_2^\infty$$

converging when $p>1$, diverging when $p\leq 1$. Hence the given series is convergent only when $p>1$.

2. <u>Comparison with other series</u>

Below we give two tests by comparison of a given series of positive terms with other such series.

<u>Theorem 1</u>. (Test by inequality)

Let $\sum a_n$ be a given series of positive terms, and let $\sum c_n$, $\sum d_n$ be two such series which are convergent and divergent respectively. Then $\sum a_n$ is convergent or divergent according as

$$a_n \leq c_n \quad \text{or} \quad a_n \geq d_n$$

for all $n>N$ for some N.

<u>Proof</u>. $a_n \leq c_n \Longrightarrow \sum\limits_N^n a_n \leq \sum\limits_N^n c_n$. By hypothesis $\sum c_n$ having a limit as $n \to \infty$, the sequence $(\sum\limits_N^n a_n)$ is bounded above by this limit. Since it is increasing, has a limit, and $\sum a_n$ is convergent.

Divergence case can be proved similarly.

A <u>generalization</u> of the above Theorem 1:

The Theorem holds true when the inequalities are replaced by

$$a_n \leq p \; c_n \quad \text{or} \quad a_n \geq q \; d_n$$

where p and q are positive numbers.

In tests for convergence, a series is compared often either with a geometric series $\sum r^n$ or with p-series $\sum 1/n^p$, since there is posibility for an easy selection of r, p for comparison.

Example 1. Test $\sum_{1}^{\infty} \frac{2n+3}{n^2}$ for convergence.

Solution. Writing $a_n = \frac{2n+3}{n^2} = \frac{2}{n} + \frac{3}{n^2}$, the series of positive terms is comparable with harmonic series:

$$a_n > \frac{2}{n} \Longrightarrow \sum a_n > 2\sum \frac{1}{n} \Longrightarrow \text{divergence.}$$

Example 2. Test $\sum_{1} \frac{n}{\ell nn + 2^n}$ for convergence.

Solution. Having

$$a_n = \frac{n}{\ell n n+2^n} \leqslant \frac{n}{2^n} \leqslant \frac{(3/2)^n}{2^n} = (\tfrac{3}{4})^n,$$

$\sum a_n$ is convergent since the geometric series $\sum (3/4)^n$ is convergent.

This Theorem 1 and its generalization may be rephrased as follows:

Corollary.

If for series $\sum a_n$, $\sum b_n$ of positive terms,

$$a \leqslant \frac{a_n}{b^n} \leqslant b \quad (\text{or} \quad a.b_n \leqslant a_n \leqslant b.b_n)$$

holds for positive numbers a, b for all $n>N$, then $\sum a_n$, $\sum b_n$, are both convergent or both divergent.

Theorem 2. (Test by limit ratio).

Let $\sum a_n$, $\sum b_n$ be two series of positive terms with

$$\lim \frac{a_n}{b_n} = \lambda(>0) \quad (\lambda \neq 0, \ \lambda=0, \ \text{or } \lambda = \infty)$$

a) If $\lambda \neq 0$, the series are of the same nature,

b) If $\lambda=0$, the convergence of $\sum b_n$ implies that of $\sum a_n$ (or divergence of $\sum a_n$ implies that of $\sum b_n$)

c) If $\lambda = \infty$, the convergence of $\sum a_n$ implies that of $\sum b_n$ (or divergence of $\sum b_n$ implies that of $\sum a_n$)

Proof.

a) If $\lambda \neq 0$, given $\varepsilon > 0$ and less then λ, there is $N > 0$ such that

$$\lambda - \varepsilon < \frac{a_n}{b_n} < \lambda + \varepsilon \quad \text{for all} \quad n > N.$$

Then by above Corollary the two series have the same nature.

b) Let $\lambda=0$. Then for $n > N$ for some N,

$$\frac{a_n}{b^n} < \varepsilon \quad \text{or} \quad a_n < \varepsilon b_n$$

holds. By Theorem 1 the assertion is true.

c) Let $\lambda = \infty$. Then there is $a > 0$ such that

$$\frac{a_n}{b^n} > a \quad \text{or} \quad a_n > ab_n$$

for all $n > N$. Again by Corollary the assertion is true. ∎

Example. Test the convergence by limit rati st:

a) $\sum\limits_1 n\, e^{-n}$ b) $\sum\limits_1^\infty \tan \frac{1}{n}$

Solution.

a) Comparing $a_n = \frac{n}{e^n}$ with $b_n = \frac{1}{n^2}$ we have

$$\lambda = \lim \frac{a_n}{b_n} = \lim \frac{n^3}{e^n} = 0 \Longrightarrow \text{conv. of } \sum_1 n\, e^{-n}$$

b) Comparing $a_n = \tan \frac{1}{n}$ with $b_n = 1/n$ we have

$$\lambda = \lim \frac{a_n}{b_n} = \lim \frac{\tan \frac{1}{n}}{\frac{1}{n}} = 1 \Longrightarrow \text{div. of } \sum \tan \frac{1}{n}$$

3. Intrinsic tests

By an intrinsic test we mean one related only to the terms of the given serie of positive terms without reference to other series. The following two theorems express such tests:

Theorem 1. (Root and ratio tests of CAUCHY)

A series $\sum a_n$ of positive terms is convergent if there is a number k such that

a) $\sqrt[n]{a_n} \leqslant k < 1$ or b) $\frac{a_{n+1}}{a_n} \leqslant k < 1$

and divergent if

a') $\sqrt[n]{a_n} \geqslant 1$ or b') $\frac{a_{n+1}}{a_n} \geqslant 1$.

for all $n > N$ for some N.

Proof.

a) Since $\sqrt[n]{a_n} \leqslant k < 1 \Longrightarrow a_n \leqslant k^n$, the series is comparable with convergent geometric series $\sum k^n$. Hence $\sum a_n$ is convergent.

b) $\frac{a_{n+1}}{a_n} \leqslant k < 1 \Longrightarrow a_{n+1} \leqslant k\, a_n$

$$\Rightarrow \quad \begin{aligned} &a_{N+1} \leqslant k\, a_N, \\ &a_{N+2} \leqslant k\, a_{N+1} \leqslant k^2\, a_N, \\ &\vdots \\ &a_n \leqslant k\, a_{n-1} < \dots < k^{n-N}\, a_N \end{aligned}$$

$$\Rightarrow \quad S_n = a_1 + \dots + a_{N-1} + a_N(1+k + \dots + k^{n-N}) \to \text{a limit}$$

a') $\sqrt[n]{a_n} \geqslant 1 \Longrightarrow a_n \geqslant 1 \Longrightarrow a_n \not\to 0$ (div.)

b') $\frac{a_{n+1}}{a_n} \geqslant 1 \Longrightarrow a_{n+1} \geqslant a_n \Longrightarrow a_n \not\to 0$ (div.)

The following tests are more useful in practice, than the above test because of the determination of k.

<u>Theorem 2</u>. (Lim root, lim ratio tests of CAUCHY)

A series $\sum a_n$ of positive series is convergent if

a) $\lim \sqrt[n]{a_n} < 1$ or b) $\lim \frac{a_{n+1}}{a_n} < 1$,

and divergent if

a') $\lim \sqrt[n]{a_n} > 1$ or b') $\lim \frac{a_{n+1}}{a_n} > 1$.

Test fails if limits are equal to 1.

<u>Proof</u>.

a) Let $\lim \sqrt[n]{a_n} = r$.

If $r < 1$ there is k such that $r < k < 1$. Since r is the limit, $\sqrt[n]{a_n} \leqslant k$ holds for all $n > N$ for some N. Then by root test, $\sum a_n$ is conv.

a') If $r > 1$, then $\sqrt[n]{a_n} > 1$ holds for all $n > N$ and $a_n \not\to 0$. (div.)

The proofs of b, b' are similar. ■

<u>Remark</u>. If one of the lim root, lim ratio test fails, the other fails too.

In the failure case, one way apply the fo ing:

<u>RAABE-DUHAMEL's Test</u>:

A series $\sum a_n$ of positive terms is converg... or divergent according as

$$\lim n\left(\frac{a_n}{a_{n+1}} - 1\right)$$

is greater or less than 1. Test fails if limit is equal to 1.

<u>Example</u>. Test the following series of positive terms for convergence:

a) $\sum\limits_{1} (\frac{n+1}{n})^n$ b) $\sum\limits_{0} \frac{2^n}{n!}$ c) $\sum\limits_{2} \frac{n!}{n^n}$ d) $\sum\limits_{-1} \frac{n}{n+2}$

Solution.

a) Since $a_n \to e \neq 0$, the series is diverges.

b) $\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \frac{2}{n+1} \to 0 < 1$ (it converges)

c) $\sqrt[n]{a_n} = \frac{\sqrt[n]{n!}}{n} \to \frac{1}{e} < 1$ (it converges)

d) $a_n \to 1 \neq 0$ (it diverges)

C. ALTERNATING SERIES

A series

$$\sum_{0}^{\infty} (-1)^n a_n = a_0 - a_1 + a_2 - \dots + (-1)^n a_n + \dots \quad (a_n > 0)$$

in which the terms alternate in sign is called an alternating series.

The series

$$1 - \frac{1}{2} + \frac{1}{3} - \dots + (-1)^{n+1} \frac{1}{n} + \dots$$

is an alternating one, known as the alternating harmonic series.

Since an alternating series is not a series of positive terms, the previous tests cannot be applied. However there is a test special to alternating series which is the following:

Theorem (LEIBNIZ)

The alternating series

$$a_0 - a_1 + a_2 - \dots + (-1)^n a_n + \dots \quad (a_n > 0)$$

is convergent if

a) $a_0 \geqslant a_1 \geqslant a_2 \geqslant \dots \geqslant a_n \geqslant \dots$

b) $a_n \to 0$.

Proof. It will suffice to prove that s_{2n} and s_{2n+1} have the same limit.

$$s_{2n} = (a_0-a_1)+ \dots +(a_{2n}-a_{2n}) \geq 0 \quad \text{(from 1)}$$

$$s_{2n} = a_0-(a_1-a_2)- \dots -(a_{2n-2}-a_{2n-1})-a_{2n} \leq a_0 \quad \text{(from 1)}$$

$$\Longrightarrow 0 \leq s_{2n} \leq a_0 .$$

Hence (s_{2n}) is bounded, and being monotone increasing it converges to a limit s. Now,

$$s_{2n+1} = s_{2n}+a_{2n+1} \to s+0 = s. \blacksquare$$

Corollary. In a convergent alternating series

$$s = a_0-a_1+a_2- \dots + (-1)\ a_n+R_{n+1}$$

with given hypothesis, the inequality

$$|R_{n+1}| < a_{n+1}$$

holds, that is the error made in taking s_n for s is less than a_{n+1}.

Proof.

$$R_{n+1} = (-1)^{n+1}(a_{n+1} - a_{n+2}+ \dots)$$

$$\Longrightarrow |R_{n+1}| = |a_{n+1} - a_{n+2} + \dots |$$

$$= a_{n+1} - (a_{n+2}-a_{n+3})- \dots < a_{n+1}$$

Example. Given the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \dots + (-1)^{n+1} \frac{1}{n} + \dots$$

a) show its convergence

b) to have max error of 10^{-2} in sum s, how many terms should be taken?

Solution.

a) Since $1 \geqslant \frac{1}{2} \geqslant \frac{1}{3} \geqslant \dots \geqslant \frac{1}{n} > \dots$ and $\frac{1}{n} \to 0$, there is convergence.

b) $\frac{1}{n+1} < \frac{1}{10^2} \Rightarrow n+1 > 100 \Rightarrow n > 99$ (100 terms)

D. SERIES OF ARBITRARY TERMS

If a series is one of positive terms, one applies a test given for such series, if the series is alternating, one applies LEIBNIZ'test (test for alternating series). For an arbitrary series the following theorem holds:

Theorem. A series

$$\sum_{1} a_n = a_1 + a_2 + \dots + a_n + \dots$$

is convergent if the series

$$\sum_{1} |a_n| = |a_1| + |a_2| + \dots + |a_n| + \dots$$

of absolute values is convergent.

Proof. Let s_n, S_n be the corresponding partial sums. The sum s_n contains non negative and negative terms and we write

$$s_n = P_n - Q_n, \quad S_n = P_n + Q_n$$

where $P_n, -Q_n$ are sums of positive and negative terms respectively.

(P_n), (Q_n) are monotone increasing sequences bounded above by $S = \lim S_n$. Then $P_n \to P$, $Q_n \to Q$. and $S_n \to P + Q$. It follows that $s_n \to P - Q$. ■

A series $\sum a_n$ such that $\sum |a_n|$ is convergent is called an absolutely convergent series, and the above theorem states that an absolutely convergent series is convergent.

As the alternating harmonic series shows, a series may be convergent without being absolutely convergent. Such series are called simply convergent[1] series:

$$\sum |a_n| \text{ (conv.)} \Longrightarrow \sum a_n \text{ (conv)} \quad \ldots \text{ abs. conv. of } \sum a_n$$

$$\sum |a_n| \text{ (div.)} \Longrightarrow \begin{cases} \Sigma a_n \text{ (conv)} & \ldots \text{ simply conv. of } \Sigma a_n \\ \text{or} & \\ \Sigma a_n \text{ (div)} & \end{cases}$$

There is an essential difference between the absolutely convergent series and simply convergent ones. The absolutely convergent series have the following two properties among others:

1. The terms can be rearranged in any order (rearrangement does not alter the sum).
2. Finitely or infinitely many terms may be replaced by their sum.

These properties may not be shared by simply convergent series, that is, a rearrangement of terms in a simply convergent series may give a different sum as illustrated by the following example:

Consider the simply convergent alternating harmonic series

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + (-1)^{n+1}\frac{1}{n} + \ldots$$

Let us rearrange the terms to have the series

$$S' = \left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{10} - \frac{1}{12}\right)$$
$$+ \ldots + \left(\frac{1}{2n+1} - \frac{1}{4n+2} - \frac{1}{4n+4}\right) + \ldots$$

[1] In many texbooks conditional convergent or semi-convergent terminologies are used instead of simply convergent.

Observe that every term in one series is contained in the other exactly once:

$$s' = \sum_0 \left(\frac{1}{2n+1} - \frac{1}{4n+2} - \frac{1}{4n+4} \right)$$

$$= \sum_0 \left(\frac{1}{4n+2} - \frac{1}{4n+4} \right) = \frac{1}{2} \sum_0 \left(\frac{1}{2n+1} - \frac{1}{2n+2}\right)$$

$$= \frac{1}{2} \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + (-1)^{n+1} \frac{1}{n} + \ldots\right) = \frac{1}{2} s.$$

E. EVALUATION OF SERIES

Each series can be evaluated by neglecting the remainder R_{n+1} for certain n with some approximation.

Exact evaluation is impossible in general, except for convergent geometric series and some series whose general term a_n is rational function of n. There are other possibilities by the use of power series (§1. 3).

Example: Evaluate the geometric series

a) $\sum_0 (\frac{1}{2})^n$ b) $\sum_0 (-1)^n \frac{2^n}{3^n}$

Solution. Recalling

$$a(1 + r + \ldots + r^n + \ldots) = \frac{a}{1-r} \quad (|r| < 1)$$

we have

a) $r = \frac{1}{2}$ $(|\frac{1}{2}| < 1) \Longrightarrow s = \dfrac{1}{1 - \frac{1}{2}} = 2,$

b) $r = -\frac{2}{3}$ $(|-\frac{2}{3}| < 1) \Longrightarrow s = \dfrac{1}{1 + \frac{2}{3}} = 3/5.$

Example. Given $t = 2{,}1\overline{37}$,

a) write it as a geometric series,

b) disscuss the convergence and find t as a ratio of two integers.

Solution.

a) $t = 2 + \frac{1}{10} + \frac{37}{1000} + \frac{37}{100000} + \ldots + \frac{37}{1000.100^{n-1}} + \ldots$

$$= \frac{21}{10} + \frac{37}{1000}\left(1 + \frac{1}{100} + \ldots + \frac{1}{100^{n-1}} + \ldots\right)$$

b) The series within the paranthesis is a geometric series with $r = 1/100$ which is absolutely less than 1. Then it is convergent:

$$t = \frac{21}{10} + \frac{37}{1000} \frac{1}{1 - \frac{1}{100}} = \frac{21}{10} + \frac{37}{990} = \frac{2116}{990} \in Q$$

Example. Find the sums:

a) $\frac{1}{1.2} + \frac{1}{2.3} + \ldots + \frac{1}{n(n+1)} + \ldots$

b) $\frac{1}{2^2-1} + \frac{1}{4^2-1} + \ldots + \frac{1}{(2n)^2-1} + \ldots$

Solution.

a) $a_n = \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} \Longrightarrow A = 1, \quad B = -1$

$$\Longrightarrow a_n = \frac{1}{n} - \frac{1}{n+1}$$

$$\Longrightarrow s_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1} \Longrightarrow S = 1$$

b) $a_n = \frac{1}{(2n)^2-1} = \frac{A}{2n-1} + \frac{B}{2n+1} \Longrightarrow A = \frac{1}{2}, \quad B = -\frac{1}{2}$

$$\Longrightarrow a_n = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

$$\Longrightarrow s_n = \frac{1}{2}\left[(1-\frac{1}{3})+(\frac{1}{3}-\frac{1}{5})+\dots+(\frac{1}{2n-1}-\frac{1}{2n+1})\right]$$

$$= \frac{1}{2}\ (1-\frac{1}{2n+1}) \Longrightarrow s = 1/2$$

EXERCISES (I, 2)

11. Are the following define a series? Give reason

a) $1+2^3+3^3+4^3+\ \dots$ b) $\sum \frac{1}{2^n}$ c) $\sum_0^\infty n!$

12. In the following series forming s_{2n} and s_{2n+1} show convergence or divergence

$$1 - 1 + \frac{1}{2} - \frac{1}{2} + \dots + \frac{1}{n} - \frac{1}{n} + \dots$$

13. Show the following for divergence by the use of general term test:

a) $\sum_1 \frac{2^n}{n^3}$ b) $\sum_1 \sin n^2 \frac{\pi}{2}$ c) $\sum_1 (1+\frac{2}{n})^n$ d) $\sum_0 (\frac{3}{2})^n$

14. Test the following for convergence by integral test:

a) $\sum_1 \frac{n}{n^2+1}$ b) $\sum_2 \frac{1}{n\ \ell n^2\ n}$

c) $\sum_3 \frac{1}{n\ \ \ell n\ n\ \ \ell n(\ell n\ n)}$ d) $\sum_0 \frac{1}{\sqrt{n^2+1}}$

15. Test for convergence by comparison.

a) $\sum_0 \frac{2^n+1}{3^n+n}$ b) $\sum_1 \frac{2n}{n^2+1}$

c) $\sum_1 \frac{2n}{2^n}$ d) $\sum_1 \frac{2n-1}{5n^2-31n+8}$

16. Same question for:

a) $\sum_5 \frac{3n}{n+2^n}$ b) $\sum_1 \frac{n!}{n^3}$

17. Test the series given in Exercises 5 and 6 by the use of CAUCHY's root or ratio test.

18. Same question for the following series:

a) $\sum_0 \frac{1}{n!}$ b) $\sum_1 \frac{n!}{10^{2n-1}}$

c) $\sum_2 (1 - \frac{2}{n})^n$ d) $\sum_0 (\frac{n}{n+1})^{n^2}$

19. Test the following alternating series for convergence:

a) $\sum_2 (-1)^n \frac{1}{\ln n}$ b) $\sum_0 \frac{(-1)^n}{n+1}$

c) $\sum_1 (-1)^n \sin \frac{1}{n}$ d) $\sum_1 (-1)^n \frac{\ln n}{n^2}$

20. Same question for:

a) $\sum_0 \frac{(-1)^n}{n!}$ b) $\sum_1 (-1)^n \cos \frac{1}{n}$

21. Test for convergence:

a) $\sum_2 \ln \frac{n}{n-1}$ b) $\sum_0 n^2 \ln (1 + \frac{1}{2^n})^{\frac{1}{2^n}}$

22. Test for convergence:

a) $\sum_1 \frac{n!}{n^n}$ b) $\sum_1 (\frac{n+1}{2n})^n$

23. Test for convergence:

a) $\sum_0 \frac{n^n}{(2n)!}$ b) $\sum_1 \frac{(n+1)(n+2)\ldots(n+n)}{2^n n^n}$

24. Discuss absolutely and simply convergence:

a) $\sum_1 (\frac{1}{n} - \frac{1}{n+\alpha n})$, $(0<\alpha_n<1)$ b) $\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n+1}} + \ldots$

25. Same question for:

a) $\sum_1 \frac{(-1)^{n-1}}{\ln(n+1)}$ b) $\sum_1 \frac{(-n)^n}{(2n-1)^{10}}$

26. Discuss the convergence, and find the sum when convergent:

a) $\sum\limits_0 \dfrac{5^n - 3^n}{6^n}$

b) $\sum\limits_1 (-1)^{n+1} \dfrac{1}{n}$ (use the identity:

$$1 - \frac{1}{2} + \frac{1}{3} - \dots - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+n}$$

and express the right hand side as a RIEMANN sum)

27. Find the sums:

a) $\sum \dfrac{n}{n^4+4}$ b) $\sum\limits_1 \dfrac{2n^2+6n+3}{n(n+1)(n+2)(n+3)}$

(Hint: Use decomposition into partial fractions)

28. Find s_n and evaluate lim s_n:

a) $\sum\limits_0 \sin \dfrac{4(4n^2+1)}{(4n^2-1)^2} \sin \dfrac{16n}{(4n^2-1)^2}$ b) $\sum\limits_1 \dfrac{\sqrt{n+1} - 2\sqrt{n} + 1}{2^{n+1}}$

29. Let $[A_0B_0]$ be the interval $[0, 1]$. Let $[A_0B_0]$ be bisected by B_1, $[A_0B_1]$ by A_1, ... , $[A_{n-1}B_n]$ by A_n and $[A_nB_n]$ by B_{n+1}. Show that limiting position of A_n (or B_n) trisects $[0, 1]$.

30. A ping-pong ball, when dropped from a height rebounds a distance of three fourth of that hight. Find the total distance travelled by the ball if the initial height is h cm.

ANSWERS TO EVEN NUMBERED EXERCISES

12. $s_{2n} = 0$, $s_{2n+1} = \dfrac{1}{2n+1} \to 0$, conv.

14. a) Div., b) conv., c) div., d) div.

16. a) Conv., b) div.,

18. a) Conv., b) div., c) div., d) conv.

20. a) Conv., b) div.

22. a) Conv., b) Conv.,

24. a) Simply conv.,

b) abs. conv.

26. a) Conv., S=4, b) conv., $s = \ln 2$

28. a) $s_n = -\frac{1}{2}\cos 4 + \frac{1}{2}\cos\frac{4}{2n+1}$, $s = \frac{1}{2}(1 - \cos 4)$

b) $s_n = \frac{\sqrt{n+1}}{2^{n+1}} - \frac{1}{2} + \frac{1}{4}\left(1 + \dots + \frac{1}{2^{n-1}}\right)$, $s = 0$

30. $7h$ cm

I. 3. Power Series

A. DEFINITIONS

An infinite series

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0+a_1(x-x_0)+ \ldots + a_n(x-x_0)^n+ \ldots \quad (1)$$

in powers of $x-x_0$, where x is a variable and x_0, a_n are constants, is called a power series.

If $x_0 = 0$, the power series is simply

$$\sum_{0}^{\infty} a_n x^n = a_0+a_1 x + \ldots + a_1 x^n + \ldots \quad (1')$$

Some power series are convergent for all values of the variable x, some others for no value of x except for $x=x_0$; still others are convergent for all values is an interval of non zero length and divergent outside this interval. Such an interval "I" is called the interval of convergence of the series (1).

There are series convergent (divergent) at both ends of the interval of convergence I or convergent at one end and divergent at the other.

A main problem, in connection with power series, is the determination of the interval of convergence. It is determined by CAUCHY's limit root or limit ratio test applied to series $\sum_0 |a_n(x-x_0)|$ of absolute values, yielding an inequality involving x, the solution set of which is the required interval of convergence I.

For every x in the obtained interval I, the power series will be absolutely convergent, and accordingly the interval I may also be called the interval of absolute convergence.

In the interval of (absolute) convergence I the half length R is referred to as the radius of convergence[1].

1) The reason for calling it radius of convergence is that the power series $\sum a_n(z-z_0)$ for complex numbers admits a circle of convergence.

Applying to (1) the CAUCHY's limit root or limit ratio test, say limit ratio test, for absolute convergence, we set $u_n = a_n(x-x_o)$ and have

$$\lim \left|\frac{u_{n+1}}{u_n}\right| = \lim \left|\frac{a_{n+1}(x-x_o)^{n-1}}{a_n(x-x_o)^n}\right|$$

$$= \lim \left|\frac{a_{n+1}}{a_n}\right| |x - x_o| < 1$$

for convergence, or

$$|x - x_o| < \lim \left|\frac{a_n}{a_{n+1}}\right|$$

where

$$\lim \left|\frac{a_n}{a_{n-1}}\right| = R$$

is the radius of (absolute) convergence, and

$$|x - x_o| < R \Rightarrow I = (x_o - R, x_o + R)$$

If limit root test has been applied, we would get

$$\lim \sqrt[n]{|a_n|} = R^{-1}.$$

<u>Example</u>. Find the interval of (absolute) convergent of the power series:

a) $\sum\limits_{1} \frac{n^n}{n!} (x-1)^n$ b) $\sum\limits_{0} (-1)^n \frac{\ell n^n 2}{n!} x^n$

<u>Solution</u>.

a) $u_n = \frac{n^n}{n!}(x-1)^n \Rightarrow |u_n| = \frac{n^n}{n!}|x-1|^n$,

$$\sqrt[n]{|u_n|} = \frac{n}{\sqrt[n]{n!}}|x-1| \to \frac{1}{e}|x-1| < 1$$

$$\Rightarrow |x-1| < e \Rightarrow -e < x-1 < e \Rightarrow 1-e < x < 1+e \Rightarrow$$

$$I = (1-e, 1+e)$$

b) In this case we evaluate R:

$$R = \lim \left|\frac{a_n}{a_{n+1}}\right| = \lim \left(\frac{\ell n^n 2}{n!} \Big/ \frac{\ell n^{n+1} 2}{(n+1)!}\right)$$

$$= \lim \frac{n+1}{\ell n 2} = \infty \Rightarrow I = (-\infty, \infty).$$

B. REPRESENTATION OF FUNCTIONS BY POWER SERIES

For the present the only function that is known to be representable by power series is $f(x) = 1/(1-x)$. Indeed the geometric series $\sum_0 x^n$ with common ratio x is known to be equal to $1/(1-x)$:

$$\frac{1}{1-x} = 1+x+x^2 + \ldots + x^n + \ldots$$

which is valid in $I = (-1, 1)$.

The question now arises as to which functions are representable by power series. The answer to this question is given by the following theorem borrowed form Advanced Calculus:

Theorem.

1. A function $f(x)$ admitting derivatives of all orders at a point x_0 has a unique power series representation
$$f(x) = a_0+a_1(x-x_0)+ \ldots +a_n(x-x_0)^n+ \ldots \quad (1)$$
valid in an interval I of absolute convergence.

2. The power series representing the derivative $f'(x)$ is obtained from (1) by term-by-term differentiation:
$$f'(x) = a_1+2a_1(x-x_0)+ \ldots +n\, a_n(x-x_0)^{n-1}+ \ldots \quad (2)$$
valid in the same interval as that of (1)

3. The power series representing the integral $\int_{x_0}^{x} f(x)dx$ is obtained from (1) by term-by-term integration.
$$\int_{x_0}^{x} f(t)dt = a_0(x-x_0)+ \ldots + \frac{a_n}{n-1}(x-x_0)^{n-1}+ \ldots \quad (3)$$
valid in the same interval as that of (1).

The coefficients a_0 and a_1 are immediately obtaniable from (1) and (2):

$$a_0 = f(x_0), \quad a_1 = f'(x_0).$$

Differentiating (2) n-1 times, one gets

$$f^{(n)}(x) = n(n-1) \ldots 1\, a_n + \text{powers of } (x-x_0)$$

and

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

Thus we arrive at the power series:

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \ldots$$

of $f(x)$ at x_0, called the TAYLOR Series of $f(x)$ at $x = x_0$ which is valid for $x \in I$ and non valid if x is exterior to this interval I of absolute convergence.

TAYLOR series becomes what one calls the McLAURIN series if x_0 is taken at the origin:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$

Example. Find the power series of the following functions at indicated points:

a) e^x, $x=0$, b) $\sin x$, $\cos x$, $x=0$, c) $\ln x$, $x=1$

Solution.

a) $f(x) = e^x$ has derivatives of all orders at $x=0$:

$$f^{(n)}(x) = e^x \Rightarrow a_n = \frac{f^{(n)}(0)}{n!} = 1/n! \Rightarrow$$

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \ldots + \frac{1}{n!}x^n + \ldots \qquad (-\infty, \infty)$$

Since e^x is convergent for any $x \in R$, we have

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} + \ldots$$

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + (-1)^n \frac{1}{n!} + \ldots$$

b) $f(x) = \sin x$ has derivatives of all orders, and are $\cos x$, $-\sin x$, $-\cos x$, $\sin x$ periodically, and

$$f(0) = 0,\ f'(0) = 1,\ f''(0) = 0,\ f'''(0) = -1.$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{2n+1} + \ldots \qquad \ldots(-\infty, \infty)$$

The expansion of $\cos x$ is obtained more easily by differentiating the series of $\sin x$:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{2n} + \ldots \qquad \ldots(-\infty, \infty)$$

Note that the odd function $\sin x$ (the even function $\cos x$) involves only odd (even) powers of x.

c) $f(x) = \ln x$ has derivatives of all orders at $x = 1$.

$$f'(x) = \frac{1}{x} = x^{-1},\ f''(x) = (-1)x^{-2},\ f'''(x) = (-1)(-2)x^{-3},\ \ldots$$

$$f^{(n)}(x) = (-1)(-2)\ldots(-n-1)x^{-n} = (-1)^{n-1}(n-1)!\ \frac{1}{x^n}$$

$$\Rightarrow f^{(n)}(1) = (-1)^{n-1}(n-1)! \qquad n = 1, 2, \ldots$$

$$\ln x = \sum_1 \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_1 (-1)^{n-1}\frac{(x-1)^n}{n}$$

$$\ln x = \frac{x-1}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \ldots + (-1)^{n-1}\frac{(x-1)^n}{n} + \ldots(-1,1)$$

Part 3 of the theorem is usefull in obtaining expansions of functions whose derivatives are (simple) rational functions such as $\ln(1+x)$, $\arctan x$, Argth x.

Example. Find McLAURIN series for

a) $\ell n(1+x)$ b) arctan x c) Argth x

Solution. Knowing the sum of geometric series

a) $1 - x + x^2 - \dots + (-1)^n x^n + \dots = \frac{1}{1+x} = \frac{d}{dx}\,\ell n(1+x)$,

we have by term by term integration

$$\ell n(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n \frac{x^{n+1}}{n+1} - \dots, (-1, 1)$$

In a similar way :

b) $\frac{d}{dx}\arctan x = \frac{1}{1+x^2} = 1 - x^2 + x^4 - \dots + (-1)^n x^{2n} + \dots$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^n \frac{x^{2n+1}}{2n+1} + \dots \quad (-1, 1)$$

c) $\frac{d}{dx}\,\text{Argth}\, x = \frac{1}{1-x^2} = 1 + x^2 + x^4 + \dots + x^{2n} + \dots$

$$\text{Argth}\, x = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots + \frac{x^{2n+1}}{2n+1} + \dots \qquad (-1, 1)$$

The series of $\ell n(1+x)$ being convergent at the end point 1 of $I = (-1, 1)$, we have

$$\ell n\, 2 = 1 - \frac{1}{2} + \frac{1}{3} - \dots + (-1)^{n+1}\frac{1}{n} + \dots \quad \text{(why convergent?)}$$

The same being true for arctan x we get

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \dots + (-1)^n \frac{1}{2n+1} \qquad \text{(why convergent?)}$$

These are not preferable for computing $\ell n\, 2$ and $\pi/4$ as the series converge very slowly.

If $f(x)$ is a polynomial of degree n the power series representing it, is a finite series, since all derivatives of order heigher than n are zero.

Example. Expand $P(x) = x^3 - x + 1$ in powers of $x-1$.

Solution. $P(x) = x^3 - x + 1$, $P'(x) = 3x^2 - 1$, $P''(x)=6x$, $P'''(x) = 6 \Rightarrow P(1) = 1$, $P'(1) = 2$, $P''(1) = 6$, $P'''(1) = 6$

$$\Rightarrow \quad P(x) = P(0) + \frac{P'(0)}{1!}(x-1) + \frac{P''(0)}{2!}(x-1)^2 + \frac{P'''(0)}{3!}(x-1)^3$$

$$x^3-x+1 = 1+2(x-1) + 3(x-1)^2 + (x-1)^3.$$

Binomial series

The expansion of $(1+x)^\alpha$ for any $\alpha\varepsilon R$ is called a binomial series which becomes a finite series when α is a positive integer n:

$$\begin{aligned} 1+x &= 1+x \\ (1+x)^2 &= 1 + 2x + x^2 \\ (1+x)^3 &= 1 + 3x + 3x^2 + x^3 \\ &\vdots \\ (1+x)^n &= \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \dots + \binom{n}{k} x^k + \dots + \binom{n}{n} x^n \end{aligned}$$

where the coefficients are binomial coefficients given by

$$\binom{n}{k} = C_n^k = C(n, k) = \frac{n(n-1)\dots(n-k+1)}{k!}.$$

In the binomial series the coefficients of x^n will be denoted by $\binom{\alpha}{n}$ which we show to be

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}. \quad \text{with } \binom{\alpha}{0} = 1$$

Let $f(x) = (1+x)^\alpha$. Then

$$\begin{aligned} f(x) &= (1+x)^\alpha & &\Rightarrow & f(0) &= 1 \\ f'(x) &= \alpha(1+x)^{\alpha-1} & &\Rightarrow & f'(0) &= \alpha \\ f''(x) &= \alpha(\alpha-1)(1+x)^{\alpha-2} & &\Rightarrow & f''(0) &= \alpha(\alpha-1) \\ &\vdots & & & &\vdots \\ f^{(n)}(x) &= \alpha(\alpha-1)..(\alpha-n+1)(1+x)^{\alpha-n} & &\Rightarrow & f^{(n)}(0) &= \alpha(\alpha-1)\dots(\alpha-n+1) \end{aligned}$$

$$(1+x)^{\alpha} = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \dots + \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}x^n + \dots$$

As to the radius of convergence, we have

$$R = \lim \left|\frac{a_n}{a_{n+1}}\right| = \lim \frac{|\alpha(\alpha-1)\dots(\alpha-n-1)|}{n!} \cdot \frac{(n+1)!}{|\alpha(\alpha-1)\dots(\alpha-n+1)(\alpha-n)|}$$

$$= \lim \left|\frac{n+1}{\alpha-n}\right| = 1$$

Hence the interval of convergence is $(-1, 1)$ as in geometric series.

For $\alpha = -1$ we have the geometric series

$$\frac{1}{1+x} = 1-x+x^2 - \dots - (-1)^n x^n + \dots,$$

and for $\alpha = 1/2, \ -1/2, \ 1/3, \ -1/3$ the familiar algebraic functions $\sqrt{1-x}$, $\frac{1}{\sqrt{1-x}}$, $\sqrt[3]{1-x}$, $\frac{1}{\sqrt[3]{1-x}}$ and their corresponding series.

Observe that the binomial series for $x>0$ is an alternating series when $\alpha<0$ or $0<\alpha<1$.

When $\alpha > 1$ and non integer, say $\alpha = p+r$ (p is a positive integer and $0 < r < 1$) we have

$$(1+x)^{\alpha} = (1+x)^p (1+x)^r$$

where $(1+x)^p$ has finite number of terms and $(1+x)^r$ is an alternating series.

<u>Example</u>. Expand the following in powers of x:

a) $\frac{1}{\sqrt{1+x}}$ b) $(1+x)^{5/3}$

<u>Solution</u>.

a) $\frac{1}{\sqrt{1+x}} = (1+x)^{-1/2}$

$$= 1 + \frac{(-1/2)}{1!} x + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2!} x^2 + \ldots + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)\ldots(-\frac{1}{2}-n+1)}{n!} x^n + \ldots$$

$$= 1 - \frac{1}{2.1!} x + \frac{1.3}{2^2.2!} x^2 + \ldots + (-1)^n \frac{1\ldots(2n-1)}{2^n.n!} x^n - \ldots$$

b) $\alpha = \frac{5}{3} = 1 + \frac{2}{3} \Rightarrow (1+x)^{5/3} = (1+x)(1+x)^{2/3}$

In the second factor the exponent being less than 1, it can be expanded as above. or by direct expansion

$$(1+x)^{5/3} = 1 + \frac{5}{3} x + \frac{1}{2!} \ \frac{5}{3} (\frac{5}{3} - 1)x^2 + \frac{1}{3!} \ \frac{5}{3} (\frac{5}{3} - 1)(\frac{5}{3} - 2)x^3 + \ldots$$

which is an alternating series after the second term.

C. ALGEBRA OF POWER SERIES

The following theorem permits us to write the power series of $f(x) + g(x)$, $f(x) - g(x)$, $f(x)\ g(x)$ and $f(x)/g(x)$ as soon as the power series of $f(x)$ and $g(x)$ are known. For brievity the theorem will be stated for McLAURIN series:

Theorem. Let

$$f(x) = \sum_0 a_n x^n, \qquad g(x) = \sum_0 b_n x^n$$

be absolutely convergent on I_f, I_g respectively. Then the corresponding power series of $f\pm g$, fg, f/g are given by

$$f(x) \pm g(x) = \sum_0 (a_n \pm b_n) x^n$$

$$f(x) \quad g(x) = \sum_0 p_n x^n = \sum_0^\infty (a_0 b_n + a_1 b_{n-1} - \ldots + a_n b_0) x^n \text{ (CAUCHY Product)}$$

$$\frac{f(x)}{g(x)} = \sum_0 q_n x^n, \ (b_0 \neq 0)$$

where q_n $(n=0, 1, 2, \ldots)$ is given by the recurrence relations

$$a_n = b_0 q_n + b_1 q_{n-1} + \ldots b_n q_0,$$

each series being absolutely convergent at least in the intersection $I_f \cap I_g$.

If $f(x)$, $g(x)$ are expanded into power series at distinct points x_1 and x_2, then $f(x)$ must be expanded at x_2 or $g(x)$ at x_1, but if not possible expand both at a point x_3.

When it is necessary index shift will be helpful in the above operations. (Index shift: $\sum\limits_{n=0} a_n x^n = \sum\limits_{n=r} a_{n-r} x^{n-r}$)

Example 1. By the use of

$$\text{Ch } x = \frac{1}{2}(e^x + e^{-x}), \ \text{Sh } x = \frac{1}{2}(e^x - e^{-x})$$

obtain McLAURIN series for Ch x and Sh x.

Solution. Since

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^{2n}}{(2n)!} + \frac{x^{2n-1}}{(2n-1)!} + \dots \qquad (-\infty, \infty)$$

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \dots + \frac{x^{2n}}{(2n)!} - \frac{x^{2n-1}}{(2n-1)!} + \dots \qquad (-\infty, \infty)$$

by addition and subtraction, we get

$$\text{Ch}x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{x^{2n}}{(2n)!} + \dots \qquad (-\infty, \infty)$$

$$\text{Sh}x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2n-1}}{(2n-1)!} + \dots \qquad (-\infty, \infty)$$

Observe that the even (odd) function Chx(Shx) involves only the even (odd) powers of x.

EULER Formula:

Assuming that

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots + \frac{z^n}{n!} + \dots$$

is absolutely convergent for all complex numbers z, we have for $z = ix$, the expansion

$$e^{ix} = 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \ldots + \frac{(ix)^n}{n!} + \ldots$$

$$= 1 + i\,\frac{x}{1!} - \frac{x^2}{2!} - i\,\frac{x^3}{3!} + \ldots + i^n\,\frac{x^n}{n!} + \ldots$$

$$= (1 - \frac{x^2}{2!} + \frac{x^4}{4} - \ldots + (-1)^n\,\frac{x^{2n}}{(2n)!} + \ldots)$$

$$+ i(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\,\frac{x^{n-1}}{(2n-1)!} + \ldots)$$

$$= \cos x + i \sin x$$

$$e^{ix} = \cos x + i \sin x \qquad \text{(EULER)}$$

Example 2. Obtain the McLAURIN series for $\ln\sqrt{\frac{1+x}{1-x}}$.

Solution. Writing

$$\ln\sqrt{\frac{1+x}{1-x}} = \frac{1}{2}\ln\frac{1+x}{1-x} = \frac{1}{2}\left[\ln(1+x) - \ln(1-x)\right],$$

from

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + \frac{x^{2n}}{2n} - \frac{x^{2n-1}}{2n-1} + \ldots \qquad (-1,\ 1)$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots - \frac{x^{2n}}{2n} - \frac{x^{2n-1}}{2n-1} - \ldots, \qquad (-1,\ 1)$$

we get

$$\ln\sqrt{\frac{1-x}{1-x}} = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots + \frac{x^{2n-1}}{2n-1} + \ldots \qquad (-1,\ 1)$$

Example 3. Obtain the McLAURIN series for

a) $x^3 \cos x$ b) $[\ln(1+x)] \sin x$

Solution.

a) $x^3\cos x = x^3 \sum\limits_{0} (-1)^n \frac{x^{2n}}{(2n)!} = \sum\limits_{0} (-1)^n \frac{x^{2n+3}}{(2n)!}$

b) We have

$$[\ln(1+x)] \sin x = (x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1}\,\frac{x^n}{n} + \ldots)$$

$$(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\,\frac{x^{2n+1}}{(2n+1)!} - \ldots)$$

n :	0	1	2	3	4
a_n :	0	1	$-\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{4}$
b_n :	0	1	0	$-\frac{1}{6}$	0
p_n :	0	0	1	$-\frac{1}{2}$	$\frac{1}{6}$

$$[\ln(1+x)]\sin x = x^2 - \frac{1}{2}x^3 + \frac{1}{6}x^4 + \ldots$$

where the general term is omitted since it is unnecessary, and we have all properties of the series, because, those of $\ln(1+x)$ and $\sin x$ are known.

Example 4. Obtain power series expansions of the following rational functions at the indicated points:

a) $\frac{3+x}{1-x}$, $x = 0$ b) $\frac{3+x}{x}$, $x = 1$

Solution.

a) Direct division gives (since $b_0 = 1 \neq 0$)

$$\frac{3+x}{1-x} = 3 + 4x + 4x^2 + \ldots + 4x^n + \ldots .$$

Observe that it involves a geometric series with common ratio x. Then it is convergent for $|x| < 1$.

Obtain the same series by performing

$$(3+x)(1+x+ \ldots + x^n + \ldots),$$

and also by differentiating $(3+x)/(1-x)$ successively at $x = 0$.

b) The series being in powers of $x-1$, use substitution $x - 1 = t$ or $x = 1 + t$. Then

$$\frac{3+x}{x} = \frac{3+(1+t)}{1+t} = \frac{4+t}{1+t} = 4 - 3t + 3t^2 - \ldots + (-1)^n 3t^n - \ldots$$

$$= 4 - 3(x-1) + 3(x-1)^2 - \ldots + (-1)^n 3(x-1)^n + \ldots$$

convergent for $|x-1| < 1$.

When the series of $f(x)$ is given, to obtain the series of $f(u(x))$, on may set $u(x)$ for x. The following are examples to illustrate the process.

a) $\sin\frac{x}{2} = (\frac{x}{2}) - \frac{(x/2)^3}{3!} - \dots + (-1)^n \frac{(x/2)^{2n-1}}{(2n-1)!} + \dots$

b) $$e^{-x^2} = 1 + \frac{(-x^2)}{1!} + \frac{(-x^2)^2}{2!} + \dots + \frac{(-x^2)^n}{n!} + \dots$$
$$= 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \dots + (-1)^n \frac{x^{2n}}{n!} + \dots$$

c) $$e^{\sin x} = 1 + \frac{\sin x}{1!} + \frac{\sin^2 x}{2!} + \dots + \frac{\sin^n x}{n!} + \dots$$
$$\cong 1 + (x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots)$$
$$+ \frac{1}{2}(x - \frac{x^3}{3!} + \dots)^2 + \frac{1}{6}(x - \frac{x^3}{3!} + \dots)^3$$
$$\cong 1 - (x - \frac{x^3}{6} + \frac{x^5}{120}) + (\frac{x^2}{2} - \frac{x^4}{3}) + (\frac{x^3}{6} - \frac{x^4}{2})$$
$$e^{\sin x} \cong 1 + x + \frac{x^2}{2} - \frac{1}{8}x^4 - \frac{1}{15}x^5$$

D. THE REMAINDER THEOREM

Recall the MVT and its first extension for a function $f(x)$ defined on $[a, b]$, namely

$$f(b) = f(a) + R_1, \quad R_1 = \frac{f'(1)}{1!}(b-a), \qquad c_1 \varepsilon (\overline{a, b})$$

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + R_2, \quad R_2 = \frac{f''(2)}{2!}(b-a)^2, \quad c_2 \varepsilon (\overline{a, b})$$

under differentiability and continuity conditions.

The generalizations of these is expressed below as the <u>Remainder Theorem</u> or <u>Generalized MVT</u>:

<u>Theorem</u>. If $f(x)$ defined on $[a, b]$ satisfies

1. $f(x) \varepsilon D^{n+1}(a, b)$ (Hence continuous in (a, b))
2. $f(x)$ is continuous at a and b,

then there exists a number c_{n+1} in (a, b) such that

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \dots + \frac{f^{(n)}(a)}{n!}(b-a)^n + R_{n+1} \quad (1)$$

where the remainder R_{n+1} is given by

$$R_{n+1} = \frac{f^{(n+1)}(c_{n+1})}{(n+1)!}(b-a)^{n+1}$$

<u>Proof</u>. Transposing every term on the right hand side in (1), except R_{n+1}, to the left hand side and setting

$$\varphi(x) = f(b)-f(x) - \frac{f'(x)}{1!}(b-x) - \dots - \frac{f^{(n)}(x)}{n!}(b-x)^n,$$

consider the function

$$F(x) = \varphi(x) - \frac{(b-x)^{n+1}}{(b-a)^{n+1}}\varphi(a).$$

This function satisfies the three ROLLE conditions:

1) $F(x) \varepsilon D(a, b)$

2) $F(x)$ is continuous at a and b

3) $F(a) = \varphi(a) - \varphi(a) = 0$ and $F(b) = \varphi(b) - 0 = 0$.

Hence by ROLLE's Theorem there is a number $c_{n+1} \varepsilon (\overline{a, b})$ such that $F'(c_{n+1}) = 0$.

Now,

$$\begin{aligned} F'(x) &= \varphi'(x) + (n+1)\frac{(b-x)^n}{(b-a)^{n+1}}\varphi(a) \\ &= \Big\{- f'(x) - \Big(\frac{f''(x)}{1!}(b-x) - \frac{f'(x)}{1!}\Big) - \dots \\ &\qquad - \Big(\frac{f^{(n+1)}(x)}{n!}(b-a)^n - \frac{f^{(n)}(x)}{(n-1)!}(b-x)^{n-1}\Big)\Big\} \\ &\quad + (n+1)\frac{(b-x)^n}{(b-a)^{n+1}}\varphi(a) \\ &= -\frac{f^{(n+1)}(x)}{n!}(b-a)^n + (n+1)\frac{(b-x)^n}{(b-a)^{n+1}}\varphi(a) \end{aligned}$$

$$\Rightarrow \quad 0 = F'(c_{n+1}) = -\frac{f^{(n+1)}(c_{n+1})}{n!}(b-c_{n+1})^n + (n+1)\frac{(b-c_{n+1})^n}{(b-a)^{n+1}}\varphi(a)$$

$$\Rightarrow \quad \varphi(a) = \frac{f^{(n+1)}(c_{n+1})}{(n+1)!}(b-a)^{n+1}$$

which is, in view of (1'), the remainder R_{n+1}. ∎

The Remainder Theorem when written in the form

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_{n+1},$$

where

$$R_{n+1} = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}, \qquad c\varepsilon(\overline{x_0, x})$$

is called TAYLOR's Formula of $f(x)$ at x_0, and the one at $x_0 = 0$ is the McLAURIN's Formula.

If R_{n+1} is neglected, then $f(x)$ is said to be approximated by a polynomial of degree n:

$$f(x) \cong s_n(f) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

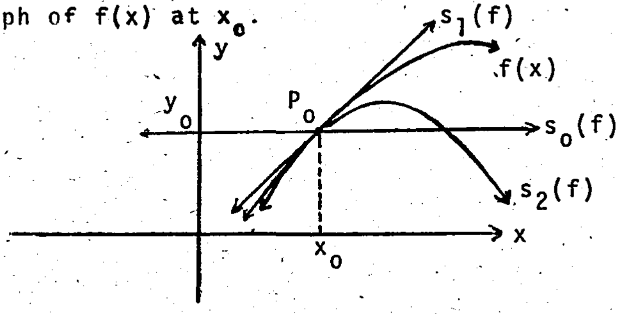
If $n = 0$, the initial approximation $s_0(f)$ is the constant polynomial

$$y = f(x_0)$$

The first and second approximations $s_1(f)$, $s_2(f)$ are the polynomial functions

$$y = f(x_0) + f'(x_0)(x-x_0)$$
$$y = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2$$

of which the first is the tangent line, and the second is a parabola tangent to the graph of $f(x)$ at x_0.



If one approximates $f(x) = s(x)$ by $s_n(f)$, then the error E is $|R_{n+1}|$. Since R_{n+1} is uncertain due to the presence of c_{n+1}, one must find an upper bound K:

$$E = |R_{n+1}| < K$$

Example. Compute the numbers.

a) e b) $\ell n \frac{4}{3}$ c) $\cos 3^0$

correct to three decimal places, that is with an error E less than 10^{-3}).

Solution.

a) We have

$$e^x = \Big(\sum_0^n \frac{x^k}{k!}\Big) + R_{n+1}, \quad R_{n+1} = \frac{e^c}{(n+1)!} x^{n+1}, \quad c \in (\overline{0, x}).$$

$x=1$ gives

$$e = \Big(\sum_0^n \frac{1}{k!}\Big) + \frac{e^c}{(n+1)!} \qquad (0 < c < 1)$$

$$E = \frac{e^c}{(n+1)!} \Rightarrow E < \frac{e}{(n+1)!} < \frac{3}{(n+1)!}$$

$$\frac{3}{(n+1)!} < \frac{1}{10^3} \Rightarrow (n+1)! > 3000 \Rightarrow n+1 > 7$$

$$e \cong 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!}$$

$$= \frac{1}{7!}(5040 + 5040 + 2580 + 840 + 210 + 42 + 7 + 1)$$

$$= 13700/5040 \cong 2{,}718$$

b) $\ell n(1+x) = \Big(\sum_1^n (-1)^{k-1} \frac{x^k}{k}\Big) + R_{n+1}, \quad |R_{n+1}| < \frac{|x|^{n+1}}{n+1}$

(since the series is alternating).

$1 + x = \frac{4}{3} \Rightarrow x = 1/3$. Then

$$\ell n \frac{4}{3} = \left(\sum_{1}^{n} (-1)^{k-1} \frac{1}{k.3^k}\right) + R_{n+1}$$

$$E = |R_{n+1}| < \frac{1}{(n+1)3^{n+1}} < \frac{1}{10^3} \Rightarrow (n+1)3^{n+1} > 1000$$

$\Rightarrow \quad n > 4$. Then

$$\ell n \frac{4}{3} \cong \frac{1}{1.3} - \frac{1}{2.3^2} + \frac{1}{3.3^3} - \frac{1}{4.3^4}$$

$$= 0{,}3333 - 0{,}0555 - 0{,}0123 - 0{,}0031$$

$$\cong 0{,}287.$$

c) We have

$$\cos x = \left(\sum_{0}^{n} (-1)^k \frac{x^{2k}}{(2k)!}\right) + R_{2n+1}, \quad |R_{2n+1}| = \frac{|\sin c|}{(2n+1)!} |x|^{2n+1}$$

or

$$\cos x = \left(\sum_{0}^{n} (-1)^k \frac{x^{2k}}{(2k)!}\right) + 0 + R_{2n+2}, \quad |R_{2n+2}| = \frac{|\cos c|}{(2n+2)!} |x|^{2n+2}$$

which are being valid for x in radian, one has $x = \frac{\pi}{60}$ rad $(= 3^0)$. Then

$$\cos 3^0 = \cos \frac{\pi}{60} = \sum_{0}^{n} (-1)^k \frac{1}{(2k)!} \left(\frac{\pi}{60}\right)^{2k} + \ldots$$

The error in computation must be less than the larger of $|R_{2n+1}|$, $|R_{2n+2}|$. Then

$$|R_{2n+1}| = \frac{|\sin c|}{(2n+1)!} \left(\frac{\pi}{60}\right)^{2n+1} < \frac{1}{(2n+1)!} \left(\frac{1}{15}\right)^{2n+1} < 10^{-3}$$

$$|R_{2n+2}| = \frac{|\cos c|}{(2n+2)!} \left(\frac{\pi}{60}\right)^{2n+2} < \frac{1}{(2n+2)!} \left(\frac{1}{15}\right)^{2n+2} < 10^{-3}$$

The first inequality (implying the second) is verified for $n > 0$, in particular for $n = 1$:

$$\cos 3^0 = 1 - \frac{1}{2}\left(\frac{\pi}{60}\right)^2 = 1 - \frac{\pi^2}{7200} \cong 1 - \frac{9.8696}{7200}$$

$$\cong 1 - 0,0014 = 0,9986 \cong 0,998.$$

Obtain $\cos 3^0$ approximately with an error less than 10^{-3} by the use of error property in alternating series.

E. SOME OTHER APPLICATIONS

1. Evaluation of limits in indeterminate forms:

$\lim\limits_{x \to x_0} \frac{f(x)}{g(x)}$ or $\lim\limits_{x \to x_0} [f(x) - g(x)]$ can be evaluated by expanding both $f(x)$ and $g(x)$ into power series at x_0.

Example. Evaluate

a) $\lim\limits_{x \to 1} (x)^{1/(x-1)}$ b) $\lim\limits_{x \to 0^-} \frac{e^x - 1 - x}{\sin x - x}$

Solution.

a) $y = x^{\frac{1}{x-1}} \Rightarrow \ln y = \frac{1}{x-1} \ln x$. Then

$$\ln x = (x-1) - \frac{(x-1)^2}{2!} + \ldots$$

$$\Rightarrow \ln y = 1 - \frac{x-1}{2!} + (\text{powers of } x-1)$$

$$\lim_{x \to 1} \ln y = 1 \Rightarrow \ln (\lim_{x \to 1} y) = 1 \Rightarrow \lim_{x \to 1} y = e.$$

b) $$\lim_{x \to 0^+} \frac{e^x - 1 - x}{\sin x - x} = \lim_{x \to 0^+} \frac{(\cancel{1} + \cancel{x} + \frac{x^2}{2!} + \ldots) - \cancel{1} - \cancel{x}}{(\cancel{x} - \frac{x^3}{3!} - \ldots) - \cancel{x}}$$

$$= \lim_{x \to 0^+} \frac{\frac{1}{2} + (\text{powers of } x)}{-\frac{x}{3!} + (\text{powers of } x)} = -\infty \text{ (no limit)}$$

2. Evaluation of integrals.

In case the evaluation of $\int f(x)dx$ for a given function $f(x)$ is impossible, or offers some difficulties, it is performed by power series.

Example. Evaluate $A = \int_0^3 e^{-x^2}dx$ approximately

Solution.

$$A = \int_0^{0,3} \left(1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \dots + (-1)^n \frac{x^{2n}}{n!} + \dots\right)dx, \quad I-(-\infty, \infty)$$

$$= x - \frac{x^3}{1!3} - \frac{x^5}{2!5} - \dots + (-1)^n \frac{x^{2n+1}}{n!(2n+1)} + \dots \Big|_0^{0,3}$$

$$= 0,3 - \frac{(0,3)^3}{1!3} + \frac{(0,3)^5}{2!5} - \dots + (-1)^n \frac{(0,3)^{2n+1}}{n!(2n+1)} + \dots$$

$$\cong 0,3 - 0,089 - 0,000343$$

$= 0,29443$ is correct with an error less than 10^{-5} (correct to five decimal places). Why?

3. Determination of the equation $y(x)$ of the curve through a given point with $y' = f(x, y)$.

Example. Find the series for $y(x)$ with the following conditions:

$$y' = \frac{x^2}{x+y}, \quad y(1) = -2 \quad \text{(initial condition)}$$

Solution. In the powers series

$$y(x) = y(1) + \frac{y'(1)}{1!}(x-1) + \frac{y''(1)}{2!}(x-1)^2 + \dots$$

of $y(x)$ at $x=1$ the coefficients are obtained by successive implicit differentiation of $(x+y)y' - x^2 = 0$ using $y(1) = -2$:

$$(x+y)y' - x^2 = 0 \Rightarrow (1-2)y'(1)-1 = 0 \Rightarrow y'(1) = -1,$$

$$(1+y')y' + (x+y)y'' - 2x = 0$$

$$\Rightarrow -y''(1) - 2 = 0 \Rightarrow y''(1) = -2.$$

$$y''y' + (1+y')y'' + (1+y')y'' + (x+y)y''' - 2 = 0 \quad \text{or}$$

$$y'y'' + 2(1+y')y'' + (x+y)y''' - 2 = 0$$

$$\Rightarrow (-2)(-1) - y'''(1) - 2 = 0, \quad y'''(1) = 0$$

$$y''^2+y'y'''+2y''^2+2(1+y')y'''+(1+y')y'''+(x+y)y''''=0$$

$$4+2.4-y''''(1) = 0 \Rightarrow y''''(1) = 12,$$

and so on. Then

$$y(x) = -2-(x-1)-(x-1)^2+\frac{1}{2}(x-1)^4+\ldots$$

4. Evaluation of series of constant terms

In §1. 2 evaluation of geometric series and some series with general term expressible as simple partial fraction was discussed. Now it is time to evaluate, identifying the series. In the first step identify $f(x)$, and in the second, find value a of x such that $f(a)$ is the given series.

Example. Evaluate

a) $A = 1+\frac{1}{1!2}+\frac{1}{2!2^2}+\ldots+\frac{1}{n!2^n}+\ldots$

b) $B = \sqrt{3}-\frac{(\sqrt{3})^3}{3}+\frac{(\sqrt{3})^5}{5}-+\ldots+(-1)^n\frac{(\sqrt{3})^{2n+1}}{2n+1}+\ldots$

c) $C = 1+2\cdot\frac{1}{3}+3\cdot\frac{1}{3^2}+\ldots+n\cdot\frac{1}{3^{n-1}}+\ldots$

Solution.

a) $f(x) = e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+\ldots$ and $x = \frac{1}{2}$
$\Rightarrow A=f(\frac{1}{2})=\sqrt{e}$.

b) $f(x) = \arctan x = x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots+(-1)^n\frac{x^{2n+1}}{2n+1}+\ldots$

and $x=\sqrt{3} \Rightarrow B = \arctan\sqrt{3} = \frac{\pi}{3}$.

c) $C = \sum_1^\infty n(\frac{1}{3})^{n-1}$; $f(x) = \sum n\,x^{n-1}$

$$= \frac{d}{dx}\sum x^n = \frac{d}{dx}(1+x+x^2+\ldots) = \frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2}$$

$$\Rightarrow C = f(\tfrac{1}{3}) = \tfrac{9}{4}.$$

EXERCISES (I. 3)

31. Find the interval of convergence of the following power series:

a) $\sum\limits_{0} \dfrac{nx^n}{n^2+1}$ b) $\sum\limits_{3} \dfrac{n^2}{n^4-1}(x+1)^n$ c) $\sum\limits_{0} \dfrac{n^2+1}{n!}x^n$

32. Same question for:

a) $\sum\limits_{0} \dfrac{n-1}{\sqrt{n^2+1}}x^{2n}$ b) $\sum\limits_{1} \dfrac{(x-2)^{2n}}{n+\sqrt{n}}$ c) $\sum\limits_{1} \dfrac{x^n}{n}$

33. In Exercises 31 and 32, discuss convergence at the end points of the intervals of convergence if finite.

34. For what values of x, there is absolute convergence:

a) $\sum\limits_{1} t^n$ where $t = \dfrac{-x}{x-2}$ b) $\sum\limits_{1} \dfrac{1}{n}t^n$ where $\dfrac{2x}{x+4} = t$

35. Determine the interval of convergence:

a) $\sum\limits_{a} P(n)x^n$ b) $\sum\limits_{0} \dfrac{P(n)}{n!}x^2$ c) $\sum\limits_{0} \dfrac{n!}{P(n)}x^n$

where $P(n)$ is a polynomial ($P(n)\neq 0$ for all n).

36. Find the interval of absolute convergence:

a) $\sum\limits_{1} \dfrac{n-1}{n}x^{n-1}$ b) $\sum\limits_{0} n^2(x-1)^n$ c) $\sum\limits_{0} x^{n^2}$

37. Same question for:

a) $\sum\limits_{1} \dfrac{(x-3)^n}{n}\ell n^n 4$ b) $\sum\limits_{0} (-1)^n \dfrac{x^{2n}}{(2n)!}$ c) $\sum\limits_{1} \dfrac{n^n}{n!}x^n$

38. Expand

a) $\cos x$, $\sin x$ at $x=\pi/2$ b) Ch x, Sh x at $x=\ell n2$

39. Expand the following at $x=0$:

a) $\tan x$, up to x^7 b) $\sec x$, up to x^6 c) $e^{\sin x}$, up to x^6

40. Expand

a) $f(x) = \dfrac{x}{x+1}$ at $x=2$, b) $\dfrac{3-2x}{x^2-3x+2}$ at 0.

(Hint for (b): decompose it into partial fractions)

41. Expand $1/(1+e^x)$ in powers of x up to x^4, using

$$1/(1+t) = 1-t+t^2 - \dots + (-1)^n t^n + \dots$$

42. Expand

a) x^3+2x^2-3x+1 in powers of $x-2$, b) x^4-3x^2+4 in powers of $x+2$

In the following exercises 43-50, use differentiation or integration of a known power series:

43. Find the sum:

a) $\sum\limits_1 n\,x^{n-1}$ b) $\sum\limits_1 n^2\,x^{n-1}$ c) $\sum\limits_1 n^3\,x^{n-1}$

44. Find the sum:

$$b+(a+b)x+(2a+b)x^2+ \dots +(na+b)x^n- \dots$$

45. Show that

$$2.1+3.2a+4.3a^2+ \dots + (n+2)(n+1)a^n+ \dots$$

is convergent for $|a| < 1$, and then find the sum.

46. Evaluate

$$\frac{1}{2} - (\frac{1}{2})^2 + \frac{1}{2!}(\frac{1}{2})^3 - \dots + (-1)^n \frac{1}{n!}(\frac{1}{2})^{n+1} - \dots$$

47. Find the sums:

a) $\sum\limits_0 \frac{(-1)^n}{(2n+1)!} \cdot \frac{\pi^{2n+1}}{3^{2n+1}}$ b) $\sum\limits_0 \frac{1}{(2n)!} \ln^{2n} 3$

48. Find the sum:

$$1 - \frac{x^4}{5} + \frac{x^8}{9} - \dots + (-1)^n \frac{x^{4n}}{4n+1} + \dots$$

49. Evaluate

$$\sum_0 \frac{nx^n}{(2n)!}$$

50. Evaluate

a) $\sum\limits_0 (2n+1)x^{2n+1}$ b) $\sum\limits_1 \frac{x^{2n}}{2n}$

51. Show the following

a) $\operatorname{Cos} ix = \operatorname{Ch} x$, $\operatorname{Ch} ix = \cos x$ b) $\operatorname{Sin} ix = i \operatorname{Sh} x$, $\operatorname{Sh} ix = i \sin x$

where $i^2 = -1$.

52. Let $f(x) = \sum_0 a_n(x-c)^n$. Prove $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$

53. Prove that if

$$f(x) = \sum_0 \frac{x^{2n}}{2^{2n}(2n)!}, \qquad F(x) = \sum_0 \frac{x^{2n}}{(2n)!}$$

then $2f^2(x) - F(x) = 1$.

54. Evaluate the following by series:

a) $\lim_{x \to 0} \frac{\ln(1+x)}{\sin x}$ b) $\lim_{x \to 0} \frac{e^x - \cos x}{x}$

c) $\lim_{x \to 0} \frac{e^x - \sin x - 1}{x}$ d) $\lim_{x \to 2} \frac{\sqrt{x+2} - 2}{x-2}$

55. Evaluate the following, correct to three decimal places:

a) $\sqrt{404}$ b) $\sqrt[3]{27090}$ c) $\sqrt[3]{0{,}375}$.

56. Graph $s_0(f)$, $s_1(f)$, $s_2(f)$ for $f(x) = e^x$ in a neighborhood $N[0]$.

57. Show by squaring that

$$(1+x+x^2+ \ldots +x^n+ \ldots)^2 = 1+2x+3x^2+ \ldots +(n+1)x^n+ \ldots$$

58. Show that the terms a_n of the FIBONACCI sequence a_n defined by $a_1=1$, $a_2=1$, $a_n=a_{n-1}+a_{n-2}$ are the coefficients of the McLAURIN series of

$$f(x) = \frac{1}{1-x-x^2}.$$

(Verify only 6 terms).

59. Show that the arc length of the ellipse $x=a\cos\theta$, $y=b\sin\theta$ with eccentricity $e = c/a$ $(c = \sqrt{a^2-b^2})$ is

$$s = 4a\int_0^{\pi/2} \sqrt{1-e^2 \sin^2\theta}\, d\theta$$

$$= (1 - \frac{1}{4} e^2 - \frac{3}{64} e^4 - \ldots)2\pi a$$

60. Evaluate correct to two decimal places:

a) $\int_0^1 \sin(x^2)dx$ b) $\int_0^\pi \frac{\text{Sin } x}{x} dx$

c) $\int_0^4 \sqrt{\frac{64+x}{x}}\, dx$ d) $2\int_0^1 e^{-x^2/4}dx$

Answers to even numbered exercises

32. a) $(-1, 1)$, b) $(1, 3)$, $(-1, 1)$

34. a) $x > 1$, b) $-4/3 < x < 4$

36. a) $(-1, 1)$, b) $(0, 2)$, c) $(-1, 1)$

38. $\cos x = \sum_0 (-1)^{n+1} \frac{1}{(2n+1)!} (x - \frac{\pi}{2})^{2n+1}$

$\sin x = \sum_0 (-1)^{n+1} \frac{1}{(2n)!} (x - \frac{\pi}{2})^{2n}$

$\text{Ch } x = \sum_0 (1 + \frac{(-1)^n}{4}) \frac{1}{n!} (x - \ln 2)^n$

$\text{Sh } x = \sum_0 (1 - \frac{(-1)^n}{4}) \frac{1}{n!} (x - \ln 2)^n$

40. a) $\frac{2}{3} + \sum_1 (-1)^{n+1} \frac{1}{3^{n+1}} (x-2)^n$, b) $\sum_0 \frac{2^{n+1}+1}{2^{n+1}} x^n$

42. a) $11 + 17(x-2) + 8(x-2)^2 + (x-2)^3$

b) $8 - 20(x+2) + 21(x+2)^2 - 8(x+2)^3 + (x+2)^4$

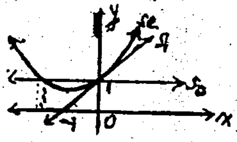
44. $\frac{a}{(1-x)^2} + \frac{b-a}{1-x}$

46. $1/(2\sqrt{e})$

48. $\frac{1}{2x}$ arctan x^2

50. a) $x\,\frac{1+x^2}{1-x^2}$, b) $\int \frac{dx}{x(1-x^2)}$

54. a) 1, b) 1, c) 0, d) 1/4.

56. 

60. a) 0,30 b) 1,85 c) 16,11 d) 1,84

A SUMMARY

(CHAPTER I)

1.1 If $f: \mathbb{Z}_p \to R$, $a_n = f(n)$, then $(a_n)_p^\infty: a_p, a_{p+1}, \ldots, a_n, \ldots$ is called an <u>sequence</u> which is convergent if $\lim a_n$ exists, <u>divergent</u> otherwise. $(a_n)_1$ is <u>monotone</u> if $a_1 \geq a_2 \geq \ldots \geq a_n \geq \ldots$ or $a_1 \leq a_2 \leq \ldots \leq a_n \leq \ldots$, and <u>bounded</u> if $-K \leq a_n \leq K$ for all $n \geq 1$ for some $K > 0$. Any monotone bounded sequence is convergent.

1.2 $\sum_1^\infty a_n = a_1 + \ldots + a_n + \ldots$ is called a <u>series</u>. It is called <u>convergent</u> if $s_n = a_1 + \ldots + a_n$ has a limit, <u>divergent</u> otherwise.

Geometric series: $\sum_0 ar^n = \frac{1}{1-r}$ if $|r| < 1$, divergent otherwise.

Harmonic series: $\sum_1 1/n$ is divergent.

p-series: $\sum_1 1/n^p$ is convergent for $p>1$, divergent for $p<1$

In a convergent series $a_n \to 0$. If $a_n \not\to 0$ the series is divergent (a test for divergence).

Tests for series of positive terms:

1) <u>Integral tests</u>: If $a_n = f(n)$, and $f(x)$ is positive and decreasing on $[p, \infty)$, then $\sum_p a_n$ and $\int_p^\infty f(x)dx$ are both convergent or both divergent.

2) <u>Comparison tests</u>: Let $d_n \leq a_n \leq c_n$. Then convergence of $\sum c_n \Rightarrow$ convergence of $\sum a_n$, and divergence of $\sum d_n \Rightarrow$ divergence of $\sum a_n$.

3) <u>Intrinsic tests of CAUCHY</u>: If $\sqrt[n]{a_n} \to \lambda$ or $a_{n+1}/a_n \to \lambda$, there is convergence for $\lambda < 1$, divergence for $\lambda > 1$. Test fails when $\lambda = 1$.

Test for alternating series:

If in the alternating series $a_0 - a_1 + a_2 - \dots + (-1)^n a_n + \dots$ ($a_n > 0$), a_n is decreasing and $a_n \to 0$, then the series is convergent, and $|R_{n+1}| < a_{n+1}$.

Tests for arbitrary series:

Convergence of $\sum |a_n| \Rightarrow$ convergence of $\sum a_n$. Such a series $\sum a_n$ is called <u>absolutely convergent.</u> $\sum a_n$ is called <u>simply convergent</u> if $\sum a_n$ is convergent but $\sum |a_n|$ is divergent.

1.3 $\sum\limits_0^\infty a_n (x-x_0)^n$ is called a <u>power series</u>, expanded in powers of $x-x_0$ (at x_0). The interval, at the interior of which the series is convergent and at exterior divergent is called the <u>interval of convergence</u>. It is determined by application of CAUCHY root or ratio test to $\sum |a_n (x-x_0)^n|$.

<u>Taylor series</u>. A function $f(x)$ having derivatives of all orders at $x=x_0$ has a unique power series expansion at x_0, given by $\sum\limits_0 f^{(n)}(x_0)(x-x_0)^n/n!$, called TAYLOR series which is McLAURIN series if $x_0 = 0$, and

$$f'(x) = \sum_1 f^{(n)}(x_0)(x-x_0)^{n-1}/(n-1)!,$$

$$\int_{x_0}^{x} f(x)\,dx = \sum_0 f^{(n)}(x_0)(x-x_0)^{n+1}/(n+1)!$$

<u>Algebra</u>: $\sum\limits_0 (a_n \pm b_n) x^n = \sum\limits_0 a_n x^n \pm \sum\limits_0 b_n x^n$

$(\sum\limits_0 a_n x^n)(\sum\limits_0 b_n x^n) = \sum\limits_0 (a_0 b_n + \dots + a_n b_0) x^n$ (CAUCHY Product)

$\dfrac{\sum_0 a_n x^n}{\sum_0 b_n x^n} = \sum\limits_0 q_n x^n$ with $a_n = b_0 q_n + \dots + b_n q_0$.

<u>Remainder Theorem</u>: If $f(x) \in D^{n-1}[a, b]$ and continuous at a and b, then there is $c \in (a, b)$ such that

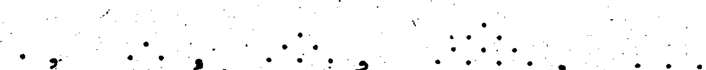
$$f(b) = \sum_{0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k + R_{n+1}, \quad R_{n+1} = \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n-1} \quad \text{or}$$

$$f(x) = \sum_{0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_{n+1}, \quad R_{n+1} = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n-1}$$

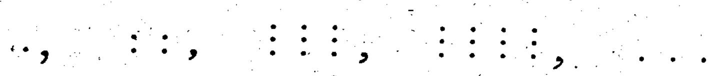
where $c \varepsilon (\overline{a, x})$.

MISCELLANEOUS EXERCISES

61 The sequence $(t_n)_1$ of triangular numbers and the sequence of square numbers are defined below:



$(t_n)_1$: 1, 3, 6, 10, ... , $\frac{n(n+1)}{2}$, ...



$(s_n)_1$: 1, 4, 9, 16, ... , n^2 , ...

a) show that $(8t_n+1)_1$ is a subsequence of (s_n)

b) discuss the numbers 2701 and 1979 to be triangular numbers

c) show that $(s_n-t_n)_2$ is $(t_n)_1$

62. Given the geometric sequence $(g_n)_o$ where $g_n = a\,r^n$, $g_n>0$ prove that the following are geometric sequences:

a) $(\frac{g_n + g_{n-1}}{2})_o$ b) $(\sqrt[n]{g_o \cdots g_n})_o$

63. Let $(a_n)_o$, $(g_n)_o$ be an arithmetic and geometric sequences respectively $(g_n>0)$. Compute the constants A and G such

that

$$A\,a_n + G\,\ell n\, g_n = 1 \quad \text{for all} \quad n \geq 0.$$

64. If $(g_n)_o$ is a positive geometric sequence, show that $(g_{n+p}/g_n)^{1/p}$ is independent of the integer p

65. Find the general term of the sequence defined by a_o, $a_n = 3a_{n-1} - 4$

66. Same question for:

$$a_1,\ a_2,\ a_{n-2} - 2a_{n-1} + a_n = 0$$

67. Use the equality

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} + \ldots = \frac{\pi^2}{6} \qquad \text{(EULER)}$$

to evaluate

$$s = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots + \frac{1}{(2n-1)^2} + \ldots$$

68. Test for convergence:

a) $\sum\limits_{4} \dfrac{2n-1}{5n^2-31n+8}$ b) $\sum\limits_{15} \dfrac{n!}{n^3}$

69. Write the numbers as geometric series, and then write them as the ratio of two integers:

a) $a = 1,\overline{2}$ b) $b = 1,\overline{23}$ c) $c = 1,2\overline{3}$ d) $d = 56,\overline{9}$

70. Test for convergence:

a) $\sum\limits_{1} \dfrac{1}{\sqrt{n(1+n^2)}}$ b) $\sum\limits_{2} \dfrac{1}{\sqrt{n}\ \ell n\ n}$

c) $\sum\limits_{2} \dfrac{1}{\ell n^n n}$ d) $\sum\limits_{1} \dfrac{a^n}{1-a^n}$

71. Test for convergence:

a) $\sum\limits_{1} \dfrac{n!}{n^p}$ b) $\sum\limits_{1} (-1)^n \dfrac{2n-1}{n^p+n}$

72. Test for convergence: $\sum_1 \frac{1^3 + 2^3 + \ldots + n^3}{(n+1)!}$

73. Discuss convergence:

a) $\sum_1 \left[\frac{1 \ldots (2n-1)}{2 \ldots (2n)}\right]^p$ b) $\sum_1 \left|\frac{1 \ldots (2n-1)}{2 \ldots (2n)} \frac{4n+3}{2n+2}\right|^2$

74. Test for convergence:

a) $\sum_1 \frac{1}{(2n-1)(2n+1)}$ b) $\sum_1 \frac{1}{1 + e^{1/n}}$

75. Test for convergence by comparison:

a) $\sum_2 \frac{1}{n^3-1}$ b) $\sum_1 \frac{\sin n}{n^3}$ c) $\sum_0 \frac{n+5}{n^2-3n-5}$ d) $\sum_2 \frac{1}{\sqrt{n}\ \ell n\ n}$

76. Show convergence of

a) $\sum_0 e^{-\alpha_n}$ b) $\sum_0 \ell n\left(1 - \frac{1}{\alpha_n}\right)$

where $\alpha_n > 0$ and $\alpha_n \to \infty$.

77. Find the sums:

a) $\sum_{n=1} \frac{1}{(2n+1)(2p+2n+1)}$ $(p \in N)$ b) $\sum_1 \frac{n(n+1)}{(n+2)(n+3)(n+4)(n+5)}$

78. Test for convergence:

a) $\sum_0 \left(\frac{n+1}{n+2}\right)^{n^2}$ b) $\sum_1 \left(\frac{(n-1)(n-2)}{n^2}\right)^{n^2}$

79. If $\sum u_n$, $\sum v_n$ are series of positive terms, show that

a) $\sum G_n$ b) $\sum H_n$

are convergent, where G_n, H_n are geometric and harmonic means of u_n, v_n respectively.

80. Test for convergence:

a) $\sum_1 \frac{1}{a^n + n^{1/a}}$ b) $\sum_1 a \sin \frac{1}{n}$, $(a \neq 0)$

81. Test for convergence:

a) $\sum\limits_1 \frac{(a+b)(a+2b)\ldots(a+nb)}{(a+c)(a+2c)\ldots(a+nc)}$ b) $\sum\limits_1 n^3\sqrt{n+\sqrt{n}}$, $(b,\ c > 0)$

82. Let $s = \sum\limits_0 a_n$ $(a_n > 0)$. Show that

a) $\sum\limits_0 a_n^2$ is convergent (with the value S), b) $S < s^2$.

83. Test for convergence:

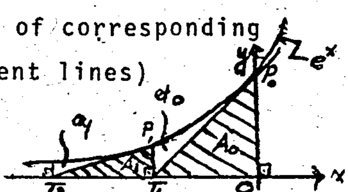
$$\sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{3n+1} + \frac{1}{3n+2} + \frac{1}{3n+3}\right)$$

84. Prove $\sum\limits_1 \frac{1}{n^p} < \frac{p}{p-1}$ if $p > 1$.

85. Show that $\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \ldots < \frac{1}{n!n}$

86. If $A_0, A_1, \ldots, a_0, a_1, \ldots$ are the areas of corresponding regions, evaluate (P_0T_1, P_1T_2, ... are tangent lines)

a) $\sum\limits_0 A_n$ b) $\sum\limits_0 a_n$



87. If $A_1, A_2, \ldots$ are the areas under the arcs of $y = (\sin x)/x$, prove that $\sum\limits_1 (-1)^{n-1}A_n$ is convergent

88. Find interval of convergence:

a) $\sum\limits_0 x^n \operatorname{Sh} n\alpha$ b) $\sum\limits_1 \frac{x^n}{n} \ln n\alpha$

where α is a positive constant.

89. If $(a_n) \to a$ and $a_n > 0$ find the interval of convergence of

$$\sum_1 \frac{x^n}{a_1 \ldots a_n}$$

90. Evaluate the sum:

$$\sum_0^{\infty} \frac{1+n}{(2n)!}$$

91. Given

$$f(x) = \frac{x^2}{1.2} + \frac{x^3}{2.3} + \ldots + \frac{x^{n-1}}{n(n-1)} + \ldots,$$

a) find the interval of convergence,

b) find $f'(x)$, $f''(x)$ and then deduce $f(x)$.

92. Identify $f(x)$: $f(x) = \sum_0 \frac{x^n}{(n+1)(n+3)}$

93. Evaluate

a) $\sum_1^\infty \ln(1 - \frac{1}{n^3})$ b) $\sum_1^\infty \frac{1}{n(n+1)(n+2)}$

94. Find the sum $\sum_{n=0}^\infty \left(\frac{1}{(n+1)!} \sum_1^n k^3\right)$

95. Find $f(x) = \frac{\alpha}{3!} x - \frac{\alpha(\alpha-1)}{4!} x^2 + \dots (-1)^n \frac{\alpha(\alpha-1)\dots(\alpha-n)}{(n+3)!} x^{n+1}$

96. Find $g(x) = \sum_0^\infty \frac{x^n}{(n+1)!}$

97. The sequences

$$f_o = 0,\ f_1 = 1,\ f_{n-2} = f_n + f_{n-1}$$

$$\ell_o = 2,\ \ell_1 = 1,\ \ell_{n-2} = \ell_n + \ell_{n-1}$$

are respectively called the FIBONACCI (1180-1250) and LUCAS (1842-1891) sequences respectively.

Obtain the terms up to f_{12}, ℓ_{12}, and then prove the following:

a) $f_{n+1}/f_n \to \tau$ and $\ell_{n+1}/\ell_n \to \tau$ where τ is the positive root of $x^2 - x - 1 = 0$ (considering existence of limits)

b) $\sum_{k=0}^n f_k = f_{n+2} - 1$, $\sum_{k=0}^n \ell_k = \ell_{n+2} - 1$

98. For the sequences defined in Exercise 97 show:

c) $f_{2n} = f_n \ell_n$ (LUCAS) d) $f_{2n+1} = f_n^2 + f_{n-1}^2$ (LUCAS)

e) $f_{n-1}f_{n+1} - f_n^2 = (-1)^n$ (SIMSON, 1687-1768)

f) $f_n = \frac{\tau^n + (-\tau)^n}{\tau + \tau^{-1}}$, $\ell_n = \tau^n + (-\tau)^{-n}$ (BINET)

99. Determine the interval of convergence and determine behavior at the end points:

a) $x + 2!x^2 + 3!x^3 + \dots + n!\,x^n + \dots$

b) $\frac{1}{a} + \frac{b}{a^2} x + \frac{b^2}{a^3} x^3 + \ldots + \frac{b^{n-1}}{a^n} x^n + \ldots \quad (a, b > 0)$

100. Prove $e - 1 < \sum_{1}^{\infty} \frac{2^n}{1.3\ldots(2n-1)} < 2e$

ANSWERS TO EVEN NUMBERED EXERCISES

66. $a_n = \frac{1}{(n-1)a_2 - (n-2)a_1}$

68. a) Div., b) div.

70. a) Conv., b) div., c) Conv., d) conv. for $|a| < 1$

72. Conv.

74. a) Conv., b) div.

78. a) Conv., b) conv.

80. a) Conv. for $a < 1$, b) div.

86. a) $\frac{e}{2(e-1)}$, b) $\frac{e-2}{2(e-1)}$

88. a) $(-e^{-2\alpha}, e^{-2\alpha})$, b) $(-1, 1)$

90. $\frac{3}{4}(e + e^{-1})$

92. $-\frac{1}{2}\left[\ln(1-x) - \frac{\ln(1-x) - x - x^2/2}{x^3}\right]$

94. $e/4$

96. e^x/x

CHAPTER 2

MATRICES

2. I. MATRICES

A. DEFINITIONS:

A rectangular array of the form

$$\begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{nj} & \cdots & a_{mm} \end{bmatrix} \qquad \begin{matrix} i=1, \ldots, m, \\ \\ j=1, \ldots, n \end{matrix}$$

of mn entries (elements) is called a (rectangular) matrix of the size (shape) mxn (m by n). Some authors use the symbols ( ) or $\| \ \|$ instead of [ ] to represent matrices. If $a_{ij} \varepsilon \mathbb{R}$ for all i, j the matrix is called a real matrix. An mxn matrix consists of m rows and n columns. The element a_{ij} lies in the ith row and the jth column. A matrix consisting of a single row (column) is called a row matrix (column matrix).

The above matrix of size mxn is abbreviated by one of the following symbols:

$$A_{mxn},\ [a_{ij}]_{mxn},\ (a_{ij})_{mxn},\ \|a_{ij}\|_{mxn}$$

In some cases the subsecipt mxn may be omitted.

If m = n, the matrix is called a square matrix and is said to be an nth ordered matrix or a matrix of order n. The elements a_{ii} in a square matrix lying on the main diagonal are called the diagonal elements.

In an mxn $(m \neq n)$ matrix the elements a_{ii} may be called similarly the diagonal elements.

If in a square matrix all entries below (above) the diagonal are zero, the matrix is called an <u>upper</u> (<u>lower</u>) <u>triangular matrix</u>. The following are triangular matrices:

$$\begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 7 & 0 \\ 3 & 4 & -5 \end{bmatrix}$$

An upper triangular matrix   A lower triangular matrix

($a_{ij} = 0$ for $i>j$)   ($a_{ij} = 0$ for $i<j$)

The above definition may be extended to any matrix, with

$$\begin{bmatrix} 1 & 0 & 8 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}_{4\times 3} \qquad \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 4 & 3 & 0 & 0 & 0 \end{bmatrix}_{2\times 5}$$

<u>Transpose of a matrix</u>:

The <u>transpose</u> of a matrix $A = [a_{ij}]_{m\times n}$ is the matrix $A^T = [a_{ji}]_{n\times m}$. According to this definition the transpose is obtained by changing rows into columns and column into rows. Why $(A^T)^T = A$?

<u>Example</u>. Write the transpose of the following matrices:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 5 \\ 0 & -2 \\ 1 & 4 \end{bmatrix}, \quad C = [3,\ 0,\ -7]$$

and give reasons for unalteration of the diagonal elements.

<u>Answer</u>.

$$A^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}, \quad B^T = \begin{bmatrix} 3 & 0 & 1 \\ 5 & -2 & 4 \end{bmatrix}, \quad C^T = \begin{bmatrix} 3 \\ 0 \\ -7 \end{bmatrix}$$

In a matrix and its transpose, $a_{ij} = a_{ji}$ when $i=j$ and these matrices have the same diagonal elements.

Some special square matrices:

In a square matrix $[a_{ij}]_n$

a) If $a_{ij} = a_{ji}$ (for all i, j) A is called a symmetric matrix.

b) If $a_{ij} = - a_{ji}$ (for all i, j) A is called a skew symmetric matrix, and A is zero axial since $a_{ii} = 0$.

c) If $a_{ij} = 0$ when $i \neq j$, A is a diagonal matrix.

d) If a diagonal matrix A have equal diagonal elements A is a scalar matrix.

e) If in a scalar matrix $\lambda=1$, A is an identity matrix (unit matrix) $I_n[=\delta_{ij}]_n$

f) If in a scalar matrix $\lambda=0$, A is a zero matrix 0_n.

The matrices

$$\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & {}_2 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} \lambda & 0 & \dots & 0 \\ 0 & \lambda & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \lambda \end{bmatrix} I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix} 0_n = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

are diagonal, scalar, identity and zero matrices of order n respectively

An identity matrix I is the matrix $[\delta_{ij}]$ where δ_{ij} is the (kronecker δ) defined by

$$\delta_{ij} = \begin{cases} 1 & \text{when } i=j \\ 0 & \text{when } i \neq j \end{cases}$$

Accordingly $[\lambda_i \ \delta_{ij}]$, $[\lambda\delta_{ij}]$ are diagonal and scalar matrices respectively.

For a square matrix $A = [a_{ij}]$, the symbols $|A|$, det A, det$[a_{ij}]$ are used to denote the determinant of A:

$$|A| = |a_{ij}| = \det A = \det[a_{ij}]$$

We note that if A is a non square matrix, $|A|$ is not defined.

B. <u>Operations with real matrices</u>:

1. <u>Equality</u>: The matrices $[a_{ij}]$, $[b_{ij}]$ are <u>equal</u> if they are of the same size and corresponding elements are equal:

$$[a_{ij}]_{mxn} = [b_{ij}]_{mxn} \Longleftrightarrow a_{ij} = b_{ij} \quad \text{for all } i, j.$$

2. <u>Addition</u>: The <u>sum</u> of two matrices $[a_{ij}]$, $[b_{ij}]$ of the same size is a matrix of the same size whose elements are the sums of their corresponding elements:

$$[a_{ij}]_{mxn} + [b_{ij}]_{mxn} = [a_{ij} + b_{ij}]_{mxn}$$

3. <u>Multiplication by a scalar</u>: The <u>product</u> of a matrix with a scalar is a matrix of the same size obtained by multiplying every element of the matrix by that scalar:

$$c[a_{ij}] = [c\ a_{ij}] = [a_{ij}]c$$

If $c = -1$, then $cA = -A$, and it follows that $A_{mxn} + (-A_{mxn}) = 0_{mxn}$

4. <u>Subtraction</u>: The difference $A-B$ of two matrices A and B of the same size is the matrix $A + (-B)$:

$$[a_{ij}]_{mxn} - [b_{ij}]_{mxn} = [a_{ij} - b_{ij}]_{mxn}$$

5. <u>Multiplication</u>: The product AB is defined only when the number of columns in A is equal to the number of rows in B.

If A has the size $m \times n$ and B has the size $n \times p$ the product AB is the matrix C of size $m \times p$ defined by

$$C = [c_{ij}]_{m \times p} \quad \text{where} \quad c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$$

In words, the element c_{ij} of the product matrix $AB = C$ is obtained by taking the ith row from A and jth column from B (having tha same number elements) and adding the products of their corresponding elements.

The matrices where product is defined are said to be <u>conformable</u>.

<u>Example</u>.

1. Compute

$$A = 2\begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 4 & 6 \end{bmatrix} - 3\begin{bmatrix} 0 & -1 \\ 2 & 1 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

<u>Solution</u>.

$$A = \begin{bmatrix} 2 & 4 \\ 6 & -2 \\ 8 & 12 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ -6 & -3 \\ -6 & -12 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 0 & -4 \\ 3 & 0 \end{bmatrix}$$

If $A, B, \ldots$ are matrices of the same size and $\alpha, \beta, \ldots \in R$ then $\alpha A + \beta B + \ldots$ is called a <u>linear combination</u> of $A, B, \ldots$

2. Compute, when matrices are conformable:

$$\text{a)} \begin{bmatrix} 2 & 3 \\ -3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & 0 & 2 \end{bmatrix} \qquad \text{b)} \begin{bmatrix} a & b \\ a & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Solution.

a) Sizes show that the matrices A_{3x2}, B_{2x4} are conformable and the product C is of size 3x4. The element c_{ij} of C is obtained by taking the ith row of A and jth column of B and adding the products of their respective elements:

$$AB = \begin{bmatrix} 2.1+3.2 & 2.2+3(-1) & 2.3+3.0 & 2.4+3.2 \\ -3.1+1.2 & -3.2+1.(-1) & -3.3+1.0 & -3.4+1.2 \\ 0.1+2.2 & 0.2+2(-1) & 0.3+2.0 & 0.4+2.2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 1 & 6 & 14 \\ -1 & -7 & -9 & -10 \\ 4 & -2 & 0 & 4 \end{bmatrix}$$

b) Since the sizes of the two matrices are 2x2 and 3x2 and inner numbers are not equal ($2 \neq 3$), multiplication is undefined.

3. Perform the multiplications of row and columns matries

a) $\begin{bmatrix} 2 & -3 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$ b) $\begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & -3 & 4 \end{bmatrix}$

Solution.

a) Since sizes are 1x3, 3x1 the product is of size 1x1:

$$\begin{bmatrix} 2(-1) + (-3)(-2) + 4.3 \end{bmatrix} = \begin{bmatrix} 16 \end{bmatrix}$$

b) The product is of size 3x3:

$$\begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & -3 & 4 \end{bmatrix} = \begin{bmatrix} -1.2 & -1(-3) & -1.4 \\ -2.2 & -2(-3) & -2.4 \\ 3.2 & 3(-3) & 3.4 \end{bmatrix} = \begin{bmatrix} -2 & 3 & -4 \\ -4 & 6 & -8 \\ 6 & -9 & 12 \end{bmatrix}$$

Comparing the above results AB and BA, note that in general $AB \neq BA$.

4. The product of two non zero matrices may be zero. Determine the number "a" such that

$$\begin{bmatrix} 2 & 4 \\ 5 & 10 \end{bmatrix} \begin{bmatrix} -2 & a \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 2 & 4 \\ 5 & 10 \end{bmatrix} \begin{bmatrix} -2 & a \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2a-12 \\ 0 & 5a-30 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} 2a-12 = 0 \\ 5a-30 = 0 \end{matrix} \Rightarrow a=6$$

5. If A is a square matrix, the following matrices are defined:

$$A^0 = I, \quad A^1 = A, \quad A^2 = AA, \quad \dots, \quad A^n = AA^{n-1} \qquad (n \in \mathbb{N}).$$

Then evaluate the following linear combunation

$$A^2 + 2A - 3I_3$$

of A^2, A, I_3 where

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution.

$$A^2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix},$$

$$2A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$$

$$-3 I_3 = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\Rightarrow \qquad A^2 - 2A - 3 I_3 = \begin{bmatrix} -1 & 3 & 3 \\ 3 & -1 & 3 \\ 3 & 3 & -1 \end{bmatrix}$$

<u>Remark</u>. Note that multiplication of a matrix by a scalar and that of a determinant by a scalar are defined differently: a matrix is multiplied by a scalar by multiplying every element by that scalar, while a determinant is multiplied by a scalar by multiplying only one row (column) by that scalar.

Thus

$$c \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} = \begin{bmatrix} 8c & c & 6c \\ 3c & 5c & 7c \\ 4c & 9c & 2c \end{bmatrix},$$

$$c \begin{vmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{vmatrix} = \begin{vmatrix} 8c & c & 6c \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{vmatrix} = \begin{vmatrix} 8 & c & 6 \\ 3 & 5c & 7 \\ 4 & 9c & 2 \end{vmatrix}$$

As a result we have for a matrix A of order n,

$$\det c\,[a_{ij}] = \det [ca_{ij}] = c^n \det [a_{ij}].$$

C. PROPERTIES

1. $A+B = B+A$ (commutativity)

2. $(A+B)+C = A+(B+C)$ (Associativity)

3. $(A+B)^T = A^T+B^T$ (The transpose of a sum is the sum of transposes)

4. $IA = AI = A$, but $AB \neq BA$ in general. If $AB = BA$, it is said that A and B commute.

5. $(AB)C = A(BC)$ (Associativity)

6. $(AB)^T = B^T A^T$ (The transpose of a product is the product of transposes in opposite order)

7. $|AB| = |A||B| = |BA|$ (The determinant of a product is the product of determinants)

8. $(A+B)C = AC+BC$ (The right distributivity)

$C(A+B) = CA+CB$ (The left distributivity)

Proofs.

1. $[a_{ij}] + [b_{ij}] = [a_{ij}+b_{ij}] = [b_{ij}+a_{ij}] = [b_{ij}] + [a_{ij}]$

2. Proved similarly.

3. $(A+B)^T = [a_{ij}+b_{ij}]^T = [a_{ji}+b_{ji}] = [a_{ji}]+[b_{ji}] = A^T+B^T$

4. $IA = [\delta_{ij}][a_{ij}] = [\sum \delta_{ik} a_{kj}] = [\delta_{ii} a_{ij}] = [a_{ij}] = A$

Similarly $AI = A$.

5. Let the symbol $(A)_{ij}$ denote the ij-entry of A. Then we prove $((AB)C)_{ij} = (A(BC))_{ij}$. Let

$A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{n \times p}$ and $C = [c_{ij}]_{p \times q}$ Then

$$((AB)C)_{ij} = \sum_{k=1}^{p} (AB)_{ik}\, c_{kj} = \sum_{k=1}^{p} \left(\sum_{\ell=1}^{n} (a_{i\ell}\, b_{\ell k}) c_{kj}\right.$$

$$= \sum_{k} \sum_{\ell} a_{i\ell}\, b_{\ell k}\, c_{kj} = \sum_{\ell} \sum_{k} a_{i\ell}\, (b_{\ell k}\, c_{kj})$$

$$= \sum_{\ell} (a_{i\ell} \sum_{k} b_{\ell k}\, c_{kj}) = \sum_{\ell} a_{i\ell} (BC)_{\ell j} = (A(BC))_{ij}$$

6. Let $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{n \times p}$. We prove

$$(B^T A^T)_{ij} = ((AB)^T)_{ij}$$

Setting $a_{ji} = a_{ij}$ and $b_{ji} = b_{ij}$ we have

$$((AB)^T)_{ij} = (AB)_{ji} = \sum_{k} a_{jk}\, b_{ki} = \sum_{k} a'_{kj}\, b'_{ik}$$

$$= \sum_{k} b'_{ik}\, a'_{kj} = (B^T A^T)_{ij}$$

7. The proof may be found in Linear Algebra.

8. $(A+B)C = [a_{ij}+b_{ij}][c_{ij}] = [\sum_{k} (a_{ik}+b_{ik})c_{kj}]$

$$= [\sum a_{ik}\, c_{kj} + \sum b_{ik}\, c_{kj}] = [\sum a_{ik}\, c_{kj}] + [\sum b_{ik}\, c_{kj}]$$

$$= AC + BC$$

The left distributivity is proved similarly.

Example 1.

1. Find all 2×2 matrices which commute with

$$A = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$$

Solution. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be such that

$$\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$$

We have

$$\begin{bmatrix} a+2c & b+2d \\ -a-c & -b-d \end{bmatrix} = \begin{bmatrix} a-b & 2a-b \\ c-d & 2c-d \end{bmatrix}$$

$$a-2c = a-b, \qquad b+2d = 2a-b$$

$$-a-c = c-d, \qquad -b-d - 2c-d$$

$$\begin{matrix} b = -2c, & b+d = a, \\ d-a = 2c, & b = -2c \end{matrix} \quad \Rightarrow \quad b = -2c, \quad a = d-2c$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d-2c & -2c \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} d - \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix} c$$

Example 2. If M, N commute and $M^2 = 0$ or $N^2 = 0$, then prove $(MN)^2 = 0$.

Solution. $(MN)^2 = (MN)(MN) = (MN)(NM) = M(N^2)M = M0M = 0$.

Is it possible to define multiplication of two determinants?

Yes! The propertry $|A||B|$ $|AB|$ can be used to define the product of two determinants as a single determinant whose elements are obtained by the same rule for multiplication of matrices.

Example 3. Verify $(AB)^T = B^TA^T$ for

$$A = [1 \quad 3], \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \end{bmatrix}$$

Solution.

$$AB = \begin{bmatrix} -2 & 2 & 15 \end{bmatrix} \Rightarrow (AB)^T = \begin{bmatrix} -2 \\ 2 \\ 15 \end{bmatrix},$$

$$B^TA^T = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 15 \end{bmatrix}.$$

D. INVERSE OF A MATRIX

If for a given matrix A_{mxn}, there exists a matrix B_{nxm} such that

$$AB = I_m,$$

B is called a right inverse of A, and if there exists a matrix C_{mxn} such that

$$CA = I_n$$

C is called a left inverse of A.

Observe that a right and a left inverse of A_{mxn} $(m \neq n)$ are not of the same size.

Example. Find a right inverse B and a left inverse C, if any, of the rectangular matrix

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & -1 & 4 \end{bmatrix}_{2x3}$$

Solution. If a right inverse exists it will be of size 3x2. Then, writing

$$\begin{bmatrix} 1 & 0 & -2 \\ 3 & -1 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

we have

$$a - 2e = 1, \qquad b - 2f = 0,$$
$$3a - c - 4e = 0, \qquad 3a - d + 4f = 1 \qquad \text{(4 equations, 6 unknowns)}$$

$$\Rightarrow \quad a = 2e-1; \qquad b = 2f$$
$$c = 3a+4e = 10e+3, \quad d = 3a+4f-1 = 10f-1$$

and
$$B = \begin{bmatrix} 2e-1 & 2f \\ 10e+3 & 10f-1 \\ e & f \end{bmatrix}, \quad e, f \in R$$

showing the existence of infinitely many right inverses, of which one is obtained by taking $e=1$, $f = -1$, namely

$$\begin{bmatrix} 3 & -3 \\ 13 & -11 \\ 1 & -1 \end{bmatrix}$$

For a left inverse, writing

$$\begin{bmatrix} a' & b' \\ c' & d' \\ e' & f' \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 3 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we have

$$\begin{array}{lll} a'+3b' = 1 & -b' = 0 & -2a'+4b' = 0 \\ c'+3d' = 0 & -d' = 1 & -2c'+4d' = 0 \\ e'+3f' = 0 & -f' = 0 & -2e'+4f' = 1 \end{array} \qquad \text{(9 equations, 6 unknowns)}$$

$$\Rightarrow \quad b' = 0, \quad a' = 1, \quad a' = 0 \quad \text{(no solution)}$$

So given rectangular matrix A has no left inverse.

The example shows that the determination of a right inverse of A_{mxn} leads, from $A_{mxn}\, B_{nxm} = I_m$, to m^2 linear equations with mn unknowns. As to the left inverse of A_{mxn}, one has n^2 equations with mn unknowns.

Square case:

The right and left inverses B and C of a square matrix A of order n, if exist, are of order n and satisfy

$$AB = I \text{ and } CA = I.$$

It is proved in Linear Algebra that (considering system of linear equations arising from the above equations) if A has a right (left) inverse it has also a left (right) inverse which are equal to each other, and this is the case when $|A| \neq 0$:

$$AB = BA = I$$

The uniqueness of such a matrix B can be proved as follows: Suppose C is another inverse of A. Then

$$AC = CA = I$$

and

$$B = BI = B(AC) = (BA)C = IC = C.$$

This unique matrix B is called the inverse of A denoted by A^{-1} and if A^{-1} exists then A is said to be an invertible matrix.

Since

$$AA^{-1} = A^{-1}A = I$$

it follows that

$$(A^{-1})^{-1} = A$$

To show that a given matrix B is the inverse of a matrix A, it will suffice to show $AB = I$ or $BA = I$ but not necessarily both.

Example. Show that

$$B = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & -6 \\ -2 & -1 & 5 \end{bmatrix} \text{ is the inverse of } A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Proof. We only need to show that $BA = I$:

$$BA = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & -6 \\ -2 & -1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Evaluate AB!

Some square matrices are invertible, but some others are not. From $II = I$ one has $I^{-1} = I$ and identity matrices are invertible, but since $OB = BO = 0 \neq I$ zero matrices are not invertible.

Corollary. If A is invertible, then

$$|A| \neq 0 \text{ and } |A^{-1}| = |A|^{-1}$$

Proof. $AA^{-1} = I \Rightarrow |AA^{-1}| = |I| \Rightarrow |A||A^{-1}| = 1$

$$\Rightarrow |A| \neq 0 \text{ and } |A^{-1}| = 1/|A|.$$

Note. The converse is also true, i.e. if $|A| \neq 0$, then A is invertible (See §1.3, Corollary of CRAMER's Rule, BOOK ONE, Part two)

Corollary. If A, B are invertible square matrices of the same order, then AB is invertible and

$$(AB)^{-1} = B^{-1} A^{-1}$$

Proof. $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$. ∎

Determination of the inverse of a square matrix:

Given a square matrix A, its inverse A^{-1} can be determined, namely

1. by the definition,
2. by adjoint matrix
3. by elementary row operations (sweepout process)
4. by CAYLEY-HAMILTON Theorem.

1. By the definition:

This is the method used in the determination of a right or left inverse of a matrix, and applicable also to square ones.

2. By the adjoint matrix:

By the classical adjoint[1] of a square matrix $A = [a_{ij}]$ is meant the transpose of the matrix $[A_{ij}]$ of the cofactors of a_{ij}'s. In this text we simply use "adjoint" to mean the classical adjoint:

$$\text{adj } A = [A_{ij}]^T = [A_{ji}]$$

Example 1. Find the adjoint of

a) $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & 1 & 2 \end{bmatrix}$ b) $B = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$

Solution.

a) Since

$$\begin{matrix} A_{11} = -3, & A_{12} = -3, & A_{13} = 3 \\ A_{21} = -1, & A_{22} = 3, & A_{23} = -1 \\ A_{31} = 1, & A_{32} = 3, & A_{33} = -5 \end{matrix} \Rightarrow [A_{ij}] = \begin{bmatrix} -3 & -3 & 3 \\ -1 & 3 & -1 \\ 1 & 3 & -5 \end{bmatrix}$$

1) In Linear Algebra the term "adjoint" is used in two meanings that are entirely different. The one we used here is named "classical adjoint" to distinguish it from the other.

$$\text{Adj } A = \begin{bmatrix} -3 & -1 & 1 \\ -3 & 3 & 3 \\ 3 & -1 & -5 \end{bmatrix} = \left[A_{ji}\right]$$

b) $\text{Adj } B = \begin{bmatrix} 9 & -3 \\ -6 & 2 \end{bmatrix}$

<u>Theorem</u>. $A^{-1} = \frac{1}{|A|} \text{Adj } A = \frac{[A_{ji}]}{|A|}$ if $|A| \neq 0$, i.e., if $A = [a_{ij}]$ is invertible.

<u>Proof</u>. We need to show that

$$A \frac{\text{Adj } A}{|A|} = I \quad \text{or} \quad A \text{ Adj } A = |A| I$$

Indeed,

$$A \text{ Adj } A = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ a_{i1} & \cdots & & a_{in} \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \vdots & A_{1j} & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & A_{nj} & \vdots \end{bmatrix} / |A|$$

$$= \left[\sum_k a_{ik} A_{kj}\right] / |A| = \left[\delta_{ij} |A|\right] / |A| = \left[\delta_{ij}\right] = I$$

by Theorem 6 on determinant.∎(Book I)

<u>Example 2</u>. Find the inverses of the matrices A and B in Example 1, if any.

<u>Solution</u>.

a) The classical adjoint of A was obtained as the matrix

$$\begin{bmatrix} -3 & -3 & 3 \\ -1 & 3 & -1 \\ 1 & 3 & -5 \end{bmatrix}$$

and the inverse is obtained by dividing this matrix by $|A| = -6$

$$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 2 \end{vmatrix} = 2(-2-1) - 1.0 = -6$$

Hence

$$A^{-1} = \begin{bmatrix} 1/2 & 1/6 & -1/6 \\ 1/2 & -1/2 & -1/2 \\ -1/2 & 1/6 & 5/6 \end{bmatrix}$$

b) Since $\det B = 0$, B is not invertible.

Example 3. Find the inverse of

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{if} \quad |A| = ad-bc \neq 0$$

Solution. Since

$$[A_{ij}] = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

we have $\quad A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} / |A| = \dfrac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad - bc}$

3. Elementary row operations

Let A be a rectangular matrix of shape $m \times n$. Let the row matrices be $R_1, \ldots, R_m$. The following operations on rows are called the elementary row operations:

$R_i \Leftrightarrow R_j$: Interchanging of ith and jth rows,

$R_i + R_j$: Adding the jth row to the ith row,

$c\,R_i$: Multiplying a row by a non zero scalar.

An elementary operation followed by another is an operation called a <u>row operation</u>. The row operation

$$R_i + \lambda R_j \quad (\lambda \varepsilon R)$$

has an obvious meaning.

A finite number of elementary operations applied to a matrix A produce a matrix A' which is said to be a <u>row equivalent</u> to A, written

$$A \sim A'$$

In transforming a matrix A to a row equivalent matrix A' useful notations for $R_i \leftrightarrow R_j$ and $\lambda R_k + R_j$ are

$$\left|\begin{array}{ccccc} \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{i1} & \cdot & \cdot & \cdot & a_{in} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{j1} & \cdot & \cdot & \cdot & a_{jn} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{k1} & \cdot & \cdot & \cdot & a_{kn} \end{array}\right| \begin{array}{c} \\ \\ \\ \\ \\ \lambda \end{array}$$

A main problem here is to transform a given matrix A to what we call a echelon matrix. By an <u>echelon matrix</u> (<u>echelon form</u>) is meant a matrix in which the number of zero elements, in each row, preceeding the first non zero element (the <u>distinguished element</u>) increases from row to row until the last row. The last row and some rows preceeding it may consists of zero elements only. This mean that:

$$a_{ij} = 0 \qquad (j=1, \ldots, k)$$

$$a_{i+1\ j} = 0 \ (j=1, \ldots, k+1 \text{ at least})$$

The following is an echelon matrix.

$$\begin{bmatrix} 0 & -2 & 3 & 0 & -2 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 & 1 & 4 & 0 & 6 \\ 0 & 0 & 0 & 0 & 8 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The distinquished elements (the first non zero elements) in rows are $-2, -1, 8, -5$, and the numbers of zero elements in the rows preceeding these are $1, 2, 4, 7$, and they increase until the last row, which, in this example, consists of zero elements only,

Three more examples are:

$$\begin{bmatrix} 2 & 1 & 0 & 2 & 5 \\ 0 & 0 & -3 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 4 & 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Zero matrices are considered to be in echelon form.

<u>Example</u>. Reduce the given matrix A to an echelon form

$$A = \begin{bmatrix} 0 & 0 & 8 & 0 & 3 \\ 2 & 4 & 0 & 4 & 1 \\ 4 & 6 & 2 & 3 & 0 \\ 0 & 1 & 3 & 2 & 1 \end{bmatrix}$$

<u>Solution</u>. A is not of echelon form. Then

$$A = \begin{bmatrix} 0 & 0 & 8 & 0 & 3 \\ 2 & 4 & 0 & 4 & 1 \\ 4 & 6 & 2 & 4 & 0 \\ 0 & 1 & 3 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 4 & 6 & 2 & 4 & 0 \\ 2 & 4 & 0 & 4 & 1 \\ 0 & 0 & 8 & 0 & 3 \\ 0 & 1 & 3 & 2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 4 & 6 & 2 & 4 & 0 \\ 0 & 1 & 3 & 2 & 1 \\ 0 & 0 & 8 & 2 & 3 \\ 2 & 4 & 0 & 4 & 1 \end{bmatrix} -\frac{1}{2} \sim \begin{bmatrix} 4 & 6 & 2 & 4 & 0 \\ 0 & 1 & 3 & 2 & 1 \\ 0 & 0 & 8 & 0 & 3 \\ 0 & 1 & -1 & 2 & 1 \end{bmatrix} -1$$

$$\sim \begin{bmatrix} 4 & 6 & 2 & 4 & 0 \\ 0 & 1 & 3 & 2 & 1 \\ 0 & 0 & 8 & 0 & 3 \\ 0 & 0 & -4 & 0 & 0 \end{bmatrix} \frac{1}{2} \sim \begin{bmatrix} 4 & 6 & 2 & 4 & 0 \\ 0 & 1 & 3 & 2 & 1 \\ 0 & 0 & 8 & 0 & 3 \\ 0 & 0 & 0 & 0 & 3/2 \end{bmatrix}$$

Method for finding A^{-1}:

Let A be a square matrix of order n. The method for finding A^{-1} consists of considering the matrix

$$\left| A \,\vdots\, I \right|_{n \times 2n}$$

and by application of row operations, reducing it the matrix

$$\left| I \,\vdots\, B \right|_{n \times 2n}$$

where B is the desired inverse; if not reducible to this form A is not invertible.

The proof of above statement will be given in §1. 3.

<u>Example.</u> Find the inverses of the matrices

a) $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & 1 & 2 \end{bmatrix}$ b) $B = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$

<u>Solution.</u>

a) $|A\ I|_{3x6} = \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & -2 & -1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] -2 \quad \sim \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 1 & 0 \\ 0 & 5 & 3 & 1 & -2 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] -1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 1 & 0 \\ 0 & 5 & 3 & 1 & -2 & 0 \\ 0 & 3 & 3 & 0 & -1 & 1 \end{array}\right] \begin{array}{c} \\ \\ (1/3) \end{array} \quad \sim \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 1 & 0 \\ 0 & 5 & 3 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 & -\frac{1}{3} & -\frac{1}{3} \end{array}\right] -\frac{1}{5}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 1 & 0 \\ 0 & 5 & 3 & 1 & -2 & 0 \\ 0 & 0 & 2/5 & -\frac{1}{5} & \frac{1}{15} & \frac{1}{3} \end{array}\right] \begin{array}{c} \\ (1/5) \\ (5/2) \end{array} \quad \sim \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 1 & 0 \\ 0 & 1 & 3/5 & \frac{1}{5} & -\frac{2}{5} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{6} & \frac{5}{6} \end{array}\right] 2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{5} & \frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 1 & 3/5 & \frac{1}{5} & -\frac{2}{5} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{5}{6} & \frac{5}{6} \end{array}\right] \begin{array}{c} \\ \\ (-1/5) \end{array} \quad \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/6 & -\frac{1}{6} \\ 0 & 1 & 3/5 & 1/5 & -2/5 & 0 \\ 0 & 0 & 1 & -1/2 & 1/6 & 5/6 \end{array}\right] -3/5$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/6 & -1/6 \\ 0 & 1 & 0 & 1/2 & -1/2 & -1/2 \\ 0 & 0 & 1 & -1/2 & 1/6 & 5/6 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} 1/2 & 1/6 & -1/6 \\ 1/2 & -1/2 & -1/2 \\ -1/2 & 1/6 & 5/6 \end{bmatrix}$$

b) $[B \vdots I]_{2\times 4} = \left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 6 & 9 & 0 & 1 \end{array}\right] \overset{(1/2)}{\sim} \left[\begin{array}{cc|cc} 1 & 3/2 & 1/2 & 0 \\ 6 & 9 & 0 & 1 \end{array}\right] (-6)$

$$\sim \left[\begin{array}{cc|cc} 1 & 3/2 & 1/2 & 0 \\ 0 & 0 & -3 & 1 \end{array}\right].$$

where the matrix on the left cannot be reduced to the identity matrix. Then B is not invartible.

4. By CAYLEY-HAMILTON Theorem

The following theorem is proved in Linear Algebra:

Theorem (CAYLEY-HAMILTON) A square matrix satisfies its own characteristic equation.

The characteristic polynomial of a square matrix $A = [a_{ij}]_n$ is defined to be the polynomial

$$P(\lambda) = |A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & & \vdots \\ \vdots & & & \vdots \\ a_{n1} & \cdots & & a_{nn}-\lambda \end{vmatrix} = \sum_{k=0}^{n} c_k \lambda^k$$

and $P(\lambda) = 0$ is the characteristic equation of A.

Clearly

$$P(0) = |A| = c_0.$$

The theorem states that

$$P(A) = \sum_{k-0}^{n} c_k A^t = 0_n,$$

which is

$$c_n A^n + c_{n-1} A^{n-1} - \ldots + c_1 A + c_0 I_n = 0_n$$

Multiplying both side of this equation by A^{-1} (if exists), we have

$$c_nA^{n-1} + c_{n-1}A^{n-2} + \dots + c_1I_n + c_0A^{-1} = 0,$$

since $A^kA^{-1} = (A^{k-1}A)A^{-1} = A^{k-1}(AA^{-1}) = A^{k-1}$.

This latter equality is solvable for A^{-1} when $c_0 = |A| \neq 0$ which is the same condition in Method 2.

If $|A| = c_0 = 0$, then A is not invertible and such matrix is called a singular square matrix. An invertible matrix is non singular.

Example. Find the inverse of

a) $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & 1 & 2 \end{bmatrix}$ b) $B = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$

Solution.

a) The characteristic equation is

$$P(\lambda) = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & -2-\lambda & -1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

Expansion gives

$$P(\lambda) = -\lambda^3 + 2\lambda^2 + 5\lambda - 6 = 0$$

and by CAYLEY-HAMILTON Theorem we have

$$-A^3 + 2A^2 - 5A + 6I_3 = 0$$

$$\Rightarrow \quad -A^2 + 2A + 5I - 6A^{-1} = 0$$

$$\Rightarrow \quad -6A^{-1} = A^2 - 2A - 5I$$

$$= \begin{bmatrix} 6 & 1 & 3 \\ -1 & 4 & 1 \\ 5 & 1 & 4 \end{bmatrix} + \begin{bmatrix} -4 & -2 & -2 \\ -2 & 4 & 2 \\ -2 & -2 & -4 \end{bmatrix} + \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix} = \begin{bmatrix} -3 & -1 & 1 \\ -3 & -1 & 1 \\ 3 & -1 & -5 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/2 & 1/6 & -1/6 \\ 1/2 & -1/2 & -1/2 \\ -1/2 & 1/6 & 5/6 \end{bmatrix}$$

b) $c_0 = |B| = \begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix} = 0$. Then B is non invertible.

EXERCISES (2. 1.)

1. Let $A = [a_{ij}]$ be a square matrix of size $n \times n$ such that

$$a_{ij} = \begin{cases} 0 & \text{when } i+j \text{ is odd} \\ \neq 0 & \text{when } i+j \text{ is even.} \end{cases}$$

What is the number of non zero elements in A?

2. If $A = \begin{bmatrix} 1 & 2 & -3 \\ -2 & -4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 4 \\ 3 & 1 \\ 4 & 2 \end{bmatrix}$, find

a) AB b) BA

3. Given $A = [a_i \ \delta_{ij}]_{n \times m}$ and $B = [a_i^{-1} \ \delta_{ij}]_{m \times n}$, show that

$$AB = BA = I_n$$

4. Let $A = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & -2 \\ 2 & 5 & 3 \end{bmatrix}$. Then find

a) $3A-2B$ b) AB^T c) A^TB

5. If

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix}, \quad \text{find}$$

a) AB　　b) BA　　c) $2A^2+3B^2-5A+4B$

6. Verify the associative law for the product

$$\begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 3 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 2 \\ 0 & 1 \end{bmatrix}$$

7. Show that the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}^2$$

is a zero matrix.

8. If

$$P = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \qquad Q = \begin{bmatrix} -1 & 2 & 4 \\ 1 & -2 & -4 \\ -1 & 2 & 4 \end{bmatrix}$$

then prove

a) $P+Q = I$　　b) $P^2=P$, $Q^2=Q$　　c) $PQ=0$, $QP=0$

9. If A is a matrix and

$$A \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

find the shape of A.

10. If

$$A\begin{bmatrix}2\\ \\1\end{bmatrix}=\begin{bmatrix}0\\1\\2\end{bmatrix} \quad \text{and} \quad A\begin{bmatrix}1\\ \\2\end{bmatrix}=\begin{bmatrix}-3\\-1\\1\end{bmatrix}$$

find the matrix A.

11. Find all real matrices which commute with the matrix

$$A=\begin{bmatrix}1 & 2\\ -1 & -1\end{bmatrix}$$

12. Find all 2x2 matrices A satisfying

$$A^2+4A+3I_2=0_2$$

13. Show that

a) A satisfies $A^2 - A + 2I = 0$

b) B satisfies $B^3 - 2B - I = 0$

where

$$A=\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}, \quad B=\begin{bmatrix}0&1&0\\1&0&1\\1&1&0\end{bmatrix}$$

c) and apply CAYLEY-HAMİLTON Theorem to ge

14. If $A^TA = I$ then A is called an <u>orthogor</u>

a) Find λ in

$$A=\begin{bmatrix}\text{Sin }\theta & \lambda\\ \cos\theta & \sin\theta\end{bmatrix}$$

for A to be orthogonal.

b) Prove that the inverse of an orthogonal matrix is an orthogonal matrix.

15. If

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

find a set of numbers x, y, z and an orthogonal matrix B such that

$$BA = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} B$$

16. Show that

a) A is symmetric if $A^T = A$

b) A is skew symmetric if $A^T = -A$.

17. If A and B are both $2x2$ skew symmetric matrices, prove that $AB = BA$. Is the property true for $3x3$ matrices,

18. Prove that if A is square matrix, then

a) $A + A^T$ is symmetric, b) $A - A^T$ is skew symmetric

19. Use Exercise 18 to show that every square matrix can be written as the sum of a symmetric and skew symmetric matrix, and apply this to

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 2 & 1 \\ 3 & 3 & 2 \end{bmatrix}$$

20. If S and T are symmetric matrices which commute, show that ST is symmetric.

21. Prove that the product of two

a) Scalar matrices is a scalar matrix,

b) diagonal matrices is a diagonal matrix.

22. If A and B are both diagonal, prove $AB = BA$

23. Find the left inverses of the following matrices, if any:

a) $\begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 1 & 0 \\ -1 & 4 & 5 \end{bmatrix}$ c) $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$

24. Find the right inverse of the matrices in Exercise 23, if any.

25. Find the inverses of the following square matrices by the use of all given methods:

a) $\begin{bmatrix} 3 & 5 \\ -6 & 7 \end{bmatrix}$ b) $\begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 5 & 1 & -1 \end{bmatrix}$ c) $\begin{bmatrix} 4 & 3 & 2 \\ 1 & 0 & -1 \\ 5 & 3 & 1 \end{bmatrix}$

26. Find the inverses of

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 7 \end{bmatrix}$$

only by CAYLEY-HAMILTON Theorem.

27. Find the inverses of the matrices by inspection.

a) $\begin{bmatrix} 0 & \cdots & 0 & d_n \\ \vdots & & \iddots & 0 \\ 0 & \iddots & & \vdots \\ d_1 & 0 & \cdots & 0 \end{bmatrix}$ $(d_1 \ldots d_n \neq 0)$. b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & 1 \end{bmatrix}$

28. If

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 1 \\ -3 & 0 & -2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & 1 & 2 \\ -3 & -1 & 1 \\ 2 & 5 & 0 \end{bmatrix}$$

evaluate $A^{-1}B$, $B^{-1}A$ and show that each is the inverse of the other.

29. Prove

a) $(A^{-1})^T = (A^T)^{-1}$ (transpose each side of $AA^{-1} = I$).

b) the inverse of an orthogonal matrix is its transpose.

30. Under what condition for x, the matrix

$$\begin{bmatrix} x & \sqrt{2} & 0 \\ \sqrt{2} & x & \sqrt{2} \\ 0 & \sqrt{2} & x \end{bmatrix}$$

is non invertible.

ANSWERS TO EVEN NUMBERED EXERCISES

2. a) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, b) $\begin{bmatrix} -2 & -4 & 6 \\ 1 & 2 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

4. a) $\begin{bmatrix} 6 & -11 & 7 \\ 5 & -10 & 6 \end{bmatrix}$ b) $\begin{bmatrix} -5 & -8 \\ -8 & 18 \end{bmatrix}$ c) $\begin{bmatrix} 6 & 17 & 5 \\ 0 & -3 & 6 \\ 8 & 21 & 10 \end{bmatrix}$

10. $\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$

12. $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$, $\begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$ or

$$\begin{bmatrix} a & b \\ \dfrac{-(a+1)(a+3)}{b} & -a-4 \end{bmatrix}$$

14. $\lambda = -\cos\theta$

24. a) No right inverse,

b) $\begin{bmatrix} t & s \\ 1-2t & -2s \\ \frac{9t-4}{5} & \frac{9s-1}{5} \end{bmatrix}$ c) No right inverse

26. $\frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -10 & 4 & 2 \\ 7 & -3 & -1 \end{bmatrix}$

28. $A^{-1} = \frac{1}{2}\begin{bmatrix} -6 & -2 & -4 \\ 1 & 1 & 1 \\ 9 & 3 & 5 \end{bmatrix}$, $B^{-1} = \frac{1}{44}\begin{bmatrix} 5 & -10 & -3 \\ -2 & 4 & 10 \\ 13 & 18 & 1 \end{bmatrix}$ (continue!)

30. $x \varepsilon \{0, -2, 2\}$.

2. 2. SOLUTION OF SYSTEM OF LINEAR EQUATIONS BY MATRICES

Let

$$\begin{matrix} a_{11}\, x_1 + \ldots + a_{1n}\, x_n = b_1 \\ \vdots \qquad\qquad \vdots \qquad \vdots \\ a_{m1}\, x_1 + \ldots + a_{mn}\, x_n = b_m \end{matrix} \qquad (1)$$

be a system of linear equations involving m equations with n unknowns.

Setting

$$A = [a_{ij}]_{m\times n}, \quad X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

(1) becomes

$$AX = B \qquad (1')$$

To solve this equation we have two methods:

1) By inverse matrix,

2) By GAUSS' Method (by elementary row operations)

1. <u>Solution by Inverse matrix</u>:

Square case $(m=n)$:

If A^{-1} exists multiplying both sides of $(1')$ from left by A^{-1} we have

$$X = A^{-1}B$$

a unique solution matrix (column matrix).

A^{-1} can be determined by any method given in §1. 2 D, and the method of solution has the same name as that of the used method to find A^{-1}.

If A is non invertible, i.e. if $|A| = 0$, the system has no solution or has a solution matrix involving one parameter at least depending on the degree of freedom.

<u>Example</u>. Solve the square system

$$\begin{aligned} x + 2y - z &= 2 \\ x + y + z &= 6 \\ 2x - y + z &= 3 \end{aligned}$$

<u>Solution</u>. Writing the system in matrical form:

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix}$$

and finding the inverse of A by any method, we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 & -1 & 3 \\ 1 & 3 & -2 \\ -3 & 5 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \quad x = 1, \quad y = 2, \quad z = 3.$$

Rectangular case: (m, n are arbitrary)

The system is

$$AX = B$$

where A, X, B are matrices of size $m\times n$, $n\times 1$, $m\times 1$ respectively.

Since the square case has been discussed, there remain the cases:

1) $m>n$: One takes n equations, obtaining a square system and solves.

2) $m<n$: One chooses suitable $n-m$ unknowns as parameters $t_1, \ldots, t_{n-m}$ and solves.

It is advisable in all cases to solve any system of linear equations by GAUSS Method (by elementary operations):

2. Solution by GAUSS Method:

This method is applicable successfully to any rectangular system

$$AX = B \tag{1}$$

which we represent, conveniently, by the matrix

$$\left[A \,\vdots\, B\right]_{m\times(n+1)}$$

called the augmentel matrix of A.

The method consists of reducing by row operations the matrix $\left[A \,\vdots\, B\right]$ to an echelon form $\left[A' \,\vdots\, B'\right]$ which represents a system (1') of linear equations equivalent to the system (1). Since $\left[A' \,\vdots\, B'\right]$ is an echelon form, the number of non zero terms decrease as we go down until the last equation, and solving (1') is much more simpler.

Example 1. Solve the system

$$\begin{aligned} x + 2y - z &= 2 \\ x + y + z &= 6 \\ 2x - y + z &= 3 \end{aligned} \qquad (1)$$

Solution.

$$[A \vdots B] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 1 & 1 & 1 & 6 \\ 2 & -1 & 1 & 3 \end{array}\right] (-1) \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -1 & 2 & 4 \\ 2 & -1 & 1 & 3 \end{array}\right] (-2)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -1 & 2 & 4 \\ 0 & -5 & 3 & -1 \end{array}\right] (-5) \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -1 & 2 & 4 \\ 0 & 0 & -7 & -21 \end{array}\right]$$

This echelon matrix represents the system

$$\begin{aligned} x + 2y - z &= 2 \\ -y + 2z &= 4 \\ -7z &= -21 \end{aligned} \qquad (1')$$

which is equivalent to (1) since the row operations (interchanging two equations, adding two equations, miltiplying an equation by a non zero scalar) do not alter the solution of the system.

Starting from bottom and going upward we have

$$\begin{aligned} -7z = -21 &\Rightarrow z = 3 \\ -y + 2.3 = 4 &\Rightarrow y = 2 \quad \Rightarrow \quad S = [1, 2, 3] \\ x - 2.2 - 3 = 2 &\Rightarrow x = 1 \end{aligned}$$

<u>Example 2</u>. Solve

$$\begin{aligned} x_2 - x_3 + x_4 &= 2 \\ x_1 - 2x_2 + 2x_3 - x_4 &= 1 \\ 2x_1 + x_2 - x_3 + 2x_4 &= 8 \end{aligned}$$

<u>Solution</u>.

$$\left[\begin{array}{cccc|c} 0 & 1 & -1 & 1 & 2 \\ 1 & -2 & 2 & -1 & 1 \\ 2 & 1 & -1 & 2 & 8 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -2 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 & 2 \\ 2 & 1 & -1 & 2 & 8 \end{array}\right] {-2}$$

$$\sim \left[\begin{array}{cccc|c} 1 & -2 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 & 2 \\ 0 & 5 & -5 & 4 & 6 \end{array}\right] {-5} \sim \left[\begin{array}{cccc|c} 1 & -2 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 & 2 \\ 0 & 0 & 0 & -1 & -4 \end{array}\right]$$

$$\Rightarrow \quad \begin{aligned} x_1 - 2x_2 + 2x_3 - x_4 &= 1 \\ x_2 - x_3 + x_4 &= 2 \\ - x_4 &= -4 \end{aligned}$$

$\Rightarrow \quad x_4 = 4$

$\Rightarrow \quad x_2 - x_3 + 4 = 2 \quad \Rightarrow \quad x_2 = -2 + x_3 \quad -2 + t, \; (x_3 = t)$

$\Rightarrow \quad x_1 - 2(-2 + t) - 2t - 4 = 1 \Rightarrow x_1 = 1$

$S = [1, \; -2 + t, \; t, \; 4]$

<u>Remark</u>. If the last row in an echelon matrix consists of zero elements only, this row is to be discarded. If the last row is

$$0 \; . \; . \; . \; 0 | \, a \qquad (a \neq 0)$$

the system is inconsistent.

After discarding zero rows, if the remaining is consistent, the given system is consistent. In the consistency case starting from the bottom and going upward considering the equation corresponding to each row one can find the unknowns successively $(x_n, x_{n-1}, \ldots)$, some of which are taken as parameter when possible.

When the echelon form of the system is, for instance

$$\left[\begin{array}{cccc|c} 2 & 0 & -3 & 4 & 1 \\ 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

the system is inconsistent (no solution)

If the echelon fo[illegible], for instance,

$$\left[\begin{array}{cccc|c} [illegible] & [illegible] & -3 & 4 & 1 \\ 0 & 0 & 0 & 2 & 6 \end{array}\right]$$

we have consistency. Then

$$2x_4 = 6 \Rightarrow x_4 = 3$$

$$2x_1 + x_2 - 3x_3 + 4.3 = 1$$

$$\Rightarrow \quad 2x_1 + x_2 - 3x_3 = -11$$

$$x_1 = s, \quad x_3 = t \Rightarrow x_2 = -11 - 2s + 3t$$

$$S = [s, -11 - 2s + 3t, t, 3]$$

When the echelon form is

$$\left[\begin{array}{cc|c} 1 & -3 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

the system is inconsistent

If it is

$$\left[\begin{array}{cc|c} 1 & -3 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

we have consistency. Then . Then

$$2x_2 = 3 \Rightarrow x_2 = 3/2$$

$$x_1 - 3 - \frac{3}{2} = 4 \Rightarrow x_1 = 17/2$$

$$S = [17/2, 3/2]$$

If the echelon form is

$$\left[\begin{array}{cc|c} 1 & -3 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

there is consistency. Then

$$x_1 - 3x_2 = 4 \Rightarrow x_2 = t \Rightarrow x_1 = 4 + 3t$$

$$S = [4 + 3t, t]$$

<u>Proof of the validity of the sweepout process (by row operations) for finding the inverse of a matrix</u>:

Let A be a matrix of order n. To find its inverse A^{-1} consider a square system of

$$AX = Y \tag{1}$$

of linear equations. If A is invertible the unique solution is given by

$$X = A^{-1}Y \tag{2}$$

Introducing the identity matrix I of order n, (1) and (2) can be written as

$$AX = IY \tag{1'}$$

$$IX = A^{-1}Y \tag{2'}$$

where the solution (2') can be obtained from (1') by row operations. This means that the matrix $A \mid I$ representing (1') is reduced by row operations to the matrix $I \mid A^{-1}$ representing (2'):

$$A \mid I \sim I \mid A^{-1}$$

So starting with the matrix $A \mid I$ elementary row operations reduce it to $I \mid A^{-1}$ where the second side of the latter is the inverse of A.

If A is not invertible $A \mid I$ cannot be reduced to the form $I \mid A$ where the left hand side is the identity matrix.

EXERCISES (2.2)

31. Given that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & -1/2 \end{bmatrix} W \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

solve it for the matrix W.

32. Solve

a)
$$\begin{aligned} x^2+5y^2-z^2 &= 3 \\ 2x^2+3y^2-z^2 &= 7 \\ x^2-y^2+2z^2 &= 6 \end{aligned}$$

b)
$$\begin{aligned} \frac{2}{x} + \frac{1}{y} + \frac{3}{z} &= 1 \\ \frac{6}{x} - \frac{2}{y} + \frac{6}{z} &= -1 \\ \frac{1}{x} + \frac{1}{2y} - \frac{3}{z} &= 2 \end{aligned}$$

33. Solve

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= 5 \\ 2x_1 + x_2 + 4x_3 + x_4 &= 2 \\ 3x_1 + 4x_2 + x_3 + 5x_4 &= 6 \\ 2x_1 + 3x_2 + 5x_3 + 2x_4 &= 0 \end{aligned}$$

34. Discuss the solution of

$$\begin{aligned} 3x - 2y + z &= -9 \\ x + y + z &= 1 \\ 2x + y - 5z &= -7 \end{aligned}$$

35. Solve

$$\begin{aligned} x \cos^2 A + y \cos^2 B + z \cos^2 C &= \cos^2 \lambda \\ x \sin^2 A + y \sin^2 B + z \sin^2 C &= \sin^2 \lambda \\ x \tan A \div y \tan B \div z \tan C &= \tan \lambda \end{aligned}$$

a) in the general case,

b) when $A + B + C = \pi/2$

36. Discuss the solution of the system:

$$y+z = 1, \quad 2x-2y-z = 0, \quad 4x+3y+5z = 7$$

37. Discuss the solution by the use of augmented matrix:

$$\begin{bmatrix} 2 & -3 & 1 \\ 3 & 0 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}$$

38. Find right inverses, if any, of the following matrices:

a) $\begin{bmatrix} 2 & 0 \\ -1 & 3 \\ 1 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 3, & 7 \end{bmatrix}$

39. Find left inverses, if any, of the matrices given in Exercise 38.

40. Find the inverse of $\begin{bmatrix} a & b & c \\ o & d & e \\ o & o & f \end{bmatrix}$ $(adf \neq 0)$

41. Solve

$$\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2-1 & b^2-1 & c^2-1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ p \\ p^2-1 \end{bmatrix}$$

and evaluate

$$(a^2-a)x+(b^2-b)y+(c^2-c)z-(p^2-p)$$

42. Find the inverse of:

a) $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 0 & 1 & 8 \end{bmatrix}$

43. Find $x, y \in \mathbb{R}$, if any:

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} x & y \\ 2x & -y \\ -x & 3y \end{bmatrix} = \begin{bmatrix} 14 & 2 \\ -16 & 14 \end{bmatrix}$$

44. If $A_{2x2} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$, we say that the matrix. A <u>maps</u> the point (x, y) to the point (x', y'). Find A which maps (2, 3) to (1, 0), and (-1, 1) to (2, -5).

45. Find x, yεR, if any:

$$\begin{bmatrix} 3 & -1 \\ 0 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x & -y & y-x \\ 2y & 3x & x-y \end{bmatrix} = \begin{bmatrix} -6 & -9 & 12 \\ 12 & 0 & -6 \\ 6 & 6 & -9 \end{bmatrix}$$

ANSWERS TO EVEN NUMBERED EXERCISES

32. a) $[\pm 2, 0, \pm 1]$ (four solutions), b) $[2, 1, -3]$

34. $[-2, 2, 1]$

36. $[1-k, 1-2k, 2k]$

38. a) $\begin{bmatrix} t & \frac{1}{2}t - \frac{1}{4} & -\frac{3}{2}t + \frac{3}{4} \\ s & \frac{1}{2}s + \frac{1}{4} & -\frac{3}{2}s + \frac{1}{4} \end{bmatrix}_{2x3}$, b) $\frac{1}{7} \begin{bmatrix} 75 \\ 1 - 3s \end{bmatrix}$

40. $\begin{bmatrix} \frac{1}{a} & -\frac{b}{ad} & \frac{be-cd}{adf} \\ 0 & \frac{1}{d} & -\frac{e}{df} \\ 0 & 0 & \frac{1}{f} \end{bmatrix}$

42. a) $\begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$ b) $\frac{-1}{7}\begin{bmatrix} -11 & 2 & 2 \\ -16 & 8 & 1 \\ 2 & -1 & -1 \end{bmatrix}$

44. $\begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}$

A SUMMARY

2. 1. Operations with matrices:

$$A+B = [a_{ij}]+[b_{ij}]=[a_{ij}+b_{ij}]= B+A$$

$$cA = c[a_{ij}] = [ca_{ij}]$$

$$A_{mxn}\,B_{nxp} = C_{mxp} = [c_{ij}], \quad c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

Definitions:

$A^T = [a_{ij}]^T = [a_{ji}]$ (transpose of the matrix A)

$\mathrm{Adj}\ A = [A_{ij}]^T = [A_{ji}]$ (adjoint of A)

$AA^{-1} = A^{-1}A = I$ (A^{-1} is the inverse of A)

A formula for A^{-1} is $A^{-1} = \mathrm{Adj}A/|A|$

where $|A| = \det A$

Echelon matrix: Is a matrix $[a_{ij}]_{mxn}$ such that

$$\left.\begin{array}{l} a_{ij} = 0 \quad (j = 1, \ldots, k) \\ a_{i+1\ j} = 0 \quad (j=1, \ldots, k+1 \text{ at least}) \end{array}\right\} \text{for any } i$$

An example of echelon matrix:

$$\begin{bmatrix} 0 & 3 & 1 & 0 & 5 & 0 & -7 & 8 & 3 \\ 0 & 0 & 0 & 2 & -4 & 0 & 6 & -3 & 4 \\ 0 & 0 & 0 & 0 & -2 & 0 & 1 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2. 2. Solution of a system of linear equans: $AX=B$

a) Square case:
By inverse matrix: $X=A^{-1}B$ when A is invertible

b) General case:
By GAUSS Method: Obtaining an echelon form of the augmeted matrix $|A\,\vdots\,B|$ and solving step by step from bottom to top.

MISCELLANEOUS EXERCISE

46. Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix}$ Find

a) AB b) $A^3 + A^2 - 6A - 17I_3$

47. Prove

a) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n & \frac{1}{2}n(n+1) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$

48. Find a matrix B such that $B^{-1}AB$ is diagonal, where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

49. List all 2x2 echelon matrices.

50. For an nxn matrix A, prove $|A|\,|\text{adj } A| = |A|^n$

51. Obtain an echelon form of:

$$\begin{bmatrix} 2 & 1 & 0 & 3 & -2 \\ 2 & 4 & 1 & 3 & 2 \\ 4 & 2 & 1 & 3 & -2 \\ 2 & 1 & 1 & 0 & 5 \\ -4 & 1 & 1 & 3 & -2 \end{bmatrix}$$

52. If $ad - bc = 1$, then

$$\begin{bmatrix} ad & cd & -ab & -bc \\ -ac & -c^2 & a^2 & ac \\ bd & d^2 & -b^2 & -bd \\ -bc & -cd & ab & ad \end{bmatrix}^{-1} = \begin{bmatrix} ad & bd & -ac & -bc \\ -ab & -b^2 & a^2 & ab \\ cd & d^2 & -c^2 & -cd \\ -bc & -bd & ac & ad \end{bmatrix}$$

53. If $A_1, \ldots, A_n$ are invertible matrices of the same order,

a) prove $(A_1 \ldots A_n)^{-1} = A_n^{-1} \ldots A_1^{-1}$

b) prove $(A^n)^{-1} = (A^{-1})^n$

54. Find the inverses of:

a) $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

55. Show that the product of two upper (lower) square triangular matrices is an upper (lower) triangular matrix.

56. Prove that the inverse of a non singular diagonal matrix is a non singular diagonal matrix.

57. Evaluate

$$\begin{bmatrix} \frac{a \pm \sqrt{ad-bc}}{\sqrt{D}} & \frac{b}{\sqrt{D}} \\ \frac{c}{\sqrt{D}} & \frac{a \pm \sqrt{ad-bc}}{\sqrt{D}} \end{bmatrix}^2$$

where $D = a + d + 2\sqrt{ad - bc} > 0$

58. If $(A^{-1})^n$ is denoted by A^{-n}, then evaluate U^{-2}, V^{-2} where

$$U = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & 1 & 2 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 5 & 2 & 3 & -1 \\ -1 & 1 & -5 & 2 \end{bmatrix}$$

59. Prove $[\Delta(\theta)]^n = \Delta(n\theta)$, where

$$\Delta(\theta) = \begin{bmatrix} \cos^2\theta & -\sin 2\theta & \sin^2\theta \\ \cos\theta \sin\theta & \cos 2\theta & -\sin\theta \cos\theta \\ \sin^2\theta & \sin 2\theta & \cos^2\theta \end{bmatrix}$$

60. Find all 2x2 matrices which commute

a) with $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, b) with every 2x2 square matrix.

61. For which values of t, will the system

$$\begin{aligned} 4x + y - 4z &= 0 \\ 2x + ty + z &= 0 \\ (t-1)x - y + 2z &= 0 \end{aligned}$$

be consistent?

62. Solve the systems:

a) $2x - 3y = 0, \quad x + 3z = 0, \quad y + 2z = 0$

b) $x + y - z = 0, \quad x - y + z = 0, \quad x + 3y + 2z = 0$

63. Show that the solution of

$$\begin{aligned} a_1x + a_2y + a_3z &= 0 \\ b_1x + b_2y + b_3z &= 0 \end{aligned}$$

is $\left[\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} t, \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} t, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} t \right]$

64. Solve:

$$\begin{bmatrix} 1 & 2 & -3 & 5 \\ 1 & 3 & -3 & 5 \\ 1 & 3 & -1 & 5 \\ 3 & 8 & -7 & 15 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \\ 7 \end{bmatrix}$$

65. Show that the following system is inconsistent:

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + 4x_5 &= 2 \\ x_1 + 2x_2 - x_3 + x_4 + 7x_5 &= 3 \\ 2x_1 + 4x_2 - 4x_3 - 5x_4 + 8x_5 &= 8 \\ 3x_1 + 6x_2 - 5x_3 + 6x_4 + 15x_5 &= 11 \end{aligned}$$

ANSWERS TO EVEN NUMBERED EXERCISES

46. a) $\begin{bmatrix} 17 & 5 & 1 \end{bmatrix}^T$ b) 0_3

48. $B = \begin{bmatrix} 1 & w & w^2 \\ 1 & w^2 & w \\ 1 & 1 & 1 \end{bmatrix}$ where $w^3 = 1$, $w \neq 1$.

54. a) $\begin{bmatrix} 1 & -1 & 0 \\ -1 & & 0 \\ 0 & 0 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

58.

$$U^{-1} = \frac{1}{6}\begin{bmatrix} 3 & 1 & -1 \\ 3 & -3 & \\ -3 & 1 & 5 \end{bmatrix} \qquad \frac{1}{36}\begin{bmatrix} 15 & -1 & -11 \\ 9 & 9 & -9 \\ -15 & -1 & 25 \end{bmatrix}$$

$$^{-1} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -13 & 4 & 2 & 1 \\ -31 & 9 & 5 & 3 \end{bmatrix}, V^{-2} = \begin{bmatrix} 5 & -3 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ -87 & 8 & 9 & 5 \\ -229 & 87 & 25 & 14 \end{bmatrix}$$

60. a) $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ b) $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$

62. a) $[-3t, -2t, t]$, b) $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$

64. $[-5k+7,\ 0,\ 2,\ k]$

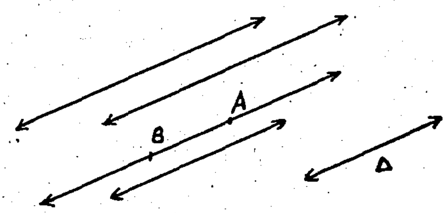
CHAPTER 3
ANALYTIC GEOMETRIC IN $\mathbb{R}^3$

3. 1. VECTORS

A. DEFINITIONS

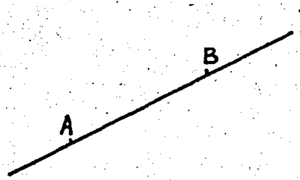
In Physical science are quantities such as velocity and acceleration that are determined not only by magnitude (length) but also by direction and sense. Such quantities have the common name vector. Vectors are very powerful tools in treating analytic geometry.

The set of all parallel lines in space define a <u>direction</u>, Δ, and each line AB of the set has this indicated direction and defines two <u>senses</u>, one from A to B, the other from B to A.



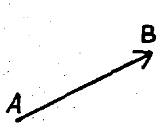
A line segment [AB] with end points A and B has a direction (that of its support line), and a <u>length</u> |AB|.

A line segment [AB] oriented from one end to the other, say from A to B, is called a <u>line vector</u>, written $\vec{AB}$. A is the initial point or the point of application and B the extremity of $\vec{AB}$.



line AB
(has a direction)

line segment [AB]
(has a direction and a length)



Vector $\vec{AB}$
(has a direction length and sense)

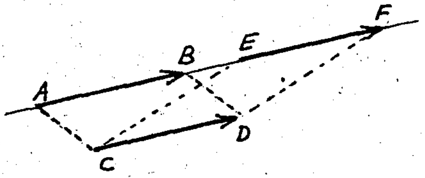
A vector having coincident end points or having a zero length is the <u>zero vector</u> denoted by $\vec{PP}$ or $\vec{0}$ or simply 0 with indefinite direction and sense.

Vectors may as well be denoted by single letters with an arrow on top: $\vec{a}$, $\vec{u}$, $\vec{e}$.

<u>Equality</u>:

Two vectors $\vec{AB}$, $\vec{CD}$ having to same direction, the same sense and the same length are considered <u>equal</u>, written $\vec{AB} = \vec{CD}$.

If $\vec{AB} = \vec{CD}$, then ABCD is a parallelogram (which may be degenerate one) and one of these vectors can be obtained from the other by a translation (under which direction, sense and length are preserved).



In the figure, ABCD and CDFE are parallelograms. It follows that $\vec{AB} = \vec{CD} = \vec{EF}$. Note that ABFE is a degenerate parallelogram.

From this definition, vectors can always be drawn to have the same initial point.

Two vectors are unequal if they differ from each other either in direction or in sense or in length.

A vector with fixed initial point is a <u>bound vector</u> (or a <u>position vector</u>), one restricted to lie on a fixed line is a <u>sliding vector</u> and ones with no such restrictions are <u>free vectors</u>.

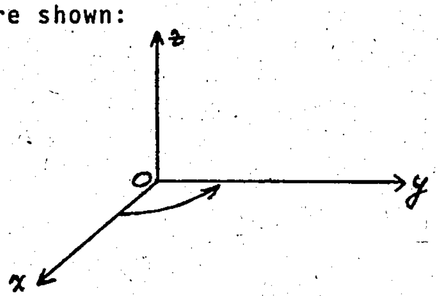
A free vector remains invariant under any translation, while a sliding vector remains invariant under a translation in the direction of the vector.

Line vectors can be expressed analytically by the use of coordinate systems. This representation will be called a coordinate vector. Coordinate vectors are more convenient in all analytical treatments than the line vectors. We introduce below rectangular coordinate system to define coordinate vectors.

Rectangular coordinate systems:

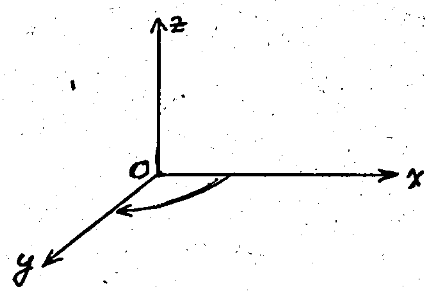
Consider in 3-space three mutually perpendicular axes Ox, Oy, Oz with a common origin O, called a rectangular (or cartesian) coordinate system, written Oxyz. A 3-space provided with such a system will be called an analytic 3-space.

In the following two figures two distinct coordinate systems are shown:



A positive system

(A right hand system)



A negative system

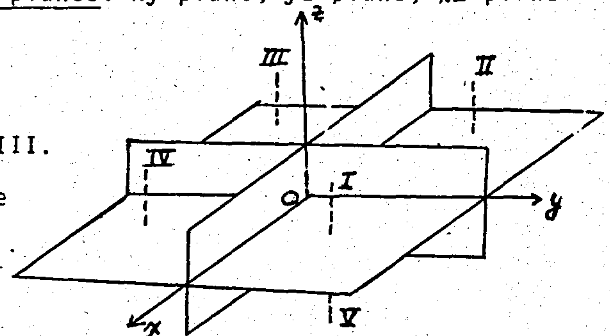
(A left hand system)

On a positive system an observer, standing on Oxy plane at O along positive z-axis observes that the angle from positive x-axis to positive y-axis is positive (counterclockwise). In a negative system the same observer observes that the same angle is negative (clockwise)

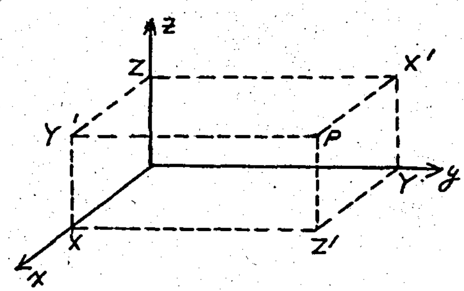
The positiveness and negativeness is invariant under any translation and rotation of the coordinate axes.

In this book we will always use positive systems.

In an Oxyz system the axes taken two at a time determine three planes called <u>coordinate planes</u>: xy-plane, yz-plane, xz-plane. These planes separates from $\mathbb{R}^3$ eight regions, called <u>octants</u>, numbered I, II, ... , VIII. The one containing the positive parts of the axes is taken as the <u>I. octant</u>.

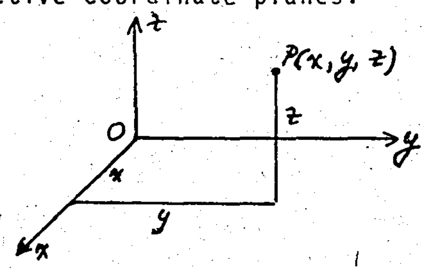


Referring to the Figure, let in the analytic 3-space, P be any point (usually taken in the I. octant) and let the projections of P on coordinates axes and planes be X, Y, Z and X', Y', Z'. These points and O, P are vertices of a rectangular parallelepiped.



The abscissas x, y, z of X, Y, Z on respective axes are called the <u>rectangular</u> (or <u>cartesian</u>) <u>coordinates</u> of P and one writes P(x, y, z). It is seen that these coordinates are directed distances of P from the respective coordinate planes.

A point P(x, y, z) in usually sketched as shown in the Figure.



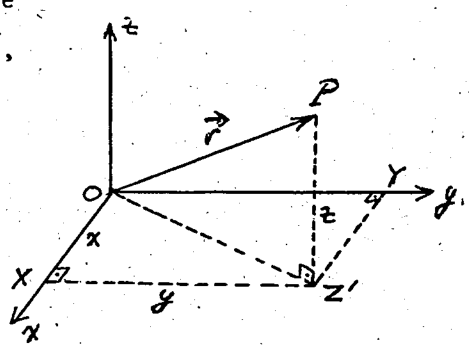
The set

$$\{(x, y, z): \ x \geqslant 0, \ y \geqslant 0, \ z \geqslant 0\}$$

represents clearly the I. octant.

Though are not in use, we call x the <u>abcissa</u>, y the <u>ordinate</u>, and z the <u>cote</u> of P.

A bound (position) vector $\vec{r} = \vec{OP}$ having its initial point at the origin and extremity at $P(x, y, z)$ defines uniquely the point P and conversely, and we make no difference between the symbols $\vec{OP}$, $\vec{P}$, P, $\vec{r}$, (x, y, z) and write



$$\vec{r} = \vec{OP} = \vec{P} = P = (x, y, z)$$

which is a <u>coordinate vector</u>.

Any free vector can be expressed as a coordinate vector.

Since (x, y, z) is an ordered triple, it can be considered as an element of the cartesian product $\mathbb{R}\times\mathbb{R}\times\mathbb{R} = \mathbb{R}^3$.

We represent the position vector (x, y, z) as the column matrix

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [x, y, z]^T$$

<u>Length of a vector:</u>

Referring to above Figure we have as the length $|\vec{OP}|$ of the vector $\vec{OP}$:

$$|\vec{OP}|^2 = |OP|^2 = |OZ'|^2 + |Z'P|^2$$

$$= (|x|^2 + |y|^2) + |z|^2 = x^2+y^2+z^2$$

$$\Rightarrow \quad |P| = |\vec{OP}| = \sqrt{x^2 + y^2 + z^2} \quad (\geqslant 0)$$

<u>Example.</u> Find the lengths of

$A = (1, -2, 2)$, $\quad B = (2, 3, 6)$, $\quad C = (\frac{1}{9}, \frac{4}{9}, \frac{8}{9})$

<u>Solution.</u>

$|A| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{1 + 4 + 4} = 3$

$|B| = \sqrt{4 + 9 + 36} = 7$

$$|C| = \sqrt{(\tfrac{1}{9})^2 + (\tfrac{4}{9})^2 + (\tfrac{8}{9})^2} = \tfrac{1}{9}\sqrt{1 + 16 + 64} = 1$$

A vector having length equal to 1 is called a <u>unit vector</u>. In the above example, C is seen to be a unit vector.

B. <u>ALGEBRA OF VECTORS</u>

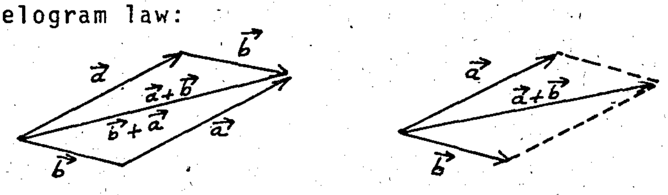
1. <u>Addition</u>:

The sum $\vec{a}+\vec{b}$ of two vectors $\vec{a}$ and $\vec{b}$, in this order, is the vector whose initial point is that of the first vector and the extremity is that of the second when the latter is translated to have its initial point at the extremity of the first vector:

The following figur[illegible] the commutative law

$$\vec{a} + \vec{b} \ [illegible] + \vec{a}$$

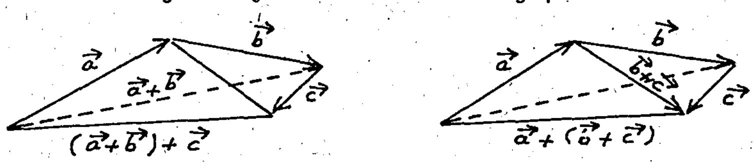
and t[illegible] parallelogram law:



The associative law

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$$

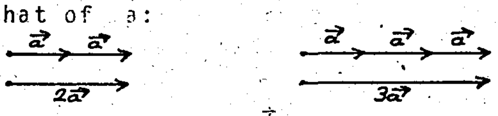
is the result congruency of the following piramids:



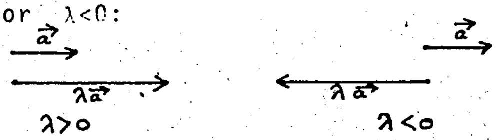
Because of this law the sum $\vec{a}+\vec{b}+\vec{c}$ has a meaning as defined by $(\vec{a}+\vec{b})+\vec{c}$ or by $\vec{a}+(\vec{b}+\vec{c})$.

2. Multiplication by scalars

Denoting the sums $\vec{a}+\vec{a}$, $\vec{a}+\vec{a}+\vec{a}$, ... by $2\vec{a}$, $3\vec{a}$, ... and defining $1\vec{a}$, $0\vec{a}$ as $\vec{a}$ and 0, the vector $n\vec{a}$ $(n\varepsilon\mathbb{N})$ will denote a vector having the same direction and sense as $\vec{a}$ and length n times that of $\vec{a}$:



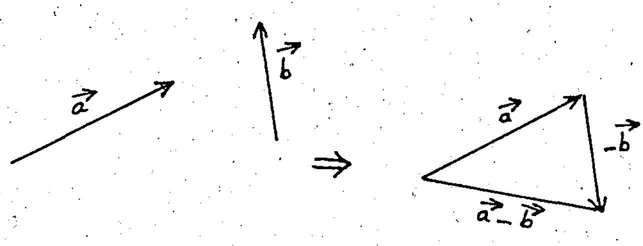
For any $\lambda\varepsilon R$, we define $\lambda\vec{a}$ as a vector of length $|\lambda||\vec{a}|$ parallel[(1)] to $\vec{a}$ and agreeing or disagreeing in sense with $\vec{a}$ according as $\lambda>0$ or $\lambda<0$:



In particular for $\lambda = -1$, we have the vector $(-1)\vec{a} = -\vec{a}$ which is opposite to $\vec{a}$, called the <u>additive inverse</u> of $\vec{a}$, since $\vec{a}+(-\vec{a}) = 0$.

$\lambda\vec{a}$ and $\vec{a}$ have parallel supports, and are called <u>collinear vectors</u>.

The difference $\vec{a} - \vec{b}$ is by definition the sum $\vec{a}+(-\vec{b})$:



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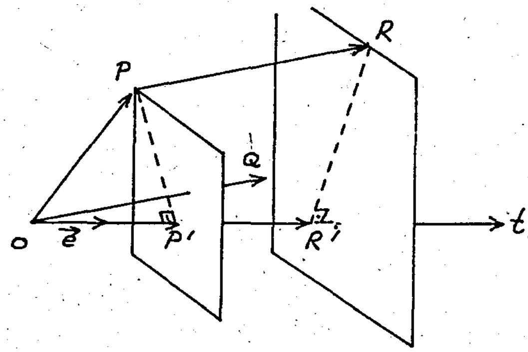
(1) Two coplanar lines (lying on the same plane) having no common point are called <u>parallel</u>. But in this book parallellism of lines is defined to include also the coincidence of lines.

Projections of vectors:

Let $\overrightarrow{OP}$ be a vector and Ot be an axis with a unit vector $\vec{e}$ along it. If P' is the projection of P on Ot (which is the intersections of Ot with the plane through P perpendicular to Ot), then $\overrightarrow{OP'}$ is the vector projection (Vector compoment), and the coordinate of P' is the scalar projections (scalar compoment) of $\overrightarrow{OP}$ on Ot. (See Fig.)



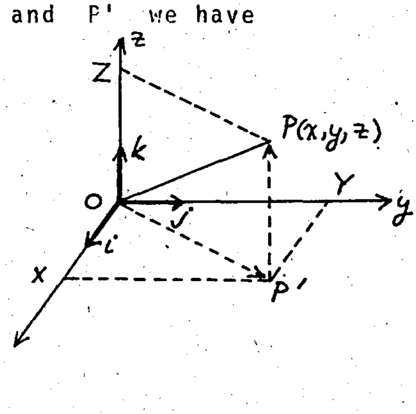
Now consider a second vector $\overrightarrow{OQ}$ and the vector $\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{OQ}$ where $\overrightarrow{OQ} = \overrightarrow{PR}$. Then the projection of $\overrightarrow{OR}$ on Ot is $\overrightarrow{OR'} = \overrightarrow{OP'} + \overrightarrow{P'R'}$ which is the sum of projectio (compoments).

Obviously the scalar component of the sum $\overrightarrow{OR}$ is the sum of scal components of $\overrightarrow{OP}$, $\overrightarrow{OQ}$.

Now, taking coordinate axes Ox, Oy, Oz instead of Ot with respective unit vectors

$$i = (1, 0, 0), \ j = (0, 1, 0), \ k = (0, 0, 1)$$

along positive parts, and denoting the projections of P(x, y, z) on the axes and xy-plane by X, Y, Z and P' we have



$$\overrightarrow{OP} = \overrightarrow{OP'} + \overrightarrow{P'P}$$
$$= \overrightarrow{OX} + \overrightarrow{OY} + \overrightarrow{OZ}$$

since $\overrightarrow{OX} + \overrightarrow{OY} = \overrightarrow{OP'}$ and $\overrightarrow{OZ} = \overrightarrow{P'P}$

Then

$$\overrightarrow{OP} = xi + yj + zk$$

where xi, yj, zk are vector

components, while x, y, z (coordinates of P) are scalar components of $\vec{OP}$.

Let

$$\vec{OP}_1 = x_1 i + y_1 j + z_1 k = [x_1\ y_1\ z_1]^T.$$

$$\vec{OP}_2 = x_2 i + y_2 j + z_2 k = [x_2\ y_2\ z_2]^T$$

Then we have

$$\vec{OP}_1 + \vec{OP}_2 = (x_1+x_2)i + (x_2+y_2)j + (x_3-y_3)k$$

$$= [x_1+x_2\ \ x_2+y_2\ \ x_3+y_3]^T$$

by properties of projections.

Also

$$\vec{OP}_1 - \vec{OP}_2 = (x_1-x_2)i + (y_1-y_2)j + (z_1-z_2)k$$

$$= [x_1-x_2\ \ y_1-y_2\ \ z_1-z_2]^T$$

Accordingly any free vector $\vec{AB}$ extending from the point $A(a_1\ a_2\ a_3)$ to $B(b_1, b_2, b_3)$ can be written as the position vector

$$\vec{OB} - \vec{OA} = (b_1-a_1,\ b_2-a_2,\ b_3-a_3)$$

since $\vec{OA} + \vec{AB} = \vec{OB}$ or $\vec{AB} = \vec{OB} - \vec{OA}$.

When a vector is multiplied by a scalar, its all components being multiplied by the same scalar, we have

$$\lambda\vec{P} = \lambda(x, y, z) = (\lambda x, \lambda y, \lambda z)$$

by the properties of projections.

Observe analogy between matrices and vectors in the operation of addition and multiplication by scalars:

$$[a_1\ a_2\ a_3] \pm [b_1\ b_2\ b_3] = [a_1 \pm b_1\ \ a_2 \pm b_2\ \ a_3 \pm b_3],$$

$$\lambda[a_1\ a_2\ a_3] = [\lambda a_1\ \ \lambda a_2\ \ \lambda a_3].$$

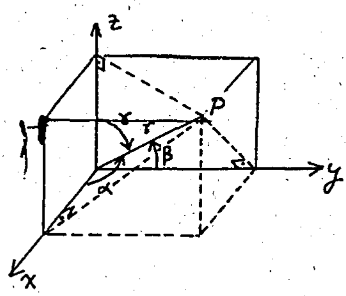
Generalizing, we have

$$c_1\vec{P}_1 + c_2\vec{P}_2 + \ldots + c_n\vec{P}_n = \sum_{k=1}^{n} c_k\vec{P}_k$$

which is called a linear combination of n Vectors.

Direction-angles, -cosine and -numbers:

Let $\vec{r} = \vec{OP}$ be a position vector with length r. The angles α, β, γ that are made by $\vec{r}$ with positive sides of axes are direction angles, their cosines $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are direction cosines and numbers proportional to direction cosines are direction numbers of $\vec{OP}$.



Since scalar projections of $P(x, y, z)$ on the axes are x, y, z one has

$$\cos\alpha = \frac{x}{r}, \quad \cos\beta = \frac{y}{r}, \quad \cos\gamma = \frac{z}{r}$$

and consequently x, y, z and for $k \neq 0$, kx, ky, kz are direction numbers.

3. Multiplication of Vectors:

There are two kinds of multiplication for vectors, scalar and vector multiplications, the results of which being scalar and vector.

a) Scalar product of two vectors:

Let $A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$ be two vectors. Then

$$A.B = |A||B| \cos\theta$$

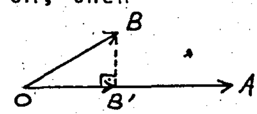
is called the scalar product of A and B, where θ is the angle between them $(0 \leqslant \theta \leqslant \pi)$.

The product is also called <u>dot product</u> or <u>inner product</u>, and also denoted by $A|B$, (A, B), $\langle A, B\rangle$ or $\langle\langle A, B\rangle\rangle$.

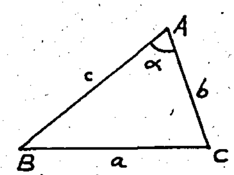
The scalar product $A.B$ certainly vanishes when $A = 0$ or $B = 0$. For non zero vectors, the product is positive, zero or negative according as θ is an acute, right or obtuse angle.

<u>Geometric interpretations</u>:

1. If $\vec{OB}'$ is the projection of $\vec{OB}$ on $\vec{OA}$, then
$\vec{OA}.\vec{OB} = \vec{OA}.\vec{OB}'$



2. The cosine law $a^2 = b^2+c^2-2bc\cos\alpha$ for a triangle ABC can be expressed in the form
$|BC|^2 = |AB|^2 + |AC|^2 - 2\vec{AB}.\vec{AC}$

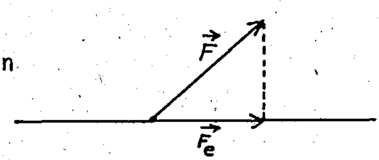


3. Two non zero vectors are perpendicular (orthogonal) if and only if their dot product is zero:
$A \perp B \Leftrightarrow \theta = \pi/2 \Leftrightarrow A.B = 0$

4. The dot product of two vectors with known lengths (of variable directions) is maximum or minimum when they are parallel in the same or opposite senses.

<u>Physical interpretation</u>:

If a particle is moving on a line in the direction of a vector $\vec{R}$, under a force $\vec{F}$, then the effective force $\vec{F}_e$ is the projection vector of $\vec{F}$ on $\vec{R}$ in the direction of motion: $\vec{F}.\vec{R} = \vec{F}_e.\vec{R}$



Properties:

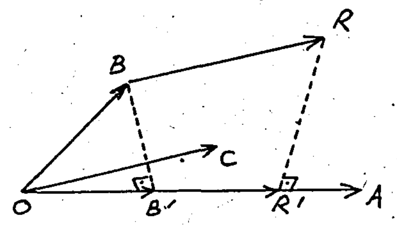
1. $A.B = B.A$ (com. law)

2. $(\lambda A).B = A.(\lambda B) = \lambda(A.B)$ $(\lambda \varepsilon R)$

3. $A.(B+C) = A.B + A.C$ (dist. law)

Proof:

The first two properties are direct consequences of the definition.



To prove the distributive law

$$A.(B+C) = A.B + A.C$$

consider $\vec{BR} = \vec{OC}$ (See Fig.), and projections B', R' of B, R on OA. Then

$$\begin{aligned} A.(B+C) &= A.R \\ &= A.R' \quad \text{(Geom. interp. 1)} \\ &= A.(B'+C') \\ &= A.B' + A.C' \quad \text{(collinearity of vectors)} \\ &= A.B + A.C \quad \text{(Geom. interp. 1)} \ \blacksquare \end{aligned}$$

Now we derive the analytic expression

$$A.B = a_1b_1 + a_2b_2 + a_3b_3$$

for $A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$.

Expanding

$$A.B = (a_1i+a_2j+a_3k).(b_1i+b_2j+b_3k)$$

by distributive law, we get nine terms, six of which are zero by properties

$$i.j = 0, \quad j.k = 0, \quad k.i = 0$$

for orthogonal vectors i, j, k, and the remaining terms are a_1b_1, a_2b_2, a_3b_3 by the properties $i.i = 1$, $j.j = 1$, $k.k = 1$ for unit vectors i, j, k

Then

$$A.B = a_1b_1 + a_2b_2 + a_3b_3$$

If the vectors are written as row and column vectors

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

then

$$A\ B = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1b_1 + a_2b_2 + a_3b_3 \end{bmatrix} = A.B$$

<u>Example 1</u>. Given the vectors

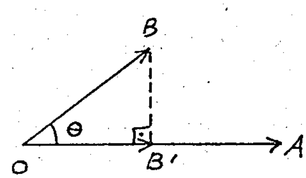
$$A = (2, -3, 4), \quad B = (5, 6, -1)$$

find

a) scalar projection of B on A

b) vector projection of B on A

c) angle between A and B



<u>Solution</u>.

a) $A.B = |A| \underbrace{|B| \cos\theta}_{\text{scalar proj.}}$

$\Rightarrow 2.5 + (-3)6 + 4(-1) = \sqrt{4+9+16}\ \ |\vec{OB}| \cos\theta$

$\Rightarrow -12 = \sqrt{29}\ |\vec{OB}| \cos\theta$

$|\vec{OB}| \cos\theta = -12/\sqrt{29}$

b) $\vec{OB}' = -\dfrac{12}{\sqrt{29}} \dfrac{A}{|A|}$ ($A/|A|$ is the unit vector in the direction and sense of A)

$$= -\frac{12}{\sqrt{29}} \frac{(2, -3, 4)}{\sqrt{29}} = \left(-\frac{24}{29} ; \frac{36}{29}, -\frac{48}{29}\right)$$

$$\cos\theta = \frac{A.B}{|A||B|} = \frac{-12}{\sqrt{29}\,\sqrt{62}} \Rightarrow \theta = \arccos \frac{-12}{\sqrt{29}\,\sqrt{62}}$$

Example 2. Given vectors

$$A = (2a, -4, a+1), \quad B = (3, a-1, 2)$$

determine $a \in R$ such that

a) $A \perp B$ b) $A // B$

Solution.

a) $A \perp B \Rightarrow A.B =$

$\Rightarrow 6a-4$ $+ 2(a+1) = 0 \Rightarrow a = -3/2$

b) $A//B \Rightarrow$. $\frac{a+1}{2}$ (no solution)

Example 3. Giv tors

$A = (1, 6,$ $9, 7)$, $C = (9, 1, 7)$

wri , if possible, C a combination of A and B.

Solution.

$= tA + SB \Rightarrow (9, 1, 7) = (1, 6, 4) + s(1, 9, 7)$

$\Rightarrow t+s = 9, \quad 6t+9s = 1, \quad 4t+7s = 7.$

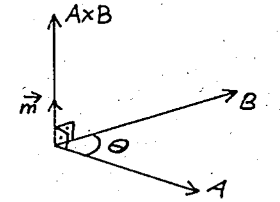
of whi the first two give $t = 80/3$, $s = -53/3$. But these do not satisfy the third one. Hence C cannot be expressed as a linear contination of A and B.

b) Vector product of two vectors:

The vector product of two vectors A and B, in this order, is the vector

$$A \times B = \vec{n}\,|A||B|\sin\theta$$

where θ is the angle between them $(0 \leqslant \theta \leqslant \pi)$, and $\vec{n}$ is the unit vector such that A, B, $\vec{n}$ form a positive system ($\vec{m} \perp A, B$).



In case of collinearity of A and B ($\theta=0$ or $\theta=\pi$) one has $A\times B = 0$, and $\vec{n}$ is uncertain.

Vector product is also called <u>cross product</u> or <u>outer product</u> and denoted also by the symbols $A\wedge B$ or $\overset{\frown}{AB}$.

Since $|A||B|\sin\theta \geqslant 0$, then $A\times B$ (for non collinear vectors) and $\vec{n}$ have the same sense.

$A\times B$ vanishes when $A=0$ or $B=0$ or when A and B are collinear.

It follows from the definition that

1) $A\times B$ is perpendicular to both A and B (direction)

2) $A\times B$ is oriented so that A, B, $A\times B$ is a positive system, not necessarily rectangular (sense)

3) $|A\times B| = |A||B|\sin\theta$ (length).

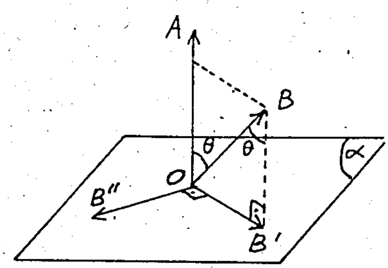
For non zero vectors, the vector product $A\times B$ vanishes if and only if $A//B$.

<u>Corollary</u>. If α is a plane perpendicular to $\vec{OA}$ and if the projection of $\vec{OB}$ on α is $\vec{OB}'$, then

1. $\vec{OA}\times\vec{OB} = \vec{OA}\times\vec{OB}'$

2. $\vec{OB}'' = \vec{OA}\times\vec{OB}$ is obtained from $\vec{OB}'$ by rotating $\vec{OB}'$ about O through 90° and magnifying it by $a = |A|$.

<u>Proof</u>.



1) The two products have obviously the same direction and sense (See Fig.).

To show the equality of lengths, we have

$$|A\times B| = |A||B|\sin\theta = |A||B'|$$
$$= |A||B'|\sin\frac{\pi}{2} = |A\times B'|$$

) $B'' = A \times B \perp A, B \Rightarrow OB'' \perp OB'$,

and $\dfrac{|B''|}{|B'|} = \dfrac{|A||B|\sin\theta}{|B|\sin\theta} = |A| = a.$

Properties.

1. $B \times A = -A \times B$ (anticommutative law)
2. $(\lambda A) \times B = A \times (\lambda B) = \lambda (A \times B)$ $(\lambda \varepsilon R)$
3. $A \times (B+C) = A \times B + A \times C$ (distributive law)

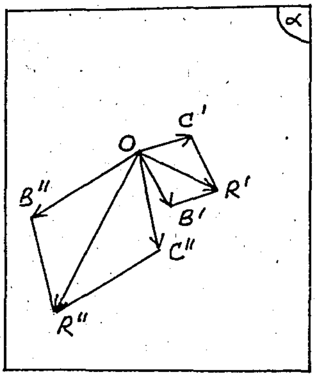
Proof.

The first two properties are direct consequences of the definition.

To prove the th[illegible] [illegible] use the above Corollary.

In the figure, [illegible] and $\vec{OR}'$ are projections of $\vec{OB}$, $\vec{OC}$ and $\vec{OR}(= \vec{OB}+\vec{OC})$ c[illegible] α perpendicular to $\vec{OA}$ (the pro[illegible]ection of $\vec{OA}$ is the [illegible])

Rotating the parallel[illegible] $OB'R$[illegible] by $\pi/2$ about O and magni[illegible]ing by $a = |A|$ we get the parallelogram $OB''R''C''$.



Now by the Corollary we have

$$\vec{OA} \times (\vec{OB} + \vec{OC}) = \vec{OA} \times \vec{OR} = \vec{OR}''$$
$$= \vec{OB}'' + \vec{OC}''$$
$$= \vec{OA} \times \vec{OB} + \vec{OA} \times \vec{OC}. \blacksquare$$

The analytic expression for

$$A \times B = (a_1 i + a_2 j + a_3 k) \times (b_1 i + b_2 j + b_3 k)$$

is obtained by expanding it by the use of distributive law. Expansion gives nine terms, three of which are zero by the relations

$$i \times i = 0, \quad j \times j = 0, \quad k \times k = 0,$$

and the sum of the remaining six terms will be

$$A\times B = (a_2b_3 - a_3b_2)i + (a_3b_1 - a_1b_3)j + (a_1b_2 - a_2b_1)k$$

or

$$A\times B = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1)$$

since

$$j\times k = -k\times j = i,\quad k\times i = -i\times k = j,\quad i\times j = -j\times i = k.$$

Observe that $A\times B$ is equal to the symbolic determinant

$$\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad (= A\times B)$$

Geometric interpretations:

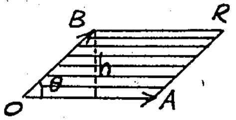
1. For non zero vectors $\vec{A} = (a_1, a_2, a_3)$, $\vec{B} = (b_1, b_2, b_3)$:
   $\vec{A}/\!/\vec{B} \Leftrightarrow \vec{A}\times\vec{B} = 0 \Leftrightarrow \dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3}$
2. $|A\times B| =$ (Area of the parallelogram OARB with adjacent sides $|OA|$, $|OB|$) $= |OARB|_2$

   where the index "2" represents the dimension of the measure.

Proof.

The first one is a direct consequence of the definition. For the second one, we have

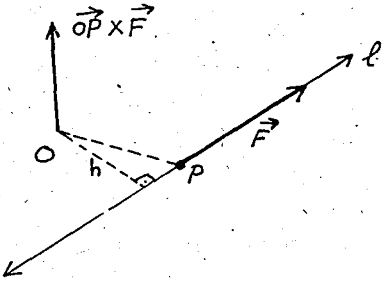
$$|A\times B| = |\vec{n}||A||B|\operatorname{Sin}\theta = |A|\underbrace{|B|\operatorname{Sin}\theta}_{h} = |OARB|_2$$



Observe that the distance between the end point of one of the vectors to the second vector can be obtained by the use of cross product.

Physical interpretation.

Given a sliding force vector $\vec{F}$ and a fixed point O



(See Fig.), then the vector product $\vec{OP} \times \vec{F}$ is the <u>moment vector</u> of $\vec{F}$ with respect to the point O, which is independent of the point of application P on ℓ.

The independence of $\vec{OP} \times \vec{F}$ from P (on ℓ) is shown as follows:

1. $\vec{OP}\times\vec{F}$ is perpendicular to the plane determined by O and ℓ (fixed direction),
2. Sense of $\vec{n}$ is unaltered,
3. $|\vec{OP}\times\vec{F}| = |F|\underbrace{|OP|\sin\theta}_{h} = |F|h.$

<u>Example 1.</u> Given the vectors $A = (t, 4, t^2)$, $B = (-2, 4t, 1)$,

a) find $t\in R$ such that $A\times B$//yz-plane,

b) find the area $|OAB|_2$ for t determined in (a)

<u>Solution.</u>

a) $A\times B = \begin{vmatrix} i & j & k \\ t & 4 & t^2 \\ -2 & 4t & 1 \end{vmatrix} = (4-4t^3)i - (2t^2+t)j + (4t^2+8)k,$

$A\times B$//yz-plane $\Rightarrow A\times B \perp i$

$\Rightarrow (A\times B)\cdot i = 0 \Rightarrow 4-4t^3 = 0 \Rightarrow t=1$

b) For $t=1$, $A = (1, 4, 1)$, $B = (-2, 4, 1)$. Then

$$|OAB|_2 = \frac{1}{2}\,|A\times B| = \frac{1}{2}\,|-3j + 12k| = \frac{1}{2}\sqrt{153} = \frac{3}{2}\sqrt{17}.$$

<u>Example 2.</u> Given the point A(1, 9, 7), B(9, 4, 2), C(0, 1, 3),

a) find the area $|ABC|_2$

b) find a vector $\vec{V}$//ABC and perpendicular to z-axis, if any.

Solution.

a) $|ABC|_2 = \frac{1}{2}\,|\vec{AB} \times \vec{AC}|$

$$\vec{AB} = B-A = (8, -5, -5)$$

$$\vec{AC} = C-A = (-1, -8, -4)$$

$$\vec{AB}\times\vec{AC} = (-20, 37, -69)$$

$$|ABC|_2 = \frac{1}{2}\sqrt{20^2 + 37^2 + 69^2}$$

b) $\vec{V} = \lambda(8, -5, -5) + \mu(1, 8, 4)$

$$= (8\lambda+\mu,\ -5\lambda+8\mu,\ -5\lambda+4\mu\ )$$

$$V.k = 0 \Rightarrow -5\lambda+4\mu = 0 \Rightarrow \mu = 5\lambda/4$$

$$V = \lambda(\frac{37}{4}\ ,\ 5,\ 0).$$

Triple products.

If a cross product of two vectors is multiplied scalarly or vectorally by a third vector, one obtains what one calls a triple product.

If A, B, C are three vectors, then $(AxB).C$, $(AxB)xC$, $Cx(BxA)$, $B.(CxA)$ are examples of triple products.

$AxB.C$, being a scalar and involving two operations, is called a triple scalar product or a mixed product, while the vector $(AxB)xC$ is called a triple vector product.

Now we obtain analytic expressions for a mixed product and a triple vector product.

i. Mixed product:

Let

$$A = (a_1, a_2, a_3),\ B = (b_1, b_2, b_3),\ C = (c_1, c_2, c_3)$$

be any three vectors. Then

$$AxB.C = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Indeed,

$$A \times B . C = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} . (c_1, c_2, c_3)$$

$$= (C_{11}\ C_{12}\ C_{13}).(c_1, c_2, c_3)$$

where C_{11}, C_{12}, C_{13} are cofactors of i, j, k. Then

$$A \times B . C = C_{11} c_1 + C_{12} c_2 + C_{13} c_3$$

$$= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Corollary. In a mixed product the interchange of the "x" and "." does not alter the mixed product:

$$A \times B . C = A . B \times C$$

Proof.

$A . B \times C = B \times C . A$ (commutativity of dot product)

$$= \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = A \times B . C$$

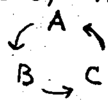
We denote $A \times B . C$ or $A . B \times C$ by the symbol $(A\ B\ C)$ so that we have

$$(A\ B\ C) = A \times B . C = A . B \times C = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Since $(A\ B\ C)$ is a determinant it obeys the properties:

1. $(\alpha A,\ \beta B,\ \gamma C) = \alpha\beta\gamma(A\ B\ C)$
2. $(A_1+A_2\ B\ C) = (A_1\ B\ C)+(A_2\ B\ C)$
3. $(A\ A\ B) = 0$
4. $(A\ C\ B) = -(A\ B\ C)$
5. $(A\ B\ C) = (B\ C\ A) = (C\ A\ B)$

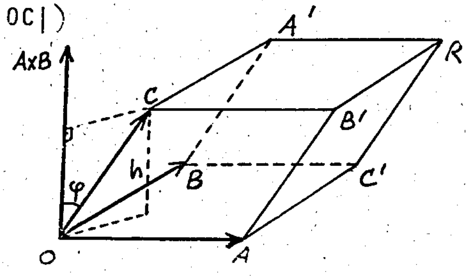
The last property states that $(A\ B\ C)$ is unaltered under a circular permutations of the letters:



Geometric interpretations

If $A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$, $C(c_1, c_2, c_3)$, then

$$|(A\ B\ C)| = (\text{volume of the parallelepiped built on } |OA|, |OB|, |OC|)$$
$$= 6|OABC|_3$$



Proof.

$$(ABC) = A\times B.C$$
$$= |A\times B|\underbrace{|C|\cos\varphi}_{h}$$
$$= |OAC'B|_2\ h = |OAC'RB'CA'B|_3$$
$$= 6|OABC|_3$$

It follows that $(A\ B\ C) = 0 \Leftrightarrow \vec{OA}$. are coplanar.

ii. Triple Vector product:

The triple vector product $(A\times B)\times C$, being a vector perpendicular to $A\times B$, is parallel to the plane OAB, and hence can be expressed as a linear combination of A and B:

$$(A\times B)\times C = (A.C)B - (B.C)A$$

Indeed, let

$$W = (A\times B)\times C = (W_1, W_2, W_3).$$

$$A\times B = (a_2b_3-a_3b_2,\quad a_3b_1-a_1b_3,\quad a_1b_2-a_2b_1)$$
$$C = (\quad c_1 \quad,\quad c_2 \quad,\quad c_3 \quad)$$

We have

$$W_1 = (a_3b_1-[illegible]\, b_3)c_3 - (a_1b_2-a_2b_1)c_2$$
$$= (a_2c [illegible] {}_3)b_1 - (b_2c_2+b_3c_3)a_1$$
$$[illegible]\, a_2c_2+a_3c_3)b_1 - (b_1c_1+b_2c_2+b_3c_3)a_1$$
$$= [illegible] \cdot C)a_1$$

S[illegible]ilarly,

$$W [illegible] = (A. [illegible]$$
$$W [illegible] = (A.C)b_3$$

T[illegible]

$$W = [illegible])(b_1\ b_2\ b_3) - [illegible]\, a_1\ a_2\ a_3)$$
$$= (A.[illegible])B - (B\cdot C)A. \blacksquare$$

Simil[illegible]

$$A\times(B\times C) = (A.C)B - (A.B)C$$

Indeed,

$$A\times(B\times C) = -(B\times C)\times A$$
$$= -\left[(A.B)C - (A.C)B\right] = (A.C)B-(A.B)C$$

The two multiplications have the same rule of expansion, namely

remote

$$(A\times B)\times C = (C.A)B - (C.B)A$$

clo[illegible]

remote

$$A\times(B\times C) = (A.C)B - (A.B)C$$

Generalization of triple scalar and tr[illegible] vector products are the following with their expansions:

1. $(A\times B).(C\times D) = (A.C)(B.D) - (B.C)(A.D)$
2. $(A\times B)\times(C\times D) = (ABD)C-(ABC)D = (ACD)B - (BCD)A$

<u>Proof</u>.

1. $(A\times B).(C\times D) - (A\times B)\times C.D$ (interchange of signs)

$$= [(A.C)B - (B.C)A].D$$

$$= (A.C)(B.D) - (B.C)(A.D)$$

2. $$\underbrace{(A\times B)}_{U}\times(C\times D) = U\times(C\times D)$$

$$= (U.D)C - (U.C)D$$

$$= (A\times B.D)C - (A\times B.C)D$$

$$= (ABD)C - (ABC)D$$

$$(A\times B)\times\underbrace{(C\times D)}_{V} = (A\times B)\times V$$

$$= (A.V)B - (B.V)A$$

$$= (A.C\times D)B - (B.C\times D)A$$

$$= (ACD)B - (BCD)A. \blacksquare$$

C. <u>VECTOR SPACES</u>

The concept of vector space is based on th[illegible]cepts of group and field.

A <u>group</u> is a non empty set G of element[illegible] a single operation, denoted by "$\circ$", is defined (that is [illegible] $a, b\varepsilon G$, then $a\circ b\varepsilon G$) satisfying the three axioms below:

G_1. For any three elements a, b, c of G, not necessarily distinct:

$$(a\circ b)\circ c = a\circ(b\circ c) \qquad \text{(associative law)}$$

G_2. There is an element "e" in G, called the <u>identity element</u>, such that

$$e \circ a = a \circ e = a \quad \text{for any } a \in G.$$

. Given any element "a" in G, corresponding to it there is unique element, denoted by a^{-1}, called the <u>inverse</u> of a, such that

$$a^{-1} \circ a = a \circ a^{-1} = e$$

If furthermore

G_4. $aob = boa$ for very $a, b \in G$ (commutative law)

holds, then th oup i ed a <u>commutative group</u> or an <u>abelian group</u>.

If the op is the addition "+", the group is an additive group, ation ".", the group is a multiplicat group.

The ide ity n additive group is denoted by and in multiplic y "1" or "I".

n additive group th element a^{-1} of "a" is written -a, the <u>addi e inverse</u> of "a" in a multiplicative one a s called the <u>multiplicative inverse</u> of "a"

rizing:

For additi ommutative group the axioms are

1. (a+ a+(b+c) (associative law)
2. $0+a = a+0 = a$ (existence of zero element)
3. $(-a)+a = a+(-a) = 0$ (existence of inverse)
4. $a+b = b+a$ (commutative law)

while for a multiplicative commutative group;

1. $(ab)c = a(bc)$
2. $1a = a1 = a$
3. $a^{-1}a = aa^{-1} = 1$
4. $ab = b$

It is easy to verify that the set $\mathbb{R}$ of all real numbers is an additive commutative group, while $\mathbb{R}^*=\mathbb{R}-\{0\}$ is a multiplicative commutative group.

The set M_{mxn} of all real matrices of the same size is a commutative additive group, and the set M_{nxn} of all non singular square real matrices is a non commutative multiplicative group.

Examples are numerous.

Now we define a field: A set $F = \{\alpha, \beta, \ldots\}$ is called a <u>field</u>, if

i. F is an additive abelian group,

ii. F^* is a multiplicative abelian group,

iii. Distributive law holds: $\alpha(\beta+\gamma) = \alpha\beta+\alpha\gamma$

The elements of a field are called <u>scalars</u>.

The familiar examples of fields are the set $\mathbb{R}$ of all real numbers, and the set $\mathbb{C}$ of all complex numbers.

<u>Vector Space</u>:

Let $V = \{u, v, \ldots\}$ be a non empty set, and let $F = \{\alpha, \beta, \ldots\}$ be a field. Suppose in V there is defined an operation of addition. Then V is called a <u>Vector Space</u> over the field F, if

I. V is an additive abelian group:

1. $(u + v) + w = u + (v + w)$
2. $0 + u = u + 0 = u$
3. $(- u) + u = u + (- u) = 0$
4. $u + v = v + u$

II. For any $u, v \varepsilon V$ and any $\alpha, \beta \varepsilon F$:

1. $1u = u$
2. $(\alpha\beta)u = \alpha(\beta u)$
3. $(\alpha+\beta)u = \alpha u + \beta u$
4. $\alpha(u+v) = \alpha u + \alpha v$

This field is denoted by $V(+, \mathbb{R})$. The elements of F are scalars while the elements V are called <u>vectors</u>.

Among many vectors spaces we mention the following as examples:

<u>Example 1</u>. The set

$$R^3 = \{(x, y, z): \ x, y, z \varepsilon \mathbb{R}\}$$

of all ordered triples or vectors in 3-space is the vector space $R^3(+, \mathbb{R})$ which become an inner product space $R^3(+, ., R)$ when inner product (dot product) is defined.

<u>Example 2</u>. The set

$$C[a, b] = \{f: \ f \text{ is continuous on } [a, b]\}$$

is a vector space $C[a, b](+, \mathbb{R})$ which becomes an inner product space $C[a, b](+, ., R)$ if an inner product $<f, g>$ is defined as

$$<f, g> = \int_a^b f(t)\, g(t)dt$$

The functions f, g are said to be <u>orthogonal</u> on $[a, b]$ if $<f, g> = 0$, and

$$\|f\| = \sqrt{<f, f>} = \left(\int_a^b f^2(t)dt\right)^{1/2}$$

is called the <u>norm</u> of f on $[a, b]$.

<u>Example 3</u>. The set

$$M_{mxn} = \{[a_{ij}]_{mxn}: \ a_{ij} \varepsilon R\}$$

is a Vector space $M_{mxn}(+, \mathbb{R})$ where the inner product (dot product) is defined as

$$\begin{aligned}\langle A, B\rangle = A.B = a_{11}b_{11} + \dots + a_{1n}b_{1n} \\ + a_{21}b_{21} + \dots + a_{2n}b_{2n} \\ \vdots \\ + a_{n1}b_{n1} + \dots + a_{nn}b_{nn}\end{aligned}$$

and the norm of A, by $\|A\| = \sqrt{A.A}$

The vector space $R^3(+, \mathbb{R})$ has the natural generalization

$$R^n(+, ., R)$$

where

$$\mathbb{R}^n = \{(x_1, \dots, x_n): x_i \varepsilon R\}$$

is the set of all ordered n-tuples or vectors in n-space in which the operation of addition, multiplication by scalars and inner product are defined as:

$$(x_1, \dots, x_n)+(y_1, \dots, y_n)=(x_1+y_1, \dots, x_n+y_n)$$

$$\lambda(x_1, \dots, x_n)=(\lambda x_1, \dots, \lambda x_n)$$

$$(x_1, \dots, x_n).(y_1, \dots, y_n) = x_1y_1 + \dots + x_ny_n$$

The vectors

$$e_1=(1, 0, \dots, 0), e_2=(0, 1, 0, \dots 0), \dots, e_n=(0, \dots, 0,1)$$

in this space are unit vectors and pairwise orthogonal as seen by application inner product:

$$e_i.e_j = \begin{cases} 1 & \text{when } i=j \\ 0 & \text{when } i\neq j \end{cases}$$

They are said to lie on n (> 1) mutually orthogonal axes $0x_1, \dots, 0x_n$ sketch of which cannot be realized when $n>3$.

Any vector $P = (x_1, \ldots, x_n)$ can be written as a linear combination of $e_1, \ldots, e_n$. Indeed

$$(x_1, \ldots, x_n) = x_1e_1 + \ldots + x_ne_n$$

Linear dependence and independence in $\mathbb{R}^n$:

The k non-zero vectors $u_1, \ldots, u_k$ are called linearly dependent if there exist k scalars $c_1, \ldots, c_k$, not all zero, such that

$$c_1u_1 + \ldots + c_kn_k = 0,$$

otherwise $u_1, \ldots, u_k$ are called linearly independent. In other words $u_1, \ldots, u_k$ are linearly independent or linearly dependent according as a relation

$$c_1u_1 + \ldots + c_ku_k = 0$$

implies or does not imply $c_1=0, \ldots, c_k=0$.

Theorem. In the vector space $\mathbb{R}^n$

1. n vectors

$$\begin{aligned} u_1 &= (u_{11}, \ldots, u_{1n}) \\ u_2 &= (u_{21}, \ldots, u_{2n}) \\ &\vdots \\ u_n &= (u_{n1}, \ldots, u_{nn}) \end{aligned}$$

are linearly dependent or independent according as

$$\det[u_{ij}]$$

is zero or non zero,

2. There is a set of n linearly independent vectors,

3. Any $n+1$ vectors are linearly dependent.

Proof.

1. Setting

$$c_1u_1 + \dots + c_nu_n = 0, \quad (c_i \varepsilon R)$$

one has

$$c_1(u_n, \dots, u_{1n}) + \dots + c_n(u_{n1}, \dots, u_{nn}) = 0$$

or

$$(c_1u_{11} + \dots + c_nu_{n1}, \dots, c_1u_{1n} + \dots + c_nu_{nn}) = 0$$

implying the homogeneous square system

$$\begin{aligned} u_{11}c_1 + \dots + u_{n1}c_n &= 0 \\ \vdots \qquad\quad & \vdots \\ u_{1n}c_1 + \dots + u_{nn}c_n &= 0 \end{aligned}$$

of linear equations in the unknowns $c_1, \dots, c_n$ of which the determinant is $D = \det|u_{ij}|$.

If $D \neq 0$, the system admits only the trivial solution $c_1 = 0, \dots, c_n = 0$ meaning that the vectors are linearly independent.

If $D = 0$, the system admits solution other than the trivial one, meaning that not all c's are zero, and the vectors are linearly dependent.

2. $\mathbb{R}^n$ contains the unit vectors

$$e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$$

which are linearly independent since $\det|u_{ij}|$ is $|I_n| = 1 \neq 0$.

3. Let $u_1, \dots, u_n, u_{n+1}$ be non zero vectors in $\mathbb{R}^n$. The theorem is proved if n of them, say $u_1, \dots, u_n$ are linearly dependent, in which case $u_1, \dots, u_n, u_{n+1}$ are linearly dependent.

Let then $u_1, \dots, u_n$ be linearly independent. It will then suffice to prove that $u = u_{n+1}$ is a linear combination of $u_1, \dots, u_n$:

$$c_1u_1 + \dots + c_nu_n = u$$

This equality is equivalent to

$$\begin{matrix} c_1u_{11} + \dots + c_nu_{n1} = u_1 \\ \vdots \qquad\qquad \vdots \qquad \vdots \\ c_nu_{1n} + \dots + c_nu_{nn} = u_n \end{matrix}$$

with $\det|u_{ij}| \neq 0$. Hence there is a unique solution in c's, meaning that $n+1$ vectors $u_1, \dots, u_n, u$ are linearly dependent.

Example 1. Given the vectors $A = (1, 2, 3)$, $B = (4, 5, 6)$ $C = (-2, 3, 0)$ and $D = (0, 11, 6)$ in $\mathbb{R}^3$ show that A, B, C are linearly independent, and then express D as a linear combination of them.

Solution. Since

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -2 & 3 & 0 \end{vmatrix} = 24 \neq 0$$

A, B, C are linearly independent.

Writing

$$(0, 11, 6) = a(1, 2, 3) + b(4, 5, 6) + c(-2, 3, 0)$$

we have

$$\begin{aligned} a + 4b - 2c &= 0 \\ 2a + 5b + 3c &= 11 \\ 3a + 6b \qquad &= 6 \end{aligned}$$

solution of which being $a=0$, $b=1$, $c=2$, we have

$$D = 0A + B + 2C$$

Example 2. Given the vectors $A = (1, 9, 0, 0)$, $B = (9, -1, 0, 0)$, $C = (0, 0, 7, 9)$, $D = (0, 0, -9, 7)$ and $E = (1, 2, 3, 4)$ in $\mathbb{R}^4$,

a) show that A, B, C, D are mutually orthogonal,

b) Test A, B, C, D for linear independence, and if it is the case, express E in terms of A, B, C, D.

Solution.

a) $A.B = 0$, $A.C = 0$, $A.D = 0$, $B.C = 0$, $B.D = 0$, $C.D = 0$

b)
$$\begin{vmatrix} 1 & 9 & 0 & 0 \\ 9 & -1 & 0 & 0 \\ 0 & 0 & 7 & 9 \\ 0 & 0 & -9 & 7 \end{vmatrix} = -10660 \neq 0 \Rightarrow \text{linear independence.}$$

$$E = aA + bB + cC + dD$$

$\Rightarrow$ $a+9b = 1$, $9a-b = 2$, $7c-9d = 3$, $9c+7d = 4$

$\Rightarrow$ $a = 19/82$, $b = 7/82$, $c = 57/130$, $d = 1/130$.

Bases:

A set $\mathcal{B} = \{b_1, \ldots, b_k, \ldots\}$ of vectors in a vector space $V(+, F)$ is called a basis for $V(+, F)$ if

1) $b_1, b_2, \ldots$ are linearly independent
2) $b_1, b_2, \ldots$ generate (span) V, that is any vector in V is a linear combination of b's.

If b_i's are mutually orthogonal, then $\mathcal{B}$ is called an orthogonal system and if furthermore b_i's are unit vectors $\mathcal{B}$ is said to be an orthonormal system.

Clearly $\{i, j, k\}$ is a basis for $\mathbb{R}^3(+, \mathbb{R})$, and $\{e_1, \ldots, e_n\}$ is a basis for $\mathbb{R}^n(+, \mathbb{R})$, and each set is orthonormal.

For the Vector space $M_{2\times2}$, a basis is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Indeed,

1) these four matrices are linearly independent, since

$$a\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}+b\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}+c\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}+d\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}=0$$

$$\Rightarrow \begin{bmatrix}a & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix}0 & b\\ 0 & 0\end{bmatrix}+\begin{bmatrix}0 & 0\\ c & 0\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & d\end{bmatrix}=0$$

$$\Rightarrow \begin{bmatrix}a & b\\ c & d\end{bmatrix}=0 \Rightarrow a=0,\ b=0,\ c=0,\ d=0$$

2) Any matrix $\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ in M_{2x2} is a linear combination of the mentioned four matrices:

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}=a\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}+b\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}+c\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}+d\begin{vmatrix}0 & 0\\ 0 & 1\end{vmatrix}$$

Recall that in M_{2x2} the inner product of $A=\begin{bmatrix}a_1 & a_2\\ a_3 & a_4\end{bmatrix}$ $B=\begin{vmatrix}b_1 & b_2\\ b_3 & b_4\end{vmatrix}$ is

$$\langle A, B\rangle = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4$$

<u>Example 1</u>. Show the following:

a) Any non zero vector in $\mathbb{R}^1$ is a basis for $\mathbb{R}^1(+, \mathbb{R})$.

b) Any two non zero non parallel vectors in $\mathbb{R}^2$ is a basis for $\mathbb{R}^2(+, \mathbb{R})$

c) Any three non zero non coplanar vectors in $\mathbb{R}^3$ is a basis for $\mathbb{R}^3(+, \mathbb{R})$.

<u>Solution</u>.

a) Let $A = (a) \neq 0$ be any vector in $\mathbb{R}^1$ with $a \neq 0$. Since $\det|a| \neq 0$, A is linearly independent, and any vector $B = (b)$ is equal to $\frac{b}{a}A$.

b) Let $A = (a_1, a_2) \neq 0$, $B = (b_1, b_2) \neq 0$ be two non parallel $(\frac{a_1}{b_1} \neq \frac{a_2}{b_2})$ vectors. They are linearly independent, since $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2-a_2b_1 \neq 0$, and any vector $C = (c_1, c_2)$ can be written as a linear combination of A, B:

$$C = \alpha A + \beta B \iff \begin{matrix} c_1 = \alpha a_1 + \beta b_1 \\ c_2 = \alpha a_2 + \beta b_2 \end{matrix},$$

the latter admits a unique solition since $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$

c) Let $A = (a_1, a_2, a_3) \neq 0$, $B = (b_1, b_2, b_3) \neq 0$, $C = (c_1, c_2, c_3) \neq 0$ be non coplanar vectors in $\mathbb{R}^3$. Since A, B, C are non coplanar we have

$$(A\ B\ C) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \neq 0$$

showing their linear independence. Any vector D in $\mathbb{R}^3$ can be written as a linear combination of A, B, C:

$$D = \alpha A + \beta B + \gamma C$$

yielding a system of linear equations in α, β, γ solution of which exists since determinant of the system is not zero.

Linear dependence in $D^{n-1}[I]\ (+, \mathbb{R})$.

Let $f_1(x), \ldots, f_n(x) \in D^{n-1}[I]$, that is, let $f_1(x), \ldots, f_n(x)$ be n functions differentiable up to the order $n-1$ on an interval I.

These functions are said to be linearly dependent if there exists scalars $c_1, \ldots, c_n$ not all zero such that

$$c_1f_1(x) + \ldots + c_nf_n(x) \equiv 0,$$

otherwise they are linearly independent.

Differentiating it successively up to the order $n-1$ we get the homogeneous system

$$\begin{aligned} c_1 f_1 + \dots + c_n f_n &\equiv 0 \\ c_1 f_1' + \dots + c_n f_n' &\equiv 0 \\ \vdots \qquad\qquad & \vdots \\ c_1 f_1^{(n-1)} + \dots + c_n f_1^{(n-1)} &\equiv 0 \end{aligned}$$

of linear equations, implying

$$W = W[f_1, \dots, f_n] = \begin{vmatrix} f_1 & \dots & f_n \\ f_1' & \dots & f_n' \\ \\ f_1^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix} \begin{cases} \equiv 0 & \text{(non triv.sol.)} \\ \not\equiv 0 & \text{(only triv.sol.)} \end{cases}$$

where W is called the WRONSKIan of the given set of functions.

Hence a set of n functions $f_1, \dots, f_n$ is linearly independent or linearly dependent according as $W[f_1, \dots, f_n]$ is identically zero or non zero.

Example 1. Show that

a) two constant functions,

b) three linear polynomials,

c) four quadratic polynomials

are linearly dependent in any interval.

Solution.

a) Let $P(x) = a_0$, $Q(x) = b_0$ be two constant functions. Then forming the WRONSKIan, we have

$$W[P, R] = \begin{vmatrix} a_0 & b_0 \\ 0 & 0 \end{vmatrix} \equiv 0 \quad \Rightarrow \quad \text{linear dependence}$$

b) Let $P(x) = a_0+a_1x$, $Q(x) = b_0+b_1x$, $R(x) = c_0+c_1x$ be three linear polynomials. Then

$$W[P,\ Q,\ R] = \begin{vmatrix} a_0+a_1x & b_0+b_1x & c_0+c_1x \\ a_1 & b_1 & c_1 \\ 0 & 0 & 0 \end{vmatrix} \equiv 0 \Rightarrow$$

linear dependence.

c) Proof is similarly done.

Example 2. Test the following for linear dependence

a) e^x, e^{-x}, Ch x b) 5, x, x^2

c) sin x, cos x d) $\sin^2 x$, $\cos^2 x$

Solution.

a) $$W[e^x,\ e^{-x},\ \text{Ch } x] = \begin{vmatrix} e^x & e^{-x} & \text{Ch } x \\ e^x & -e^{-x} & \text{Sh } x \\ e^x & e^{-x} & \text{Ch } x \end{vmatrix} \equiv 0$$

(Compare the first and third rows).

They are linearly dependent.

b) $$W[5,\ x,\ x^2] = \begin{vmatrix} 5 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 10 \neq 0 \quad \text{(linear ind.)}$$

c) $$W[\sin^2 x,\ \cos^2 x] = \begin{vmatrix} \sin^2 x & \cos^2 x \\ 2\sin x\cos x & -2\cos x\ \sin x \end{vmatrix}$$

$$= 2\ \sin x\ \cos x \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin 2x \neq 0 \ \text{(linear ind.)}$$

EXERCISES (3, I)

1. Plot the following points on a positive rectangular coordinate system:

a) $A(2, 0, 1)$ b) $B(0, 3, 2)$ c) $C(2, 2, 0)$

d) $D(1, 1, 1)$ e) $E(1, 2, 1)$ f) $F(0, 0, 3)$

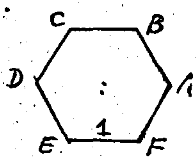
2. Examine positiveness (negativeness) of a cartesian system which is symmetric of a positive cartesian coordinate system with respect to:

   a) a point b) a line c) a plane

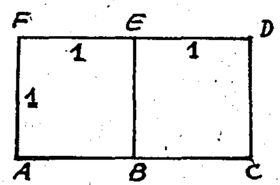
3. Find the symmetrics of the point $A(1, 3, 2)$ with respect to the

   a) origin b) x-axis c) y-axis d) z-axis

   e) yz-plane f) zx-plane g) xy-plane.



4. ABCDEF is a regular hexagon with unit side. write $\vec{AB} + \vec{AC} + \vec{AD} + \vec{AE} + \vec{AF}$ as a multiple of a vector and find its length.



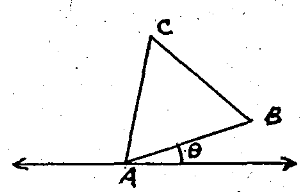
5. ABEF and BCDE are unit squares. Construct the following line vectors and compute their lengths:

   a) $\vec{FE}+\vec{FC}$ b) $\vec{FC}-\vec{FE}$ c) $2\vec{FE}+\vec{DC}$ d) $\vec{FC}-\vec{DB}+\vec{DC}$

6. By projecting the sides of an equilateral triangle ABC onto a line inclined at an angle θ to one of them, show that

$$\cos\theta = \cos(\theta + \frac{\pi}{3}) + \cos(\theta - \frac{\pi}{3}),$$

$$\sin\theta + \sin(\theta + \frac{2\pi}{3}) - \sin(\theta + \frac{4\pi}{3}) = 0$$



7. Construct two vectors whose sum and difference are the given vectors $\vec{u}$ and $\vec{v}$.

8. Theree forces of lengths 3, 5, 4 units are in the directions 50°, 120°, 215° from East. Find components of these forces in the direction of North and East. Then find the direction of the

resultant force.

9. Compute

a) $(3i + 2j - 4k).(3i - 2j + 7k)$ b) $(7, 8, 4).(5, -9, 4)$

10. Find $\lambda \varepsilon R$ for the vectors $i + 2j + 3k$ and $4i + 5i + \lambda k$ to be orthogonal.

11. Find $(A-B).(A-B)$

12. Given a vector from $A(2, 3, -4)$ to $B(-4, 5, 6)$, find the magnitude and direction cosines of the vector.

13. Expand and simplify:

$$u = (2\vec{a} - 3\vec{b}).(\vec{a} + 2\vec{b})$$

where $|\vec{a}| = 3$, $|\vec{b}| = 2$, $\measuredangle \vec{a}, \vec{b} = \pi/3$

14. Evaluate cosines of the angles α, β, γ of the triangle having vertices at $A(1, 0, 0)$, $B(0, 2, 0)$, $C(0, 0, 3)$

15. Show: $(\vec{a} + \vec{b})^2 = \vec{a}^{\,2} + \vec{b}^{\,2} - 2\,\vec{a}.\vec{b}$

16. Given $A = (1, 2, 2)$, $B = (3, -1, 1)$, find

a) $A.B$ b) $|A|$ c) vector component of B in the direction of A.

17. a) Given $A = [3, -12, 15] + t[-5, 20, -25]$ find $t \varepsilon R$ such that $A.A$ is a minimum,

b) Find s, t not both zero such that

$$s[3, -12, 15] + t[-5, 20, -25] = 0$$

18. Given $A = (1, 9, 7)$, $B = (9, 4, 2)$, find t for $|A + tB|$ to be a minimum.

19. a) Given $A = (1, 2, 0) - s(4, 5, 2) - t(1, -1, 2)$, find $s, t \varepsilon R$ such that $A.A$ is a minimum,

b) find $a, b, c \in R$ not all zero such that

$$a(1, 2, 3) + b(4, 5, 2) + c(1, -1, 2) = 0$$

20. Given $A = (-1, 2, -2)$, $B = (1, -4, 8)$, find unit vectors in the direction of

a) A b) B c) A-B

21. Place three vectors of lengths 7, 24, 25 units to have a zero resultant.

22. Three mutually orthogonal vectors $\vec{a}, \vec{b}, \vec{c}$ have lengths $\ell, \ell, \sqrt{\ell}$ respectively.

a) find $|\vec{a} + \vec{b} + \vec{c}|$

b) find angles between $\vec{a} + \vec{b} + \vec{c}$ and $\vec{a}, \vec{b}, \vec{c}$

23. A certain force $3i - 4j - 2k$ kg pushes an object in the direction of the vector $A = 2i - 3j + 5k$ by a distance equal to $|A|$

a) find the work done

b) find magnitude of the force and displacement

c) what is the angle between two vectors?

24. A vector makes an angle of 60^0 with the x-axis, 60^0 with y-axis. Then find the angles that it makes with z-axis.

25. Let $A = (2, 4, 4)$, $B = (3, 3, 0)$, $C = (1, 5, -1)$ and $G = B-Ar$, $H = C-As-Bt$ where $r, s, t \in R$ are such that $A.G = 0$, $A.H = 0$, $B.H = 0$, Find r, s, t.

26. a) show that in a parallelogram the sum of the squares of the sides is equal to the sum of the squares of diagonals.

b) show that in a triangle, the sum of the squares of the sides is equal to 3/4 of the sum of the squares of the medians

27. Expand and simplify: $(2A - 3B)x(A + 2B)$

28. Find $(A-B)x(A+B)$ and interpret its absolute value geometrically.

29. Show: $A+B+C = 0 \Rightarrow BxC = CxA = AxB$

30. Find AxB where $A = i+2j-3k$, $B = 4i-5j-6k$ and find $\sin\theta$ for these vectors.

31. Find unit vectors perpendicular to both $i+j$ and $j+k$.

32. For what value of λ will the vector from $A(2, 1, 4)$ to $B(3, -1, 2)$ be parallel to the vector from $C(4, -2, 1)$ to $D(2, 2, \lambda)$?

33. Let $A(0, -2, 1)$, $B(1, 0, 3)$ and $C(3, 1, 1)$ be the vertices of a triangle. Find

a) $\vec{AB} \times \vec{AC}$ b) $|ABC|_2$

34. Given $A = (2, -1, 3)$ and $B = (-3, 0, 2)$, find a unit vector which is orthogonal to each of the vectors A and B. Is there more than one solution?

35. Let $A = (2, -14, 8)$, $B = (-3, 21, -12)$, $C = (0, 1, -2)$, $D = (1, -1, 1)$.

a) compute AxB

b) find a, b not both zero such that $aA + bB = 0$

c) find a, b, c not all zero, such that $aA + bB + cC = 0$

d) find a, b, c, d not all zero such that $aA+bB+cC+dD = 0$

36. Let $A = (3, 2, -1)$, $B = (1, -3, 5)$, $C = (4, -1, 1)$

a) find AxB b) $AxB.C$

c) find the volume of the parallelepiped, three of whose edges are $[OA]$, $[OB]$, $[OC]$

37. Given $A = (2, -3, 1)$, $B = (1, 2, -1)$ find a vector X, if any, as the solution of the equation:

a) $A.X = 5$ b) $A_\Lambda X = B$ c) $(A_\Lambda X).B = -2$

38. Prove that the triangle with vertices $P(-1, 4, 3)$, $Q(2, 0, 3)$ and $R(-1, 4, 5)$ is a right triangle, and the area $|PQR|_2 = 5$

39. Let $\overline{ABC}$ denote the oriented area of a triangle ABC in xy-plane that is, $\overline{ABC}$ is positive or negative according as A, B, C are in counterclockwise or clockwise order. Then if $A(a_1, a_2)$, $B(b_1, b_2)$, $C(c_1, c_2)$, prove

$$\overline{ABC} = \frac{1}{2} \begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix}$$

40. Prove:

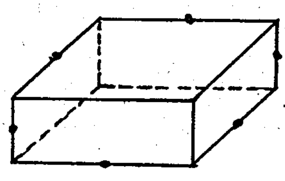
a) $|ABC|_2 = \frac{1}{2} |\vec{OB} \times \vec{OC} + \vec{OC} \times \vec{OB} + \vec{OA} \times \vec{OB}|$

b) $|ABC|_2 = \frac{1}{6} (|\vec{AB} \times \vec{AC}| + |\vec{BC} \times \vec{BB}| + |\vec{CB} \times \vec{CB}|)$.

41. Prove the identity

$$|A.B|^2 + |AxB|^2 = |A|^2|B|^2 \quad \text{(LAGRANGE)}$$

42. Show that the indicated six points as midpoints of the edges in a right parallelepiped are the vertices of a plane hexagon.



43. Evaluate $(AxB)xC$ and $Ax(BxC)$ where $A = i+j-k$, $B = 2i-j+k$, $C = i+2j-k$ and compare the results. Does there exist the associative law for cross multiplication?

44. Prove: $Ax(BxC) + Bx(CxA) + Cx(AxB) = 0$

45. Determine scalar $\lambda \in \mathbb{R}$ such that

$$(Ax(AxB)\ \ A\ C) = \lambda(ABC)$$

46. Prove

a) $AxB.(AxC)xD = (A.D)(ABC)$

b) $AxB\ .\ Ax(CxD) = (A.D)(ABC) - (A.C)(ABD)$

47. Expand $[(r_1 \times r_2) \times r_3]xr_4$

48. Prove

a) $(ABC)\ UxV = \begin{vmatrix} A.U & A.V & A \\ B.U & B.V & B \\ C.U & C.V & C \end{vmatrix}$

b) $(ABC)(UVW) = \begin{vmatrix} A.U & A.V & A.W \\ B.U & B.V & B.W \\ C.U & C.V & C.W \end{vmatrix}.$

49. If A, B, C are non collinear vectors, prove

$$P = \alpha A + \beta B + \gamma C \implies \alpha = \frac{(BCP)}{(ABC)},\ \beta = \frac{(CAP)}{(ABC)},\ \gamma = \frac{(ABP)}{(ABC)}$$

50. For $(ABC) \neq 0$, $(a\ b\ c) \neq 0$, prove that

if $a = \frac{BxC}{(ABC)}$, $b = \frac{CxA}{(ABC)}$, $c = \frac{AxB}{(ABC)}$,

then $A = \frac{bxc}{(abc)}$, $B = \frac{cxa}{(abc)}$, $C = \frac{axb}{(abc)}$

51. Prove that the following sets are vector spaces for the given operation, over the given field:

a) $\mathbb{R}$; +, $\mathbb{R}$ b) R^*; ., $\mathbb{R}$ c) $\mathbb{C}$; +, R d) C^*; ., R

52. Same question for:

a) M_{2x2}; +, $\mathbb{R}$ b) $C[a, b]$; +, R

c) $M'_{2x2} = \left\{ \begin{vmatrix} a & b \\ c & d \end{vmatrix} : \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0 \right\}$; ., R

53. Given $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 \\ 5 & 6 \end{bmatrix}$ in $M_{2x2}(+, ., \mathbb{R})$

a) find $<A, B>$ b) find $||A||$, $||B||$

c) evaluate $\cos\theta = \dfrac{<A \;.\; B>}{||A|| \quad ||B||}$ for the matrices A, B.

54. Same question for

$$C = \begin{bmatrix} 1 & 2 \\ 8 & 4 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 2 \\ -4 & 10 \end{bmatrix}$$

55. Test $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ for a basis for $M_{2x2}(+, \mathbb{R})$, and if it is a basis express $\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}$ as a linear combination of basis vectors.

56. Show that

a) $\{1, i\}$ b) $\{1-i, 1+i\}$ $(i^2=-1)$

are bases for the vector space $\mathbb{C}(+, \mathbb{R})$, and express $z = 5-3i$ in terms of the elements in each basis.

57. Show that the three vectors A, $A+B$, $A+B+C$ are linearly independent if A, B, C are so.

58. Find a linear combination of $[0 \;\; 1 \;\; 1]^T$ and $[1 \;\; 1 \;\; 0]^T$ which is orthogonal to $[0 \;\; 1 \;\; 1]^T$.

59. Given $V_1 = \frac{1}{2} i - \frac{\sqrt{3}}{2} j$, $V_2 = -\frac{\sqrt{3}}{2} i - \frac{1}{2} j$ in $\mathbb{R}^2$ show that V_1, V_2 is an orthonormal system.

60. Show that there exist scalars a, b and c not all zero such that

$$a \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + b \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix} + c \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

61. Given

$$A = \frac{1}{\sqrt{3}} i + \frac{1}{\sqrt{3}} j + \frac{1}{\sqrt{3}} k, \quad B = \frac{1}{\sqrt{2}} i - \frac{1}{\sqrt{2}} j, \; C = \frac{1}{\sqrt{6}} i + \frac{1}{\sqrt{6}} j - \frac{2}{\sqrt{6}} k$$

show that A, B, C is an orthonormal system.

62. Let $A = (a, 1, 2)$, $B - (1, b, 2)$ and $C = (1, 2, c)$. What must be relation between a, b, c for A, B, C to be linearly dependent,

63. Prove with out any computation that $[1 \;\; 2 \;\; 2]^T$, $[0 \;\; 1 \;\; 2]^T$, $[0 \;\; 0 \;\; 3]^T$ constitute a basis for $\mathbb{R}^3$. Then

a) express $[3 \;\; 6 \;\; 9]^T$ as a linear combination of the vectors of this basis,

b) which of the vectors of this basis may be replaced by $[3 \;\; 6 \;\; 5]^T$ and the resulting set still being a basis?

64. Given A(0, 1, 1), B(1, -1, 1) and C(11, 2, 0), find a linear combination of A and B that is orthogonal to C.

65. Let $A = (3/5, 4/5, 0, 0)$, $B = (-4/5, 3/5, 0, 0)$.
$C = (0, 0, 1/\sqrt{2}, 1/\sqrt{2})$, $D = (0, 0, -1/\sqrt{2}, 1/\sqrt{2})$.
Show that A, B, C, D is an orthonormal system.

66. Test for linear dependence, and determine the dimension of the space generated (spaned) by these vectors:

a) (1, 3, 5), (1, 2, -1), (4, 6, 3)

b) (1, -1, 2), (3, 4, -2), (4, 3, 0), (1, 6, -6)

c) (1, 0, 4, 3), (2, 1, 8, -6), (3, 1, 4, 8)

d) (2, 0, 3, -1), (4, 1, 1, 2), (0, 1, -5, 4)

67. The distance between the points (a, 2a, -a, 3, 1) and (3a, 2a, 2a, 1, 4) in $\mathbb{R}^5$ is $\sqrt{26}$. What are the possible values for a?

68. The vector $[a \quad 2a \quad 2a \quad 4]^T$ and $[4 \quad 2a \quad 2a \quad a]^T$ in $\mathbb{R}^4$ are orthogonal. What are the possible values of a?

69. Test the functions for linear independence:

a) $\sin x, \sin 2x$ b) $1, \cos x, \cos 2x$

c) $1, x, \ldots, x^n$ d) $1, e^x, e^{2x}, \ldots, e^{nx}$

70. Given $\sin x, \sin 2x$ in $C[0, 2\pi](+, \mathbb{R})$

a) test them for orthogonality on $[0, 2\pi]$

b) find their norms

ANSWERS TO EVEN NUMBERED EXERCISES

2. a) negative b) positive c) negative

4. $3\vec{AD}$, 6

8. $3\cos 40^0$, $5\cos 30^0$, $-4\cos 55^0$; $3\cos 50^0$, $5\cos 120^0$, $4\cos 215^0$;

$$\tan\theta = \frac{3\cos 40^0 + 5\cos 30^0 - 4\cos 55^0}{3\cos 50^0 + 5\cos 120^0 + 4\cos 215^0}$$

10. $-14/3$

12. $2\sqrt{35}$; $-3/\sqrt{35}$, $1/\sqrt{35}$, $5/\sqrt{35}$

14. $\cos\alpha = \frac{1}{5\sqrt{2}}$, $\cos\beta = \frac{4}{\sqrt{65}}$, $\cos\gamma = \frac{9}{\sqrt{130}}$

16. a) 3, b) 3, c) $(1/3, 2/3, 2/3)$

18. $-59/101$

20. a) $\pm(1, -2, 2)/3$, b) $\pm(1, -4, 8)/9$, c) $\pm(1, -3, 5)/\sqrt{35}$

22. a) $\sqrt{2\ell^2 + \ell}$;

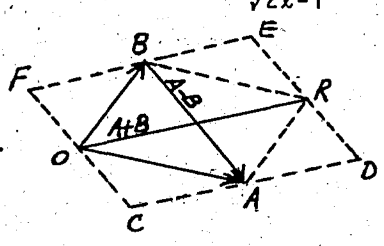
b) α, β, γ : $\alpha = \beta = \arccos \frac{\ell}{\sqrt{2\ell^2 - \ell}}$, $\gamma = \arccos \frac{1}{\sqrt{2\ell - 1}}$

24. $\pi/4$, $3\pi/4$

28. $2\, A \times B$, $|CDEF|_2 = 2|OARB|_2$

30. $(-27, -6, -13)$, $\sin\theta = \sqrt{467}/(7\sqrt{11})$.

32. 5



34. $\vec{n} = (2, 13, 3)/\sqrt{182}$, yes, $-\vec{n}$

36. a) (7, -16, -11), b) 33, c) 33

54. a) 13, b) $\sqrt{85}$, 11, c) $13/(11\sqrt{85})$

58. $a[0, 1, 1]^T - 2a[1, 1, 0]^T$ for $a \in R$

62. $4a + 2b + c - abc = 6$

64. $9a(0, 1, 1) - 2a(1, -1, 1)$

66. a) ind., 3, b) dep., 2, c) ind., 3, d) dep., 2

68. 0, -1

70. orthogonal, $\sqrt{\pi}$, $\sqrt{\pi}$

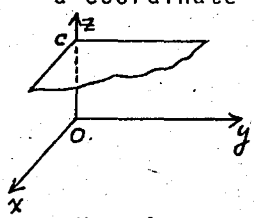
3. 2. PLANES AND LINES

A. PLANES

Planes will be denoted by small Greek letters π, α, β, ...

Equations of some special planes:

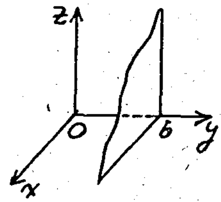
a) Plane parallel to a coordinate plane (perpendicular to a coordinate axis):



$\pi // xy$-plane

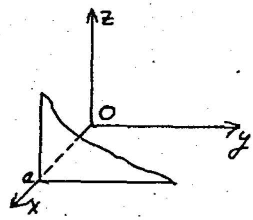
(Horizontal plane)

$z = c$



$\pi // xz$-plane

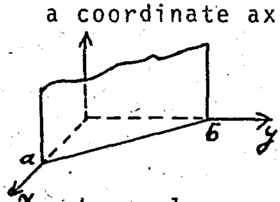
$y = b$



$\pi // yz$-plane

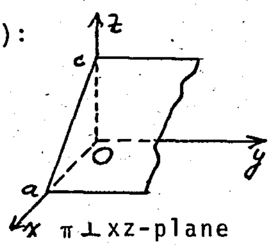
$x = a$

b) Plane perpendicular to a coordinate plane (parallel to a coordinate axis):



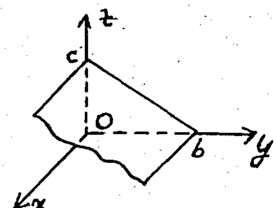
$\pi \perp xy$-plane

$$\frac{x}{a} + \frac{y}{b} = 1$$



$\pi \perp xz$-plane

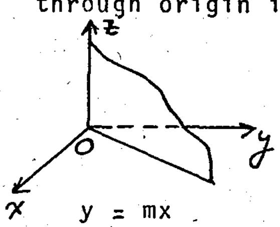
$$\frac{x}{a} + \frac{z}{c} = 1$$



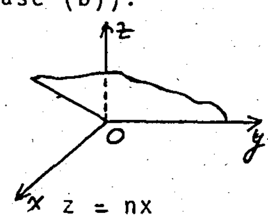
$\pi \perp yz$-plane

$$\frac{y}{b} + \frac{z}{c} = 1$$

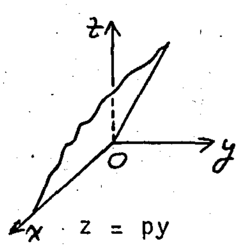
c) Plane passing through a coordinate axis (passing also through origin in case (b)):



$y = mx$

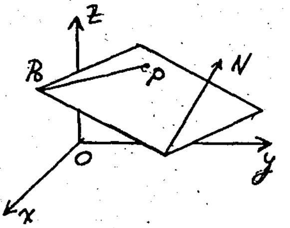


$z = nx$



$z = py$

<u>Equation of the plane through a point and perpendicular to a vector</u>:



Let $P_0(x_0, y_0, z_0)$ be the given point and $\vec{N} = (A, B, C)$ be the vector. Let $P(x, y, z)$ be any point of the plane π through P_0 and perpendicular to $\vec{N}$. Then

$$\vec{N} \perp \pi \Rightarrow \vec{N} \perp \vec{P_0P} \Rightarrow \vec{N}.\vec{P_0P} = 0 \qquad \vec{N}.(P-P_0) = 0 \left\{\underline{\text{vector equation}}\right.$$

Then the required equation is

$$\pi: A(x-x_0) + B(y-y_0) + C(z-z_0) = 0 \qquad (1)$$

or

$$\pi: Ax + By + Cz + D = 0 \text{ (\underline{general equation})} \qquad (1')$$

where $D = -Ax_0 - By_0 - Cz_0$.

The vector $\vec{N}$ is called a <u>normal vector</u> or a <u>direction</u> vector, and its components A, B, C are direction numbers of the plane π.

Observe that direction numbers A, B, C appear as coefficients in (1) or in (1'), and hence the planes

$$Ax+By+Cz+D = 0 \quad \text{and} \quad A'x+B'y+C'z+D' = 0$$

are parallel if and only if

$$\frac{A}{A'} = \frac{B}{B'} = \frac{C}{C'} \qquad (\vec{N}//\vec{N'})$$

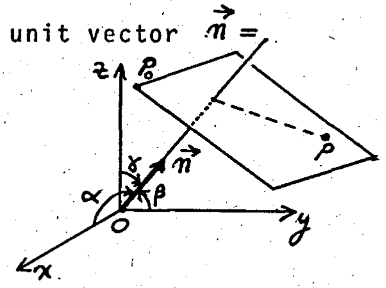
Similarly

$$AA' + BB' + CC' = 0 \Leftrightarrow \pi \perp \pi'$$

<u>Remark</u>. The plane represented by the general equation $Ax + By + Cz + D = 0$ is parallel (perpendicular) to xy-plane when $A=0$, $B=0$ $(C\neq 0)$. Similar results hold when parallel (perpendicular) to other coordinate planes.

Normal equation.

If the direction vector $\vec{N}$ of π is a unit vector $\vec{n} =$ $(\cos\alpha, \cos\beta, \cos\gamma)$, we have



$\vec{n}.(P-P_0) = 0 \Rightarrow \vec{n}.\vec{P}-\vec{n}.\vec{P}_0 = 0 \Rightarrow$

$\vec{n}.\vec{OP} = \vec{n}.\vec{OP}_0 \Rightarrow \vec{n}.\vec{OP} = \vec{n}.\vec{OH}$

where $\vec{OH}$ is the projection of $\vec{OP}$ on $\vec{n}$.

Then, if $p = |\vec{OH}|$ or $\vec{OH} = p(\cos\alpha, \cos\beta, \cos\gamma)$ we get

$$\vec{n}.\vec{OP} = p$$

or

$$\pi: x\cos\alpha + y\cos\beta + z\cos\gamma - p = 0 \qquad (2)$$

which is called the normal equation since $\cos^2\alpha+\cos^2\beta+\cos^2 = 1$. p is the distance of the origin from the plane. (2) is also called the EULER's equation or HESSIan form of π.

The equation (2) can be written in the form

$$\pi: ax + by + cz + d = 0,\ a^2 + b^2 + c^2 = 1 \qquad (2')$$

which is also a normal equation.

The general equation (1') can be normalized by dividing its every term by $\sqrt{A^2 + B^2 + C^2}$ $(\neq 0)$:

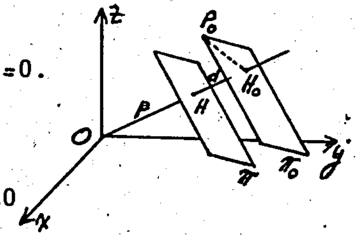
$$\pi: \frac{Ax + By + Cz + D}{\sqrt{A^2 + B^2 + C^2}} = 0 \qquad (2'')$$

so that (2), (2'), (2") are all normal equations.

The interesting fact about normal equation is the simplicity of the expression of the distance of a point $P_0(x_0, y_0, z_0)$ from the plane:

$$d(P_0, \pi) = \begin{cases} |x_0\cos\alpha + y_0\cos\beta + z_0\cos\gamma - p| \\ \\ \dfrac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}} \end{cases} \qquad (3)$$

Indeed, let π_0 be the plane $//\pi$, passing through $P_0(x_0, y_0, z_0)$:



$$\pi_0:\ x\cos\alpha + y\cos\beta + z\cos\ \ - (p\pm d)=0.$$

Since $P_0 \epsilon \pi_0$, then

$$x_0\cos\alpha + y_0\cos\beta + z_0\cos\gamma - (p\pm d)=0$$

$$\Rightarrow\quad \pm d = x_0\cos\alpha + y_0\cos\beta + z_0\cos\gamma - p$$

$$\Rightarrow\quad d = |x_0\cos\alpha + y_0\cos\beta + z_0\cos r - p|.$$

Example 1. Given the point $A(3, -2, -1)$ and $\vec{N} = (1, -8, 4)$,

a) Write the equation of the plane π through A and perpendicular to $\vec{N}$,

b) Obtain the normal equation of π,

c) Find the distance of $B(2, 2, -5)$ from π.

Solution.

a) $(x-3) - 8(y+2) + 4(z+1) = 0$

$\Rightarrow\quad x - 8y + 4z - 15 = 0$

b) Since $\sqrt{1^2 + (-8)^2 + 4^2} = 9$, we have

$$\frac{x - 8y + 4z - 15}{9} = 0$$

c) $d(B, \pi) = \dfrac{|2 - 8.2 + 4(-5) - 15|}{9} = 49/9$

Example 2. Given the planes

$\pi:\ 2x - 2y + z - 3 = 0$ and $\pi':\ 4x - 4y + 2z + 7 = 0$

a) show that $\pi // \pi'$

b) find the distance $d(\pi, \pi')$

Solution.

a) $\frac{2}{4} = \frac{-2}{-4} = \frac{1}{2} \Rightarrow \pi // \pi'$

b) The distance between these parallel planes is the distance of a point on one from the other:

$$A(0, 0, 3)\epsilon\pi, \quad \text{then} \quad d(\pi, \pi') = d(A, \pi')$$

$$= \frac{|4.0 - 4.0 + 2.3 + 7|}{\sqrt{36}} = \frac{13}{6}.$$

Observe that $d = \frac{7}{6} - \frac{-3}{3}$ (difference of distances of the origin from the planes)

<u>Example 3</u>. Given the planes

π: $2x - 2y + z + 3 = 0$ and π': $4x - 4y + az + 7 = 0$

a) find $a\epsilon R$ such that $\pi \perp \pi'$,

b) write a unit vector parallel to π.

<u>Solution</u>.

a) $2.4 + (-2)(-4) + 1.a = 0 \Rightarrow a = -16$

b) $\pi \perp \pi' \Rightarrow N' // \pi \Rightarrow \vec{n} = \frac{N'}{|N'|} // \pi$

$$\vec{n} = \frac{(4, -4, -16)}{4\sqrt{18}} = \frac{(1, -1, -4)}{3\sqrt{2}}$$

Any unit vector $\vec{U} = (\cos\alpha, \cos\beta, \cos\gamma)$ is perpenducular to $\vec{N}$ if $\vec{N}.\vec{U} = 0$ implying $2\cos\alpha - 2\cos\beta + \cos\gamma = 0$ with infinitely many solutions.

<u>Equations of a plane through three points</u>:

Let $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$ determine a plane π. The unknown equation

$$Ax + By + Cz + D = 0$$

and the three conditions

$$Ax_1 + By_1 + Cz_1 + D = 0$$
$$Ax_2 + By_2 + Cz_2 + D = 0$$
$$Ax_3 + By_3 + Cz_3 + D = 0$$

determine a HLS which admits a non trivial solution in A, B, C, D if

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0 \qquad (4)$$

which is the <u>determinantal equation</u> of the plane π.

When expanded gives an equation in the form (1').

<u>Intercept form:</u>

This is the equation of the plane determined by three points $A(a, 0, 0)$, $B(0, b, 0)$, $C(0, 0, c)$ where a, b, c are intercepts of the plane with coordinate axes.

The equation is simply

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad (5)$$

since it is satisfied by the coordinates of A, B, C.

(5) can be obtained from (4) directly.

Compare (5) with intercept form of a line is 2-space.

<u>Equation of a the plane through a given point and parallel to two non collinear vectors:</u>

Let the given point and vectors be $P_0(x_0, y_0, z_0)$, $\vec{U} = (u_1, u_2, u_3)$, $\vec{V} = (v_1, v_2, v_3)$.

If $P(x, y, z)$ is any point on the plane π, then $\vec{P_0P}$ will be a linear combination of $\vec{U}$ and $\vec{V}$:

$$\vec{P_0P} = s\vec{U} + t\vec{V}$$

or

$$P = P_0 + sU + tV \qquad (6)$$

which is a (parametric) <u>vectoral equation</u> of π.

The matrix form of (6) is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = s \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + t \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \qquad (6')$$

which is equivalent to

$$\begin{aligned} x &= x_0 + su_1 + tv_1 \\ y &= y_0 + su_2 + tv_2 \\ z &= z_0 + su_3 + tv_3 \end{aligned} \qquad (6'')$$

called the <u>parametric equations</u> of π

<u>Linear family of planes (pencil of planes)</u>:

The set of planes through a given line ℓ is called a <u>linear family of planes</u> or a <u>pencil of planes</u> of which ℓ is the <u>axis</u>.

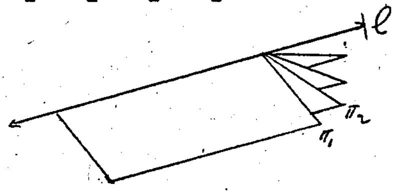
If

$$A_1x + B_1y + C_1z + D_1 = 0, \qquad A_2x + B_2y + C_2z + D_2 = 0$$

are equations of two distinct planes of the pencil, then the family is represented by the equation

$$\pi(\lambda_1, \lambda_2): \ \lambda_1(A_1x+B_1y+C_1z+D_1) + \lambda_2(A_2x+B_2y+C_2z+D_2)=0 \quad (7)$$

where $\lambda_1, \ \lambda_2 \varepsilon R$.



• Observe that if $\pi_1 // \pi_2$, the family consists of parallel planes (with no axis).

<u>Example 4</u>. Given the points $A(1, 1, 1)$, $B(2, 0, -1)$, $C(-1, 2, 0)$ and $D(0, 1, 2)$, find the equation of the plane

a) ABC in the general form and then get its intercept and parametric form,

b) through D and parallel to ABC,

c) through BC and perpendicular to ABC.

Solution.

a) $$\begin{vmatrix} x & y & z & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 0 & -1 & 1 \\ -1 & 2 & 0 & 1 \end{vmatrix} = 0 \Rightarrow 3x + 5y - z = 7 \quad \text{(general form)}$$

$$\Rightarrow \frac{x}{7/3} + \frac{y}{7/5} + \frac{z}{-7} = 1 \quad \text{(intercept form)}$$

Taking two of the coordinates, say x, y as parameters, one gets

$$x = s, \quad y = t, \quad z = 3s + 5t - 7 \quad \text{(parametric form)}$$

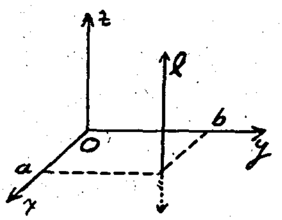
b) $3(x-0) + 5(y-1) - (z-2) = 0 \Rightarrow 3x + 5y - z = 3$

c) Let $\pi: Ax+By+Cz+D=0$.
Then ~~#~~ we have
$\pi \perp ABC \Rightarrow 3A+5B-C=0$
$B \in \pi \Rightarrow 2A-C+D=0$
$C \in \pi \Rightarrow -A+2B+D=0$.
Choosing $D=1$, we get
$\pi: \quad x+3z+1=0$

Lines (straight lines) will be denoted by small letters ℓ, a, b, d, ...

Equations of some special lines:

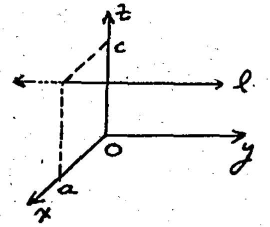
a) Line perpendicular to a coordinate plane (parallel to a coordinate axis):



$\ell \perp$ xy-plane

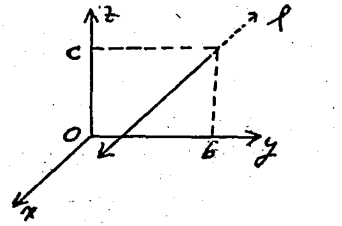
(vertical line)

$x = a,\ y = b$



$\ell \perp$ xz-plane

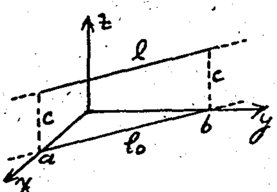
$x = a,\ z = c$



$\ell \perp$ yz-plane

$y = b,\ z = c$

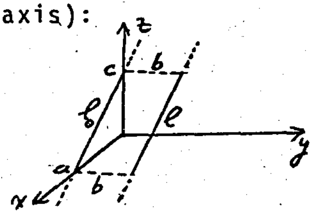
b) line parallel to a coordinate plane (perpendicular to a coordinate axis):



ℓ // xy-plane

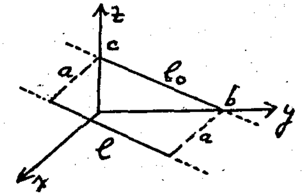
$\ell_0 : \frac{x}{a} + \frac{y}{b} = 1, \quad z = 0$

$\ell : \frac{x}{a} + \frac{y}{b} = 1, \quad z = c$



ℓ // xz-plane

$\ell: \quad \frac{x}{a} + \frac{z}{c} = 1, \ y = b$

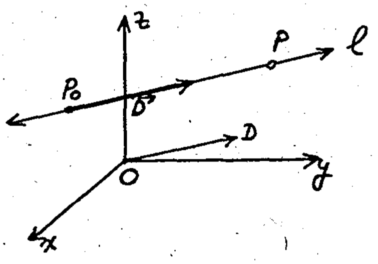


ℓ // yz-plane

$\ell: \quad \frac{y}{b} + \frac{z}{c} = 1, \ x = a$

Observe that each line is represented by a set of two equations.

<u>Equation(s) of the line passing through a point and parallel to a vector:</u>



Let $P_0(x_0, y_0, z_0)$ be a given point and $\vec{D} = (a, b, c)$ a vector. If $P(x, y, z)$ is any point of the line ℓ through P_0 and parallel to $\vec{D}$, one has

$$\vec{P_0P} \parallel \vec{OD} \Rightarrow \vec{P_0P} = t\vec{D}$$

$$\Rightarrow \quad P - P_0 = t\vec{D} .$$

$$\ell: \quad P = P_0 + tD \qquad (1)$$

called the (parametric) vectoral equation of ℓ, the matrix form of which is

$$\ell: \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad (1')$$

The equality (1′) is equivalent to

$$\ell: \quad \begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \\ z &= z_0 + tc \end{aligned} \qquad (1'')$$

which are called the parametric cartesian equations of ℓ.

Eliminating the parameter t from (1") one gets the equations

$$\ell: \quad \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} \qquad (2)$$

called the symmetric equations of ℓ.

The vector $\vec{D} = (a, b, c)$ is the direction vector and a, b, c the direction numbers of ℓ.

Observe that the symmetric equations (2) of a line ℓ are equivalent to two simultaneous linear equations, namely

$$\frac{x-x_0}{a} = \frac{y-y_0}{b}, \quad \frac{x-x_0}{a} = \frac{z-z_0}{c}$$

which are the equations of two planes containing the line ℓ. Then any two linear equations

$$Ax+By+Cz+D = 0, \quad A'x+B'y+C'z+D' = 0 \qquad (3)$$

of two intersecting planes represent their line of intersection. Consequently any line in 3-space can be represented by two simultaneous linear equations (2) in x, y, z (non parametric case).

<u>Example 1</u>. Given the points $A(1, 2, -3)$ and $B(4, -1, 2)$, obtain the

a) symmetric equations,

b) vectoral equation

c) parametric cartesian equations

d) simultaneous equations

of the line AB.

<u>Solution</u>. Since the line AB passes through the point $A(1, 2, -3)$ and admitting $\vec{AB} = B - A = (3, -3, 5)$ as a direction vector, we have

a) $\frac{x-1}{3} = \frac{y-2}{-3} = \frac{z+3}{5}$ $(= t)$

Taking B instead of A as a point on ℓ, we also have

$$\frac{x-4}{3} = \frac{y+1}{-3} = \frac{z-2}{5}$$

as symmetric equations.

b) $P = A + t\,\vec{AB}$ or $P = B + s\,\vec{AB}$.

c) Solving from (a) x, y, z in terms of t, we have

$$x = 1+3t, \quad y = 2-3t, \quad z = -3+5t$$

d) From (a) we have

$$-3(x-1) = 3(y-2), \qquad 5(y-2) = -3(z+3)$$

or

$$3x+3y-9 = 0, \qquad 5y + 3z - 1 = 0$$

<u>Remark</u>. Observe that in the symmetric equations (2) of a line, the coefficients of x, y, z on the numerators are all 1. If this is not the case, divide every term in a each fraction by the corresponding coefficient. For instance,

$$\frac{2x-1}{1} = \frac{y+2}{-3} = \frac{-3z-1}{2} \Rightarrow \frac{x-1/2}{1/2} = \frac{y+2}{-3} = \frac{z-1/3}{-2/3}$$

Example 2. Write the equation of the linear family of planes passing through the line

$$\ell: \quad \frac{x-1}{3} = \frac{y+2}{2} = \frac{z-3}{1}$$

Then find the member of the family

a) passing through the point $A(0, 2, 1)$

b) perpendicular to the plane $\pi: 2x-y = 3$

c) parallel to the line $\ell: \frac{x}{1} = \frac{y}{0} = \frac{z}{3}$

Solution.

$$\frac{x-1}{3} = \frac{y+2}{2} \Rightarrow 2x - 3y - 8 = 0$$

$$\frac{x-1}{3} = \frac{z-3}{1} \Rightarrow x - 3z + 8 = 0$$

$$\lambda(2x - 3y - 8) + \mu(x - 3z + 8) = 0$$

a) $\lambda(0 - 6 - 8) + \mu(0 - 3 + 8) = 0 \Rightarrow -14\lambda + 5\mu = 0$

Taking the values $\lambda = 5$, $\mu = 14$ we get

$$5(2x - 3y - 8) + 14(x - 3z + 8) = 0$$

$$\Rightarrow 24x - 15y - 42z + 72 = 0$$

$$\Rightarrow 8x - 5y - 14z + 24 = 0$$

b) Dot product of normal vectors $(2, -1, 0)$ and $(2\lambda+\mu, -3\lambda, -3\mu)$ of the planes must vanish:

$$2(2\lambda+\mu) + 3\lambda + 0 = 0 \Rightarrow 7\lambda + 2\mu = 0$$

Taking the values $\lambda = 2$, $\mu = -7$, we get

$$2(2x - 3y - 8) - 7(x - 3z + 8) = 0$$

$$\Rightarrow -3x - 6y - 21z - 72 = 0$$

$$\Rightarrow x + 2y - 7z + 24 = 0$$

c) Dot product of direction vectors $(1, 0, 3)$ and $(2\lambda+\mu, -3\lambda, -3\mu)$ of the line and the member must vanish:

$$2\lambda + \cancel{\mu} - 9\mu = 0 \Rightarrow \lambda - 4\mu = 0$$

Taking $\lambda = 4$, $\mu = 1$ we have

$$4(2x - 3y - 8) + (x - 3z + 8) = 0$$

$$9x - 12y - 3z - 24 = 0$$

C. INTERSECTION, ANGLE, DISTANCE

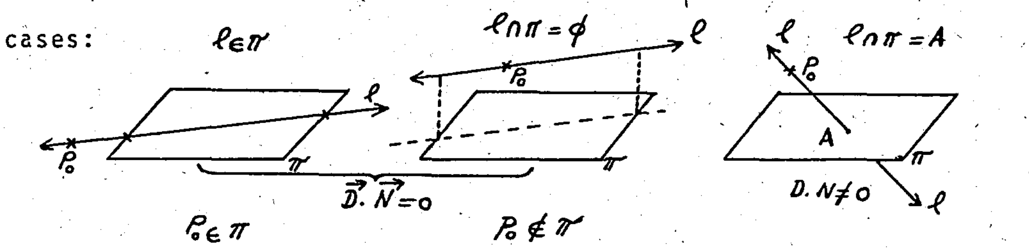
Intersections:

1. Intersection of a line and a plane: Let the line ℓ and plane π be given by the equations:

$$\ell: \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} = t \qquad (\vec{D} = (a, b, c))$$

$$\pi: Ax + By + Cz + D = 0 \qquad (\vec{N} = (A, B, C))$$

As the relative positions of ℓ, π we have three exhaustive cases:



If direction vectors $\vec{D}$ and $\vec{N}$ are perpendicular, one has $\ell // \pi$ in which case either $\ell \in \pi$ or $\ell \cap \pi = \emptyset$. In the first case $P_0 \in \pi$ and the second $P_0 \notin \pi$

If $\vec{D}.\vec{N} \neq 0$, the line intersects the plane at a point A which is determined by setting the coordinates of any point $P(x_0 + at, y_0 + bt, z_0 + ct)$ of ℓ in the equation of π, and obtaining a linear equation in t. If t_1 is the solution then $P(t_1)$ is the required intersection point.

Example 1. Find the point of intersection, if any, of

$$\ell: \frac{x-1}{2} = \frac{y}{-1} = \frac{z+2}{1} = t$$

$$\pi: \quad x + 2y - 3z = 10$$

Solution. Since $\vec{D} = (2, -1, 1)$, $\vec{N} = (1, 2, -3)$ are not perpendicular $(\vec{D}.\vec{N} = 2-2-3 \neq 0)$ intersecting point exists.

$$P \in \ell \Rightarrow P(1 + 2t, \ -t, \ -2 + t)$$
$$\Rightarrow (1+2t) + 2(-t) - 3(-2+t) = 10$$
$$\Rightarrow -3t + 7 = 10 \qquad t_1 = -1$$
$$P(-1, 1, -3).$$

Example 2. Find a relation between a and b such that

$$\ell: \frac{x-1}{2} = \frac{y}{a} = \frac{z+2}{1}, \qquad \pi: bx + 2y - 3z = 10$$

a) have no common point,

b) $\ell \epsilon \pi$.

Solution.

$$\text{a) } \ell \cap \pi = \emptyset \Rightarrow \begin{cases} \vec{D}.\vec{N} = 0 & \Rightarrow \quad 2b + 2a - 3 = 0 \\ P_0(1, 0, -2) \notin \pi & \Rightarrow \quad b + 6 \neq 10 \end{cases}$$

$$\Rightarrow b \neq 4, \quad 2a + 2b - 3 = 0 .$$

b) $2a - 2b = 3$, $b = 4 \Rightarrow a = -5/2$, $b = 4$. ■

If the line is given in non parametric form as intersection of two planes and the plane by its general equation, the intersection is obtained by solving a system of three linear equations.

If the equation are given by vectoral ones:

$$P = A + tD$$
$$P = B + \alpha U + \beta V,$$

the solution is obtained by equating P's obtaining three linear equations in t, α, β. solution of which gives $t = t_1$ and the required point is $P_1 = A + t_1 D$.

Solve Example 1 after transforming the equations into vectoral forms.

2. Intersection of two lines. Let the lines be given by

$$\ell_1: \frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} = t \qquad \ell_2: \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$$

If the direction vectors D_1, D_2 are parallel $(\frac{a_1}{b_1} = \frac{b_4}{b_2} = \frac{c_1}{c_2})$ then either $\ell_1 \equiv \ell_2$ or $\ell_1 \cap \ell_2 = \emptyset$. The former holds when $P_1(x_1, y_1, z_1) \in \ell_2$ and the latter when $P_1 \notin \ell_2$.

If D_1, D_2 are non parallel, either ℓ_1, ℓ_2 are intersecting at a point or else $\ell_1 \cap \ell_2 = \emptyset$. In the latter case ℓ_1, ℓ_2 are called skew lines. To determine the point of intersection or skewness, one sets the coordinates of $P(x_1+a_1t, y_1+b_1t, z_1+c_1t)$ in the equations of ℓ_2 obtaining two linear equations in t. In case of consistency (inconsistency) there is intersection (skewness).

Example 2. Given

$$\ell: \frac{x-a}{1} = \frac{y+1}{0} = \frac{z-1}{2} = t, \qquad \ell': \frac{x}{-1} = \frac{y-3}{2} = \frac{z}{-2}$$

determine "a" for ℓ, ℓ'

a) to intersect at a point A, and find A

b) to be skew.

Solution. The direction vectors being non parallel, they either intersect or are skew:

a) Setting $x = a+t$, $y = -1$, $z = 1+2t$ from ℓ into ℓ', we have

$$\frac{a+t}{-1} = \frac{-1-3}{2} = \frac{1-2t}{-2} \Rightarrow 2a+2t = 1+3, \quad -2a-2t = -1-2t$$

$$\Rightarrow \quad 2t = 4-2a, \quad a = 1/2 \Rightarrow t = 3/2 \Rightarrow A(2, -1, 4).$$

b) $a \neq 1/2$.

The same problem can be solved by the following technique: Using parametric points

$$P(a+t, \ -1, \ 2t+1), \quad P'(-s, \ 2s+3, \ -2s)$$

of ℓ, ℓ', we have,

$$P \equiv P' \Rightarrow a+t = -s, \quad -1 = 2s+3, \quad 2t+1 = -2s$$

$$\Rightarrow s = -2, \quad t = 3/2, \quad a = 1/2$$

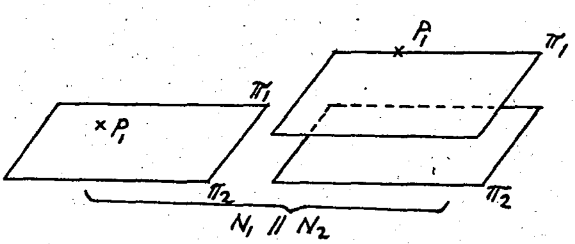
$$A(2, -1, 4). \blacksquare$$

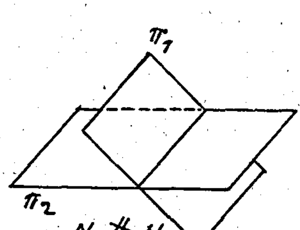
Discuss the cases where lines are given by vectoral equations, and also by pairs of general equations (four linear equations with three unknowns).

3. <u>Intersection of two planes</u>. Let the planes be given by their general equations

$$\pi_1: A_1x+B_1y+C_1z+D_1 = 0, \quad \pi_2: A_2x+B_2y+C_2z+D_2 = 0 \qquad (1)$$

We have the following three exhaustive cases:





$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} = \frac{D_1}{D_2}; \qquad \frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} \neq \frac{D_1}{D_2}$$

(i) (ii)

The two planes coincide when (i) holds, and have no common point when (ii) holds.

In the remaining case of $N_1 \nparallel N_2$, the two planes intersect along a line ℓ. The direction vector of ℓ being $D = N_1 \times N_2 = (a, b, c)$ (Since $\ell \perp N_1$, $\ell \perp N_2$), it will suffice to find a point P_0 on ℓ as a particular solution of (1): Select two unknowns having non proportional coefficients and assign any numerical value, say 0, to the third unknown, solve then (1) to get P_0 on ℓ. Then

$$\ell: \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c},$$

which could also be obtained (in parametric form) by solving the rectangular system (1).

Example 4. Find the line of intersection of the planes

$$\pi_1: 3x - y + z = 1, \qquad \pi_2: x + 2y - 2z = 5$$

in symmetric and the vector form.

Solution. Since the coefficients are not proportional, the planes intersect along a line ℓ.

Coefficients of x, y being not proportional, setting $z = 0$, we have

$$3x - y = 1, \quad x + 2y = 5 \Rightarrow x = 1, \; y = 2 \Rightarrow P_0(1, 2, 0)$$

$$D = N_1 \times N_2 = \begin{vmatrix} i & j & k \\ 3 & -1 & 1 \\ 1 & 2 & -2 \end{vmatrix} = (0, 7, 7) \parallel (0, 1, 1)$$

$$\Rightarrow \qquad \ell: \frac{x-1}{0} = \frac{y-2}{1} = \frac{z}{1} \qquad \blacksquare$$

If the planes are given by their vectoral equations

$$\pi_1: P = A_1 + s_1U_1 + t_1V_1$$

$$\pi_2: P = A_2 + s_2U_2 + t_2V_2,$$

then the vectoral equation

$$\ell: P = A + tD$$

of the intersection is obtained by equating P's and obtaining three linear equations in s_1, t_1, s_2, t_2. Solving three of them in terms of the fourth, and setting in one of the equations, one obtains the equation of ℓ.

Example 5. Find the vectoral equation of the line of intersection of the planes:

$$\pi: P = (1, 1, 1) + s(0, 0, 1) + t(1, 0, -1)$$

$$\pi': P = (1, 1, -1) + s'(0, 1, 0) + t'(0, 1, 1)$$

Solution. Equating P's

$$(1, 1, 1)+(0, 0, s)+(t, 0, -t) = (1, 1, -1)+(0, s', 0)+(0, t', t')$$

$$\Rightarrow \quad (t, 0, s-t) = (0, 0, -2)+(0, s'+t', t')$$

$$\Rightarrow \quad (t, -s'-t', s-t-t') = (0, 0, -2)$$

$$\Rightarrow \quad t = 0, \quad -s'-t' = 0, \quad s-t-t' = -2$$

$$\Rightarrow \quad t = 0, \quad s'+t' = 0, \quad s-t' = -2$$

$$t' = s+2, \quad s' = -s-2$$

$$P = (1, 1, -1)-(s+2)(0, 1, 0)+(s+2)(0, 1, 1) \quad \Rightarrow$$

$$P = (1, 1, -1) + s(0, 0, 1)$$

Angles:

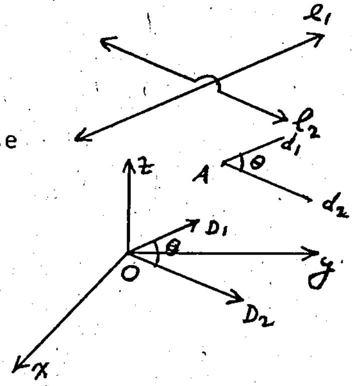
1. Angle between two lines:

The angle (ℓ_1, ℓ_2) between two lines ℓ_1, ℓ_2 is the non obtuse angle between the lines d_1, d_2 through a point A drawn parallel to the given lines.

Referring to the figure, we have

$$(\ell_1, \ell_2) \cong (d_1, d_2)$$

$$\cong \begin{cases} (\vec{D}_1, \vec{D}_2) \text{ if non obtuse} \\ (-\vec{D}_1, \vec{D}_2) = (\vec{D}_1 - \vec{D}_2) \quad \text{if obtuse} \end{cases}$$



The measure of the angle (ℓ_1, ℓ_2) will be denoted by $|(\ell_1, \ell_2)|$ which is equal to

$$\arccos \frac{|\vec{D}_1 \cdot \vec{D}_2|}{|\vec{D}_1|\ |\vec{D}_2|}$$

where $\vec{D}_1$, $\vec{D}_2$ are direction vectors of ℓ_1, ℓ_2. The absolute value of $\vec{D}_1 . \vec{D}_2$ is taken, since cosine of non obtuse angle is positive or zero.

Example. Find the cosine of the angle between the lines:

$$\ell_1: \frac{x-8}{4} = \frac{y-4}{9} = \frac{z}{5}, \quad \ell_2: x+y+z = 0, \quad 2x-y+3z = 1$$

Solution.

$$D_1 = (4, 9, 5), \qquad D_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 2 & -1 & 3 \end{vmatrix} = (4, -1, -3)$$

$$\Rightarrow \cos\theta = \frac{|16 - 9 - 15|}{\sqrt{122}\ \sqrt{26}} = \frac{4}{\sqrt{61}\ \sqrt{13}}$$

2. Angle between a line and a plane

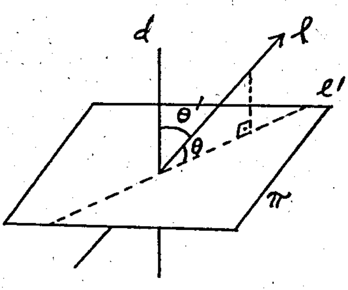
The angle (ℓ, π) between a line ℓ and a plane π is the angle between the line ℓ and its projection ℓ' on π: $(\ell,\pi) \cong (\ell,\ell')$.

Observe that the angle defined above is non obtuse:

If $d \perp \pi (d // N)$, and $\theta = |(\ell, \pi)|$,

$\theta' = |(\ell, d)|$, we have

$$\theta' = \frac{\pi}{2} - \theta \Rightarrow \cos\theta' = \sin\theta$$



$$\theta = \arcsin \frac{|\vec{N} \cdot \vec{D}|}{|\vec{N}|\,|\vec{D}|}$$

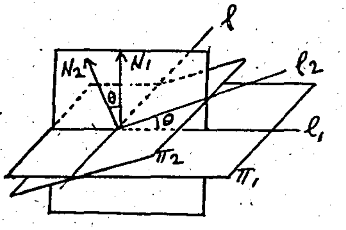
where $\vec{D}$ is a direction vector of ℓ.

3. <u>Angle between two planes</u> (<u>dihedral angle</u>):

The angle (π_1, π_2) of two intersecting planes π_1, π_2 is the angle between the lines ℓ_1, ℓ_2 of intersection of π_1, π_2 with a plane perpendicular to their line of intersection ℓ.

$$(\pi_1, \pi_2) \cong (\ell_1, \ell_2)$$

$$\Rightarrow \left|(\pi_1, \pi_2)\right| = \left|(\ell_1, \ell_2)\right| = \arccos \frac{|\vec{N}_1 \cdot \vec{N}_2|}{|\vec{N}_1|\,|\vec{N}_2|}$$



where $\vec{N}_1, \vec{N}_2$ are normal vectors of π_1, π_2.

<u>Example 1</u>. Given two lines

$$\ell_1: \begin{cases} x-y+2z = 2 \\ x+3y-z = -5 \end{cases} \quad \text{and} \quad \ell_2: \frac{x}{1} = \frac{y-2}{3} = \frac{2z-1}{-1} = t$$

a) show that $\ell_1 \cap \ell_2 \neq \emptyset$,

b) find the equation of the plane π determined by ℓ_1, ℓ_2,

c) find planes π', π'' through ℓ_1 making an angle of $\pi/6$ with π.

<u>Solution</u>.

a) Setting $x=t$, $y = 2+3t$, $z = \frac{1}{2} - \frac{t}{2}$ from ℓ_2 into $x - y + 2z = 2$, we get $t = -1 \Rightarrow x = -1$, $y = -1$, $z = 1$ which satisfy $x + 3y - z = -5$.

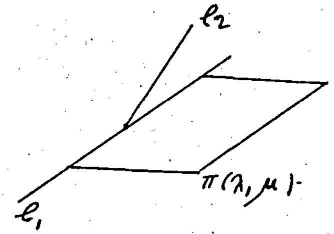
b) Planes through ℓ_1 is

$$\lambda(x - y + 2z - 2) + \mu(x + 3y - z + 5) = 0$$

$$\pi(\lambda, \mu): \quad (\lambda+\mu)x + (-\lambda+3\mu)y + (2\lambda-\mu)z + (-2\lambda+5\mu) = 0$$

$$\vec{N}(\pi) = (\lambda+\mu, \quad -\lambda+3\mu, \; 2\lambda-\mu).$$

A direction vector of ℓ_2 being $\vec{N}(\ell_2) = (1, 3, -1/2)$, and since ℓ_2 intersects ℓ_1, thenfor $\pi(\lambda\mu)$ to contain ℓ_2, one must have $\vec{N}(\ell_2) . \vec{N}(\pi) = 0$ implying



$$1(\lambda+\mu) + 3(-\lambda+3\mu) - \frac{1}{2}(2\lambda-\mu) = 0$$

$$\Rightarrow \lambda = \frac{7}{2}\mu : \qquad \lambda = 7, \; \mu = 2 \text{ say}$$

$$\pi: \; 9x - y + 12z - 4 = 0$$

c) $\frac{\pi}{6} = |(\pi, \; \pi(\lambda, \mu))|$

$$\Rightarrow \frac{\sqrt{3}}{2} = \frac{|N(\pi) \; . \; N(\pi(\lambda,\mu))|}{|N(\pi)| \; |N(\pi(\lambda,\mu))|}$$

$$= \frac{|9(\lambda+\mu) + \lambda - 3\mu + 24\lambda - 12\mu)|}{\sqrt{226}\;\sqrt{6\lambda^2 - 8\lambda\mu + 11\mu^2}}$$

$$= \frac{|34\lambda - 6\mu|}{\sqrt{226}\;\sqrt{6\lambda^2 - 8\lambda\mu + 11\mu^2}}$$

$$= \frac{1}{\sqrt{226}} \; \frac{|34 - 6k|}{\sqrt{6 - 8k + 11k^2}} \qquad (k = \mu/\lambda) \qquad \Rightarrow$$

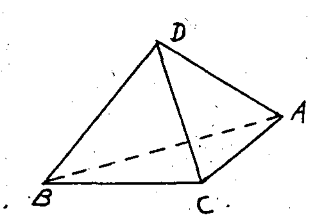
$365/k^2 - 1896k - 278 = 0$ which has two real roots say $k_1 = \frac{\lambda_1}{\mu_1}$, $k_2 = \frac{\lambda_2}{\mu_2}$. These pairs of values λ_1, μ_1 and λ_2, μ_2 give the required planes.

Example 2. Given a tetrahedron ABCD with vertices at $A(1, 1, 1)$, $B(1, -1, 0)$, $C(0, 1, 2)$, $D(2, 0, -1)$, find cosine or sine of the angle between

a) the edges [DA] and [DB]

b) the faces DBC and ABC

c) the edge [DA] and the face ABC



Solution.

a) $\vec{DA} = A-D = (-1, 1, 2), \quad \vec{DB} = B-D = (-1, -1, 1)$

$$\cos\theta = \frac{|\vec{DA} \cdot \vec{DB}|}{|\vec{DA}|\ |\vec{DB}|} = \frac{|1-1+2|}{\sqrt{6}.\sqrt{3}} = \frac{2}{3\sqrt{2}}$$

b) $$N_1 = \vec{BC} \times \vec{BD} = \begin{vmatrix} i & j & k \\ -1 & 2 & 2 \\ 1 & 1 & -1 \end{vmatrix} = (-4, 1, -3)$$

$$N_2 = \vec{BC} \times \vec{BA} = \begin{vmatrix} i & j & k \\ -1 & 2 & 2 \\ 0 & 2 & 1 \end{vmatrix} = (-2, 1, -2)$$

$$\cos\theta = \frac{|N_1 \cdot N_2|}{N_1\ \ N_2} = \frac{|8+1+6|}{\sqrt{26}\ .3} = \frac{5}{\sqrt{26}}$$

C) $$\sin\theta = \frac{|\vec{DA} \cdot \vec{N_2}|}{|DA|\ |N_2|} = \frac{|2+1-4|}{\sqrt{6}\ .3} = \frac{1}{3\sqrt{6}}$$

Distances:

1. Distance between two points

Vectorally : $d(P_1, P_2) = |\vec{P_1P_2}|$ (1')

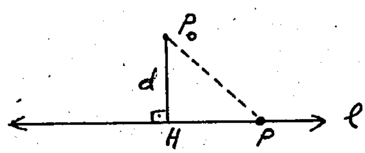
Analytically: $d(P_1, P_2) = \sqrt{(x_1-x_2)^2-(y_1-y_2)^2-(z_1-z_2)^2}$ (1")

where $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$.

2. Distance of a point from a line:

Let P_o be a given point and ℓ be a given line.

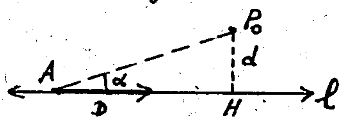
The distance $d(P_o, \ell)$ is defined to be min $d(P_o, P)$ for all $P \in \ell$, which is obtained when $P_oP \perp \ell$, and



$$d(P_o, \ell) = d(P_o, H) = |\vec{P_oH}|$$

where H is the projection of P_o on ℓ.

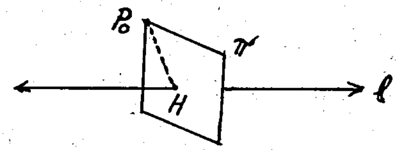
a) Vectorally:



$$|\vec{D}\times\vec{AP_0}| = |\vec{D}|\,\underbrace{|\vec{AP_0}|\sin\alpha}_{d}$$

$$d = \frac{|\vec{D}\times\vec{AP_0}|}{|\vec{D}|} \qquad (2')$$

b) analytically:

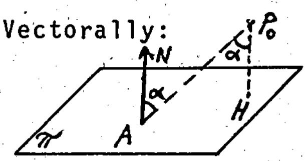


$d = |P_0H|$ is evaluated by (1") where H is obtained by finding the intersection of ℓ with the plane through P_0 and $\perp$ to ℓ. (2")

3. <u>Distance of a point from a plane.</u>

The distance of a point P_0 from a plane π is defined as in 2., and min is altained when $P_0P \perp \pi$.

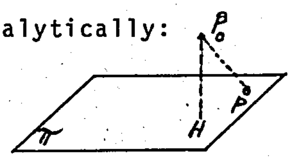
a) Vectorally:



$$|\vec{N}\cdot\vec{AP_0}| = |\vec{N}|\,\underbrace{|\vec{AP_0}|\cos\alpha}_{d}$$

$$d = \frac{|\vec{N}\cdot\vec{AP_0}|}{|\vec{D}|} \qquad (3')$$

b) analytically:



$d = |P_0H|$ is evaluated by (1") where H is obtained by finding the intersection of π with the line through P_0 and $\perp$ to π. (3")

4. <u>Distance of a line and a plane.</u>

The distance $d(\ell, \pi)$ between a line ℓ and a plane π is defined as min $d(L, P)$ for all $L\in\ell$, $P\in\pi$ which is certainly zero when ℓ intersects π $(\vec{D}.\vec{N} \neq 0)$.

If $\ell /\!/ \pi$ $(D.N \neq 0)$, min $d(L, P)$ is attained when $LP \perp \pi$, and

$$d(\ell, \pi) = d(L_1, \pi)$$

where $L_1 \in \ell$

a) Vectorally ($\ell // \pi$):

$$d(\ell, \pi) = \frac{|\vec{N}.\overrightarrow{L_1P_1}|}{|\vec{N}|} \quad (4')$$

where $L_1 \epsilon \ell_1$, $P_1 \epsilon \pi$

b) Analytically ($\ell // \pi$):

$$d(\ell, \pi) = d(L_1, \pi)$$

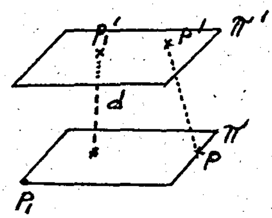
$$= \frac{|Ax_1 + By_1 + Cz_1 + D_1|}{\sqrt{A^2 + B^2 + C^2}} \quad (4")$$

where $L_1(x_1, y_1, z_1) \epsilon \ell$.

5. <u>Distance between two planes</u>

The distance $d(\pi, \pi') = \min d(P, P')$ between two planes π, π' is zero when π, π' intersect ($N \times N' \neq 0$).

If $\pi // \pi'$ ($N \times N' = 0$) the min $d(P, P')$ is attained when $PP' \perp \pi$. Then



$$d(\pi, \pi') = d(P_1', \pi) = d(P_1, \pi')$$

where $P_1 \epsilon \pi$, $P_1' \epsilon \pi'$.

a) Vectorally ($\pi // \pi'$)

$$d(\pi, \pi') = \frac{|\vec{N}.\overrightarrow{P_1P_1'}|}{|\vec{N}|} \quad (5')$$

b) Analytically ($\pi // \pi'$)

$$d(\pi, \pi') = d(P_1, \pi')$$

$$= \frac{|A'x_1 + B'y_1 + C'z_1 + D'|}{\sqrt{A'^2 + B'^2 + C'^2}} \quad (5")$$

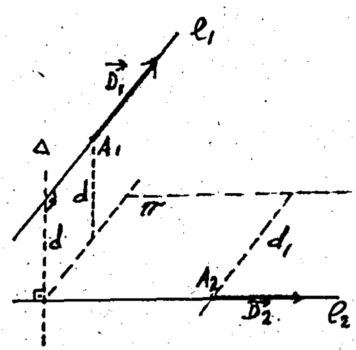
6. <u>Distance between two lines:</u>

Let the lines ℓ_1, ℓ_2 be determined by A_1, $\vec{D}_1$, A_2, $\vec{D}_2$

If $\ell_1 // \ell_2$ ($\vec{D}_1 // \vec{D}_2$), then

$$d(A_1, \ell_2) = d(A_2, \ell_1)$$

Let then $\ell_1 \not\parallel \ell_2$ (intersecting or skew). Consider the plane π through ℓ_2 and parallel to ℓ_1 determined by ℓ_2 and $d_1 // \ell_1$



Since $\ell_1 // \pi$, then

$$d(\ell_1, \ell_2) = d(\ell_1, \pi).$$

a) Vectorally:

$$d(\ell_1, \ell_2) = \frac{|\vec{N}.\vec{A_1A_2}|}{|\vec{N}|} \quad (6')$$

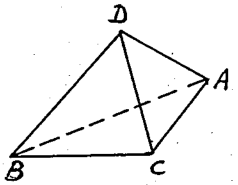
where $\vec{N} = \vec{D_1} \times \vec{D_2}$

b) Analytically:

$$d(\ell_1, \ell_2) = d(A_1, \pi) \quad (6'')$$

where π is the plane through ℓ_2 parallel to ℓ_1.

Example 5. Given the tetrahedron ABCD with $A(1, 1, 1)$, $B(1, -1, 0)$, $C(0, 1, 2)$, $D(2, 0, -1)$,



a) find D) b) find $d(D, ABC)$

c) find) d) find $d(AD, BC)$

by vector method,

e) Solve Part by analitic method.

Solution.

a) $d(A, D) = |\vec{AD}| = \sqrt{(1-2)^2 + (1-0)^2 + (1-(-1))^2} = \sqrt{6}$

b) $d(D, ABC) = \dfrac{|\vec{N}.\vec{AD}|}{|\vec{N}|}$ where $A \varepsilon ABC$,

$\vec{N} = \vec{AB} \times \vec{AC} = (0, -2, -1)\times(-1, 0, 1) = (-2, 1, -2)$

$\vec{AD} = (1, -1, -2)$

$$d = \frac{|-2 - 1 + 4|}{\sqrt{4+1+4}} = 1/3$$

c) $d(D, BC) = \dfrac{|\vec{BC} \times \vec{BD}|}{|\vec{BC}|}$

$\vec{BC} = (-1, 2, 2)$, $\vec{BD} = (1, 1, -1)$

$\vec{BC} \times \vec{BD} = (-4, 1, -3)$

$$d = \frac{\sqrt{16 + 1 + 9}}{3} = \sqrt{26}/3$$

d) $d(AD, BC) = \dfrac{|\vec{N}.\vec{AB}|}{|\vec{N}|}$

$$N = \vec{AD}\times\vec{BC} = (1, -1, -2)\times(-1, 2, 2)=(2, 0, 1)$$

$$\vec{AB} = (0, -2, -1)$$

$$d = \frac{|-1|}{\sqrt{5}} = 1/\sqrt{5}$$

e) $AD: \dfrac{x-1}{-1} = \dfrac{y-1}{1} = \dfrac{z-1}{2}$, $BC: \dfrac{x-1}{-1} = \dfrac{y+1}{2} = \dfrac{z}{2}$

$$\vec{N} = \begin{vmatrix} i & j & k \\ -1 & 1 & 2 \\ -1 & 2 & 2 \end{vmatrix} = (-2, 0, -1)$$

$$\pi: -2(x-1) + 0(y-1) - 1(z) = 0$$

$$-2x - z + 3 = 0$$

$$d = d(A, \pi) = \frac{|-2.1 - 1 + 3|}{\sqrt{4+1}} = 1/\sqrt{5}$$

D. SKETCHING OF PLANES AND LINES

A plane is sketched is general by determining its intercepts with coordinate axes or by determining its traces (lines of intersection) with coordinate planes.

Example. Sketch the planes

a) $\dfrac{x}{3} + y + \dfrac{z}{2} = 1$

b) $x - y + 2z - 2 = 0$

Solution.

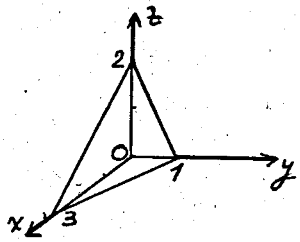
a) The equation being given by intercept form, the intercepts are $a=3$, $b=1$, $c=2$.

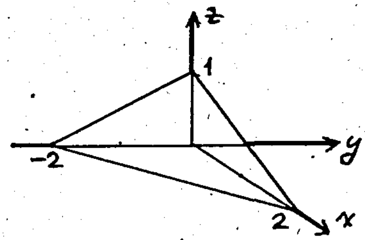
b) We obtain intercepts as follows:

x-intercept: $y=0$, $z=0 \Rightarrow a=2$

y-intercept: $x=0$, $z=0 \Rightarrow b=-2$

z-intercept: $x=0$, $y=0 \Rightarrow c=1$





A line is sketched in general by finding its traces (points of intersections with two coordinate planes), or by sketching any two of its points (or by the method given in Curve Sketching).

Example. Sketch the lines

a) $\frac{x-1}{1} = \frac{y}{2} = \frac{z+2}{-1}$

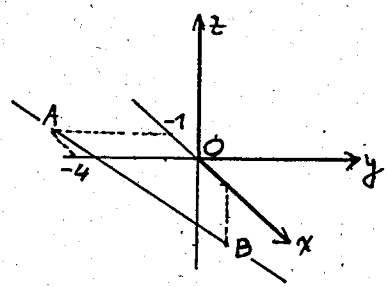
b) $x + y = 2, \quad x - y + 2z = -4$

Solution.

a) xy-trace: $z=0 \Rightarrow \frac{x-1}{1} = \frac{y}{2} = -2$

$\Rightarrow x = -1, \quad y = -4$

$A(-1, -4, 0)$



xz-trace: $y=0 \Rightarrow \frac{x-1}{1} = \frac{0}{1} = \frac{z-2}{-1}$

$\Rightarrow x = 1, \quad z = -2$

$B(1, 0, -2)$

b) xy-trace: $z=0 \Rightarrow x + y = 2, \quad x - y = -4$

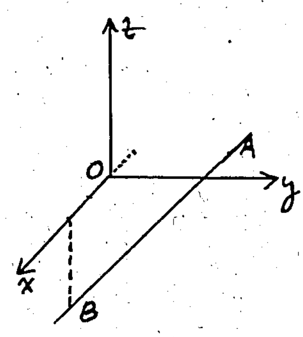
$\Rightarrow x = -1, \quad y = 3$

$A(-1, 3, 0)$

xz-trace: $y=0 \Rightarrow x = 2, \quad x + 2z = -4$

$\Rightarrow x = 2, \quad z = -3$

$B(2, 0, -3)$



EXERCISES (3, 2)

71. Find the equation of the plane ABC, where

a) $A(2, 1, 6)$, $B(5, -2, 0)$, $C(4, -5, -2)$

b) $A(0, b, c)$, $B(a, 0, c)$, $C(a, b, 0)$

72. Find the plane through $A(1, -1, 3)$ and parallel to the plane $3x + y + z = 7$

73. Which one of the points $A(8, -2, 1)$, $B(-5, 2, -1)$ is nearest to the plane $x + 4y - 8z + 9 = 0$?

74. Given the line $P = A + \lambda D$ and plane $\vec{N}.\vec{P_0P} = 0$

a) find condition for intersection at a single point.

b) find the point of intersection.

75. Determine pairwise relative positions of the given lines as parallel (concident or non intersecting), intersecting or skew

a) $\frac{x-1}{2} = \frac{y}{-4} = \frac{z}{0}$, b) $\frac{x}{-1} = \frac{y+3}{2} = \frac{z-1}{0}$, c) $\frac{x}{2} = \frac{y}{-1} = \frac{z+2}{1}$

76. Find the equation of the line passing through $A(1, 1, 1)$ and intersecting $\frac{x}{1} = \frac{y-2}{-1} = \frac{z}{2}$ orthogonally.

77. Which three of the following points are coelinear?

$A(1, 2, -3)$, $B(2, 1, -1)$, $C(3, -2, 2)$, $D(3, -2, -3)$.

78. Find the equations of the line ℓ such that

a) $A(1, 3, -2) \in \ell$, $\ell // (2, 0, 1)$

b) $B(2, 0, 1) \in \ell$, $\ell // (1, 3, -2)$

c) $A(-2\ \ 1, 1) \in \ell$, $B(1, 2, -1) \in \ell$

79. Given $A(1, 2, 3)$, $B(-2, 7, 0)$, find the distance between the lines AB and ℓ: $x+y-6 = 0$, $z-2 = 0$

80. If $A=(a_1, a_2, a_3)$ and $B=(b_1, b_2, b_3)$ are two non collinear vectors, show that the lines

$$\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3}, \quad \frac{x-b_1}{a_1} = \frac{y-b_2}{a_2} = \frac{z-b_3}{a_3}$$

intersect at a single point C and find C.

81. Given lines $P = A+\lambda U$, $P = B+\mu V$

a) find condition for their intersection

b) determine their intersection under (a)

82. Given $A(1, 2, 3)$, $B(-1, 2, 1)$, $C(4, 3, 2)$, $D(2, 2, 9)$, find

a) $d(A, BC)$, b) $d(D, ABC)$

83. Find the distance between the lines AB, CD if $A(1, -2, 1)$, $B(4, 5, 6)$, $C(-3, -2, 1)$, $D(1, 1, 4)$

84. Find the equation of the plane π if

a) $A(2, -3, 5)\in\pi$, $\pi//\pi': 3x+5y-7z = 11$

b) $B(1, 5, 9)\in\pi$, $\pi\perp\ell: A(2, 3, -4)$ $B(5, 1, -1)$

85. Show that the four planes $x+y+z-3 = 0$, $2x+2y-z+1 = 0$, $x+y-5z-8 = 0$, $3x+3y+7z-4 = 0$ are concurrent.

86. Find the equation of the line

a) through $A(3, -1, 6)$ and parallel to the planes $x-2y+z = 2$, $2x+y-z = 5$

b) through $B(-9, 4, 3)$ and perpendicular to the plane $2x+6y+9z = 0$

87. Find the parametric equations of

a) $\frac{x-1}{2} = \frac{y}{-1} = \frac{z+5}{0}$, b) $\frac{x}{3} = \frac{y-2}{1} = \frac{z}{-2}$

c) $x+y-5z = 7$ d) $3x-y+z = 5$

88. Find the distance between the skew lines

$\ell: x+y-6 = 0, \quad z=0, \qquad \ell': P_1(1, 2, 3) \; P_2(-2, 7, 0)$

89. Find the distance of $P_0(2, 3, 1)$ from the line

$$\frac{x-1}{1} = \frac{y-2}{4} = \frac{z}{2}$$

90. Examine if the point $P(1, 2, 1)$ is inside the tetrahedron where faces are the planes: $x+y-z = 1$, $x+y+z = 3$, $x-y+z = 5$, $x-y-z = 7$

91. Find the equation of the perpendicular bisector of the following segments of line:

a) $[AB]$ where $A(1, -2, 3)$, $B(3, 0, -1)$

b) $[CD]$ where $C(-1, 2, -3)$, $D(-3, 0, 1)$

92. Given $A(1, 1, 1)$, $B(1, -3, -1)$ find the locus:

a) $\{P: \frac{|PA|}{|PB|} = 2\}$ b) $\{P: \frac{|PA|}{|PB|} = 1\}$

93. Find the distance between two lines below:

a) $x-y = z$ and $x = 2y-1$, $x+y+2z+ -4 = 0$

b) $x = y+1$, $z = x+1$ and $y = x+1$, $x = 1-z$

94. Same question for:

$\ell_1: \; \frac{x}{a} = \frac{z}{c}\cos\alpha - \sin\alpha, \quad \frac{y}{b} = \frac{z}{c}\sin\alpha + \cos\alpha$

$\ell_2: \; \frac{x}{a} = \frac{z}{c}\cos\beta + \sin\beta, \quad \frac{y}{b} = \frac{z}{c}\sin\beta - \cos\beta$

95. a) Show that the equation of the common perpendicular to z-axis and the line $\ell: x = az+p$, $y = bz+q$, $z = z$ is $ax+by = 0$, $z = -\frac{ap+bq}{a^2+b^2}$

b) the distance between z-axis and ℓ is $d = \left| \frac{\begin{vmatrix} a & p \\ b & q \end{vmatrix}}{\sqrt{a^2+b^2}} \right|$

ANSWERS TO EVEN NUMBERED EXERCISES

72. $3x+y+z = 5$

74. a) $\vec{N}.\vec{D} \neq 0$, b) $P = A + \dfrac{\vec{N}.\vec{AP_o}}{\vec{N}.\vec{D}} D$

76. $\dfrac{x-1}{-1} = \dfrac{y-1}{1} = \dfrac{z-1}{1}$

78. a) $\dfrac{x-1}{2} = \dfrac{y-3}{0} = \dfrac{z+2}{1}$, b) $\dfrac{x-2}{1} = \dfrac{y}{3} = \dfrac{z-1}{-2}$

c) $\dfrac{x+2}{3} = \dfrac{y-1}{1} = \dfrac{z-1}{-2}$

80. $C = A+B$

82. a) $\sqrt{8/3}$, b) $5/3\sqrt{2}$

84. a) $3x+5y-7z = -44$, b) $3x-2y+3z = 20$

86. a) $\dfrac{x-3}{1} = \dfrac{y+1}{3} = \dfrac{z-6}{5}$ b) $\dfrac{x+9}{2} = \dfrac{y-4}{6} = \dfrac{z-3}{9}$

88. $3/\sqrt{22}$

90. Outside

92. a) $3x^2+3y^2+3z^2-6x+26y+10z+41 = 0$

b) $2y+z+2 = 0$

94. 0

3. 3. SURFACES

A. RELATIONS AND THEIR GRAPHS

A statement $p(x, y, z)$ involving variables x, y, z with $(x, y)\varepsilon\mathbb{R}^2$ and $z\varepsilon\mathbb{R}$ is called an open statement which is true for points is a subset S of $\mathbb{R}^3$ and false in $\mathbb{R}^3-S$ where S may be ϕ or $\mathbb{R}^3$.

Examples.

1. $p(x, y, z)$: $|x|+|y|+|z| \geqslant 0$
   is true for all points $P(x, y, z)$ in $\mathbb{R}^3$.

2. $q(x, y, z)$: $x^2+y^2+z^2 < 0$
   is true for no point in $\mathbb{R}^3$

3. $r(x, y, z)$: $x+y-z = 1$
   is true for all points on the plane π: $x+y-z = 1$ and false for all other points.

4. $s(x, y, z)$: $x+y-1 \leqslant z$
   is true for all points on the plane π: $x+y-1 = z$ or for points above π, and false for points below π.

5. $t(x, y, z)$: $x = 2$, $y = 3$, $z > 0$
   is true for all points on the vertical line $x = 2$, $y = 3$, above xy-plane.

If $p(x, y, z)$ is an open statement, then the set

$$\rho = \{(x, y, z): (x, y)\varepsilon\mathbb{R}^2, z\varepsilon\mathbb{R}, p(x, y, z)\} \qquad (1)$$

consisting of all points (x, y, z) for which the statement $p(x, y, z)$ is true is called a relation from $\mathbb{R}^2$ to $\mathbb{R}$ (or in $\mathbb{R}^3$), written

$$\rho : \mathbb{R}^2 \to \mathbb{R}, \quad p(x, y, z)$$

where $p(x, y, z)$ is the <u>rule</u> for the relation.

The set

$$D_\rho = \left\{(x, y): p(x, y, z)\right\} \subseteq \mathbb{R}^2$$

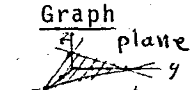
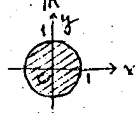
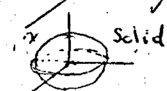
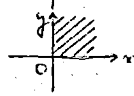
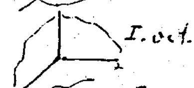
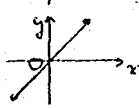
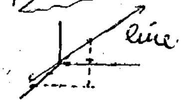
and

$$R_\rho = \left\{z: p(x, y, z)\right\} \subseteq \mathbb{R}$$

are respectively called the <u>domain</u> and <u>range</u> of the relation ρ.

When a relation is plotted in 3-space, the resulting set of points is called the <u>graph</u> of ρ. Depending upon the case either the set is empty (no graph) or the graph is a surface (for an equality relation) or a curve (for two equality relations) or a solid (for an inequality relation).

<u>Examples</u>.

	Domain	Graph
1. $\{(x, y, z): \frac{x}{2} + \frac{y}{3} + z = 1\}$	$\mathbb{R}^2$	
2. $\{(x, y, z): z = 2\}$	$\mathbb{R}^2$	
3. $\{(x, y, z): x^2+y^2+z^2 \leq 1\}$		
4. $\{(x, y, z): x \geq 0, y \geq 0, z \geq 0\}$		
5. $\{(x, y, z): x = y = z\}$		

<u>Surfaces</u>:

Since a surface is the graph of an equality relation, its equation can be written as $F(x, y, z) = 0$. For instance $x^2+y^2+z^2 = 1$ and $(x+y)^2-(2y-z)^2 = 3$ are equations of two surfaces.

Any relation $F(x, y, z) = 0$ represents a surface, and conversely any surface can be represented by such a relation including degenerate cases.

A surface may as well be represented by a system of three equalities involving two parameters u and v:

$$x = f(u, v), \quad y = g(u, v), \quad z = h(u, v)$$

which are called the <u>parametric equations</u> of the surface.

The parametric equations can be transformed to previous one by solving u, v from any two of them and setting in the third, theoretically.

<u>Example 1</u>. Given $2x - y + 3z = 6$,

a) show that it is the equation of a surface,

b) find two parametric equations.

<u>Solution</u>.

a) Since it is an equality relation involving x, y, z, represents a surface (plane)

b) $x=u$, $y=v$, $z = \frac{1}{3}(6-2u+v)$ or $x = s+t$, $y = s-t$, $z = \frac{1}{3}(6-s-3t)$.

<u>Example 2</u>. Given the system

$$2x-y+3z = 6, \quad x+y-z = 2,$$

a) show that the system represents a curve Γ,

b) write the parametric equations of Γ, if possible.

<u>Solution</u>.

a) Since the system consists of two equality relations, represents a curve (line of intersection of two planes)

b) Setting $x=u$ and solving y and z, we have $x = u$, $y = -\frac{5}{2}u+6$, $z = -\frac{3}{2}u+4$.

It is clear that any curve (system of two equality relation) can be represented by a system of three equalities with one parameter.

Linear family of surfaces:

Let

$$F(x, y, z) = 0, \qquad G(x, y, z) = 0$$

be the equations of two surfaces. Then the linear combinations of $F(x, y, z)$ and $G(x, y, z)$ equated to zero, that is

$$\lambda F(x, y, z) + \mu G(x, y, z) = 0$$

are surfaces for $\lambda, \mu \in \mathbb{R}$ passing through the curve of intersection, if they are intersecting.

Surface Sketching:

If the equation of the surface is given in the implicit form $F(x, y, z) = 0$ the latter may be thought as three relations

1) from pairs (x, y) to z
2) from pairs (x, z) to y
3) from pairs (y, z) to x

having domains on xy-, xz- and yz-planes respectively of which the first is considered generally.

Visualizing the shape of a surface is done in three steps:

1. Determination of the domain, say on xy-plane.
2. Determination of symmetries with respect to coordinate planes:

There is symmetry in xy-plane if $F(x, y, -z) \equiv F(x, y, z)$

There is symmetry in xz-plane if $F(x, -y, z) \equiv F(x, y, z)$

There is symmetry in yz-plane if $F(-x, y, z) \equiv F(x, y, z)$

Presense of symmetry with respect to another coordinate plane reduce the work in sketching.

3. Determination of cross sections of the surface with planes parallel to coordinate planes.

The cross sections parallel to xy-plane are called the <u>level curves</u> of the surface and ones lying on coordinate planes the xy-, xz- and yz-<u>traces</u>.

By means of cross sections, one gets the information about the boundedness of the surface.

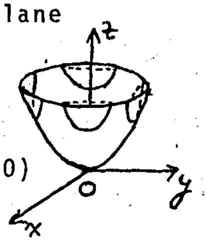
<u>Example</u>. Sketch the surface

$$S: 4x^2+y^2-z = 0$$

<u>Solution</u>.

1. The domain in xy-plane is $\mathbb{R}^2$ since the relation holds for all $(x, y)\in\mathbb{R}^2$. Having $z \geqslant 0$, the surface lies above xy-plane and passes through the origin.

2. Since the relation is unaltared when $x \to -x$ and $y \to -y$, there are symmetries in the yz- and xz-planes. So it will suffice to sketch the surface on the I. quadrant over xy-plane and get the whole surface by taking symmetries.



3. Cross sections // xy-plane (level curves):
$z = k \Rightarrow 4x^2+y^2 = k$ (ellipses for $k > 0$, origin for $k = 0$)

cross sections // xz-plane: $y = k \Rightarrow z = 4x^2+k^2$ (parabolas), cross sections // yz-plane: $x = k \Rightarrow z = y^2+4k^2$ (parabolas).

There are other coordinate systems in 3-space which are more convenient than rectangular one in some cases. The following are two examples of such systems.

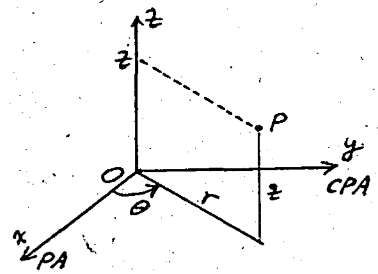
B. CYLINDRICAL AND SPHERICAL COORDINATE SYSTEMS

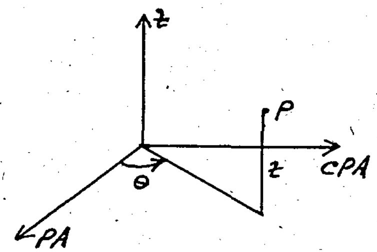
1. Cylindrical coordinate systems (Polar coordinate systems in 3-space).

A rectangular coordinate system Oxyz in which xy-plane is taken as polar plane (with x-axis as polar axis) is called a cylindrical coordinate system.

Cylindrical coordinates of a point P are θ, r, z and one writes P(θ, r, z).

When the coordinates of a point P are given, plotting of P is done in the order θ, r and z, while having a point P its coordinates are obtained in the order z, r, θ.





Transforming relations.

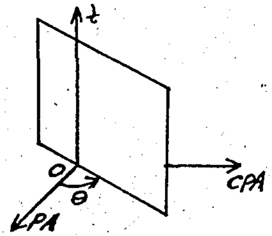
These are the relation between cartesian coordinates x, y, z and cylildrical ones θ, r, z of an arbitrary point P:

$$x = r\cos\theta \qquad \theta = \arctan\frac{y}{x}$$

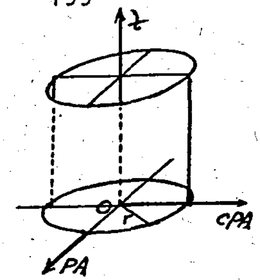
$$y = r\sin\theta \quad \text{or} \quad r = \pm\sqrt{x^2+y^2}$$

$$z = z \qquad z = z$$

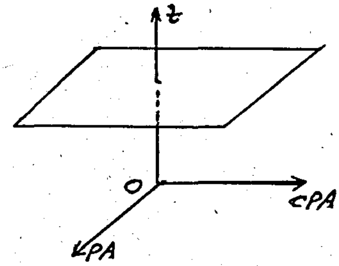
θ-constant surfaces are planes through z-axis, r-constant surfaces are circular right cylinders with Oz as axis, and z-constant surfaces are horizontal planes.



θ-constant surfaces

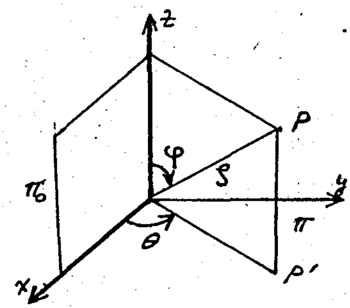


r-constant surfaces



z-constant surfaces

2. <u>Spherical coordinate system.</u>



In a rectangular coordinate system Oxyz, taking xz-plane as initial polar plane π_0 (with Oz as polar axis) and a plane through z-axis as a variable polar plane π (with Oz as polar axis), then the <u>spherical coordinates</u> of a point are defined by the directed angle θ from π_0 to π, and the polar coordinates φ, ρ on π and one writes $P(\theta, \varphi, \rho)$ where we impose the restrictions $\theta \in [0, 2\pi]$, $\varphi \in [0, \pi]$ and $\rho \geq 0$.

Plotting of a point $P(\theta, \varphi, \rho)$ is done in the order θ (determining the plane π), φ and ρ (on π).

<u>Transforming relations:</u>

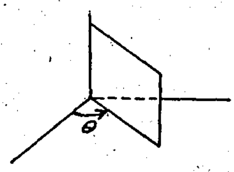
These are the relations between the cartesian coordinates x, y, z and spherical ones of an arbitrary point P. If P' is the projection of P on xy-plane, from $P'(\theta, r)$ and $r = \rho \operatorname{Sin}\varphi$ one gets

$$x = \rho \sin\varphi . \cos\theta \qquad\qquad \theta = \arctan \frac{y}{x}$$

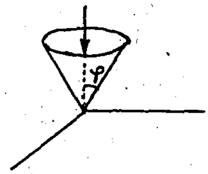
$$y = \rho \sin\varphi . \sin\theta \qquad \text{or} \qquad \varphi = \arccos \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$z = \rho \cos\varphi \qquad\qquad \rho = \sqrt{x^2+y^2+z^2}$$

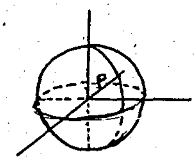
θ-constant surfaces are planes through z-axis, φ-constant surfaces are circular rihgtcones with Oz as axis, and ρ-constant surfaces are spheres with center at O.



θ-constant surface
(a plane)



φ-constant surface
(a cone)



ρ-constant surface
(a sphere)

We have one more set of transforming relations among three coordinate system which is between cylindrical and spherical one, namely

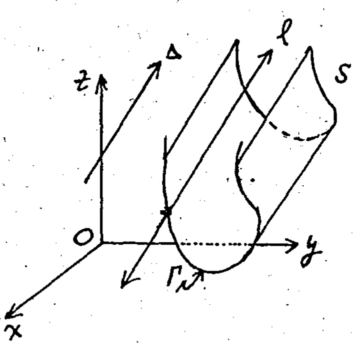
$$\begin{aligned} \theta &= \theta \\ r &= \rho \sin\varphi \\ z &= \rho\cos\varphi \end{aligned} \qquad \text{or} \qquad \begin{aligned} \theta &= \theta \\ \varphi &= \arctan\frac{r}{z} \\ \rho &= \sqrt{r^2+z^2} \end{aligned}$$

C. CYLINDERS, CONES, SURFACES OF REVOLUTION

1. Cylinders:

A surface S generated by a variable line ℓ of given direction Δ, and subject to another condition such as intersecting a curve Γ (or remaining tangent to a given surface Σ) is called a cylinder.

The line ℓ is the generatrix, Δ the direction, and Γ the directrix of the cylinder S, and we say that S in defined by Δ and Γ .



Equation of a cylinder

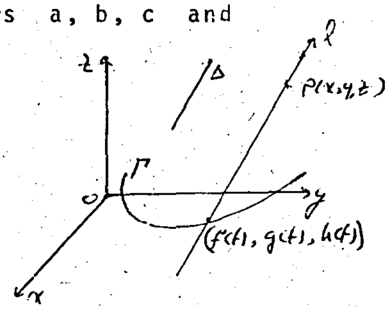
The equation of the cylinder defined by $\Delta = (a, b, c)$ and

a) $\Gamma: \ x = f(t), \ y = g(t), \ z = h(t)$,

b) $\Gamma: \ F(x, y, z) = 0, \ G(x, y, z) = 0.$

Solution.

a) Since generatrix ℓ has direction numbers a, b, c and passes through $(f(t), g(t), h(t))$ one has



$$S: \ \frac{x-f(t)}{a} = \frac{y-g(t)}{b} = \frac{z-h(f)}{c}$$

as the symmetric equations of the cylinder.

When t is eleminated one obtains

$$S: \ H(x, y, z) = 0$$

b) If $P(x, y, z)$ is any point on ℓ (a point of S) any other point Q on ℓ is

$$Q(x+at, \ y+bt, \ z+ct).$$

Taking Q on Γ, one has

$$F(x+at, \ y+bt, \ z+ct) = 0, \qquad G(x+at, \ y+bt, \ z+ct) = 0,$$

and the elimination of t gives

$$S: \ H(x, y, z) = 0.$$

Note that cases (a) and (b) are reducible into each other.

Example. Obtain the cartesian equations of the cylinaers defined by

a) $\Delta = (-1, 2, 3)$, $\quad \Gamma: \ x = t, \ y = t^2, \ z = 0$

b) $\Delta = (-1, 2, 3)$, $\quad \Gamma: \ x^2 = 2z, \ y-z = 1$

Solution.

a) $\frac{x-t}{-1} = \frac{y-t^2}{2} = \frac{t}{3}$

$\Rightarrow \quad 2x-2t = -y+t^2, \quad 3y-3t^2 = 2z$

$\Rightarrow \quad 6x-6t = -3y+3t^2, \quad 3t^2 = 3y-2z$

$\Rightarrow \quad 6x-6t = -3y+3y-2z \quad \Rightarrow \quad t = x-z/3$

$\Rightarrow \quad 3y-3(x-z/3)^2 = 2z$

$\Rightarrow \quad 9x^2+z^2-6xz-9y+6z = 0.$

b) Taking $Q(x-t,\ y+2t,\ z+3t)$ on Γ one has

$(x-t)^2 = 2(z+3t), \quad (y+2t)-(z+3t) = 1$

$\Rightarrow \quad (x-t)^2 = 2(z+3t), \quad t = y-z-1$

$\Rightarrow \quad (x-y+z+1)^2 = 2(z+3y-3z-3)$

$\Rightarrow \quad x^2+y^2+z^2-2xy+2xz-2yz+2x-8y+6z+7 = 0.$

2. Cones.

A surface S generated by a variable line ℓ passing through a fixed point P_0 and subject to another condition such as intersecting a curve Γ (or remaining tangent to a given surface Σ) is called a cone.

The line ℓ is the generatrix, Γ the directrix and P_0 the vertex of the cone S, and we say that S is defined by P_0 and Γ.

Equation of a cone

The equation of the cones defined by $P_0(x_0, y_0, z_0)$ and

a) Γ: $x = f(t), \quad y = g(t), \quad z = h(t)$,

b) Γ: $F(x, y, z) = 0, \quad G(x, y, z) = 0$

Solution.

a) Since ℓ passes through P_0 and its direction numbers are $x_0-f(t)$, $y_0-g(t)$, $z_0-h(f)$, one has

$$S: \frac{x-x_0}{x_0-f(t)} = \frac{y-y_0}{y_0-g(t)} = \frac{z-z_0}{z_0-h(t)}$$

as the symmetric equations of the cone.

When t is eliminated one obtains

$$S: H(x, y, z) = 0$$

b) If $P(x, y, z)$ is any point on the generatrix ℓ (a point of S), then any other point Q on ℓ has coordinates

$$\frac{x+tx_0}{1+t}, \quad \frac{y+ty_0}{1+t}, \quad \frac{z+tz_0}{1+t}$$

Setting these on $F=0$, $G=0$ and eliminating t one gets

$$S: H(x, y, z) = 0$$

Example. Obtain the cartesian equations of the cones defined by

a) $P_0(0, 0, 1)$, $\quad \Gamma: x = 2t, \quad y = t^2, \quad z = 0$

b) $P_0(0, 0, 1)$, $\quad \Gamma: x^2 = 2z, \quad y-z = 1$

Solution.

a) $\dfrac{x}{2t} = \dfrac{y}{t^2} = \dfrac{z-1}{-1}$

$\Rightarrow \quad t^2x = 2ty, \qquad -x = 2t(z-1)$

$\Rightarrow \quad tx = 2y, \qquad -x = 2t(z-1)$

$\Rightarrow \quad -x = \dfrac{4y}{x}(z-1) \quad \Rightarrow \quad -x^2 = 4y(z-1)$

$\Rightarrow \quad x^2+4yz-4y = 0$

b) Having $Q(\frac{x+0}{1+t}, \frac{y+0}{1+t}, \frac{z+t}{1+t})$ on the generatrix, and setting its coordinates on $F=0$, $G=0$:

$$(\frac{x}{1+t})^2 = 2\frac{z+t}{1+t}, \qquad \frac{y}{1+t} - \frac{z+t}{1+t} = 1$$

$$\Rightarrow \quad x^2 = 2(1+t)(z+t), \quad y-(z+t) = 1+t$$

$$\Rightarrow \quad x^2 = 2(1+t)(z+t), \quad 2t = y-z-1$$

$$\Rightarrow \quad x^2 = (2+y-z-1)(z+\frac{y-z-1}{2})$$

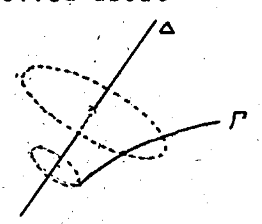
$$\Rightarrow \quad 2x^2 = (y-z+1)(y+z-1)$$

$$\Rightarrow \quad 2x^2 = y^2-(z-1)^2$$

$$\Rightarrow \quad 2x^2-y^2+z^2-2z+1 = 0.$$

3. Surfaces of revolution:

A surface S generated when a curve Γ is revolved about a line Δ is called a surface of revolution.



Γ is the generatrix and Δ the axis of the surface, and we say that S is defined by Δ and Γ.

Spheres, rihgt circular cylinders and cones are examples of surface of revolution.

Every point of Γ describes a circle with the center on the axis Δ, called a "parallel", and any plane through Δ intersects the surface along a curve called a "meridian" of S.

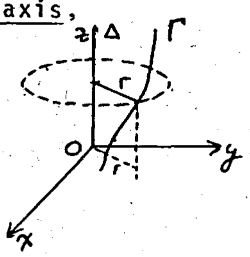
The meridians are congruent curves of S, and S may be generated by revolving any meridian about the axis Δ.

An equivalent definition of a surface of revolution is the following: A surface of revolution is the locus of a variable circle with given axis Δ and subject to another condition such as intersecting a given curve Γ (or remaining tangent to a given surface Σ).

Equation of a surface of revolution:

The equation of the surface defined by $\Delta \simeq$ z-axis,

Γ: $x = f(t), \quad y = g(t), \quad z = h(t)$



Solution: The equation of the "parallel" -circle- through $P(f(t), g(t), h(t)$ being

$$x^2+y^2 = r^2, \quad z = h(t)$$

where $r^2 = f^2(t)+g^2(t)$, we have the parametric equation

$$S: \quad x^2+y^2 = f^2(t)+g^2(t), \; z = h(t)$$

Eleminating t between the two relations, one gets

$$S: \quad F(x, y, z) = 0$$

Example. Find the cartesian equation of the surface generated by revolving

$$\Gamma: \; x = t, \quad y = t-2, \quad z = t^2$$

about z-axis.

Solution.

$$x^2+y^2 = r^2, \quad z = t^2, \text{ with } r^2 = t^2+(t-2)^2$$

$$\Rightarrow \quad x^2+y^2 = t^2+(t-2)^2, \quad z = t^2$$

$$\Rightarrow \quad x^2+y^2 = 2t^2-4t+4, \quad z = t^2$$

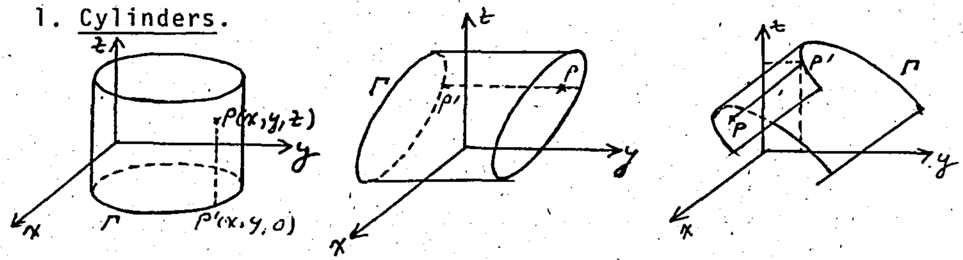
$$\Rightarrow \quad x^2+y^2 = 2z-4t+4, \quad z = t^2$$

$$\Rightarrow \quad (x^2+y^2-2z-4)^2 = 16z.$$

Particular case:

Particular cases of cylinders, cones, surfaces of revolutions are obtained by taking the generatrix Γ on a coordinate plane.

1. Cylinders.



Γ:	$f(x, y)=0$, $z=0$	$g(x, z)=0$, $y=0$	$h(y, z)=0$, $x=0$
Δ:	// z-axis	// y-axis	// x-axis

Let the directrix Γ be on the xy-plane with equation $f(x, y) = 0$, $z = 0$ and direction Δ // z-axis. Then the equation of this cylinder is

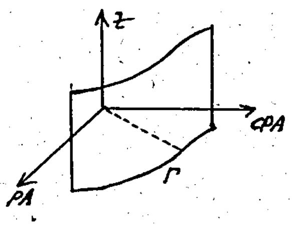
$$f(x, y) = 0 \quad \text{(for any } z\text{)}.$$

Similarly $g(x, z) = 0$, and $h(y, z) = 0$ are the equations of cylinders having directrices on xz- and yz-planes and generatrices parallel to y- and x-axis.

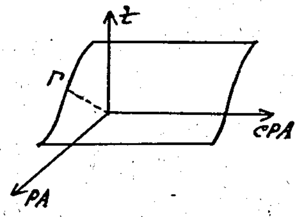
These particular cylinders are right cylinders since generatrices are perpendicular to the plane of the directrix.

Observe that in cylindrical coordinates, $F(\theta, r) = 0$ represents a right cylinder with generatrix // z-axis.

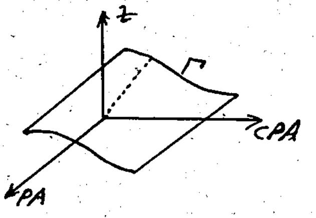
$G(r \cos\theta, z) = 0$ represents a rihgt cylinder having directrix on the plane $\theta = 0$, and $H(r \sin\theta, z) = 0$ represents a right cylinder having directrix on the plane $\theta = \pi/2$.



$F(\theta, r)=0, \quad z=0$

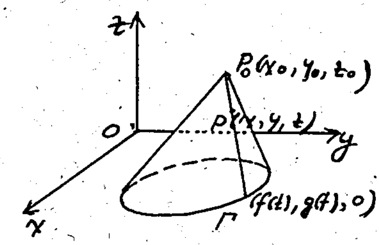


$G(r\cos\theta, z) = 0$



$H(r\sin\theta, z) = 0$

2. <u>Cones</u>



The equation of the cone with directrix

$$\Gamma: (f(t), g(t), 0)$$

on xy-plane and vertex at $P_0(x_0, y_0, z_0)$ is

$$S: \frac{x-x_0}{x_0-f(t)} = \frac{y-y_0}{y_0-g(t)} = \frac{z-z_0}{z_0}$$

which yields

$$S: F(x, y, z) = 0$$

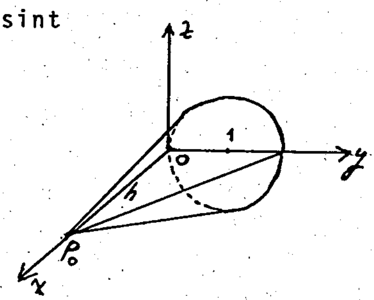
when t is eleminated.

Similar equations are obtained when Γ lies on xz-plane or yz-plane.

<u>Example</u>. Write the equation of the cone with vertex at $(h, 0, 0)$ and directrix $\Gamma: (y-1)^2+z^2 = 1, \quad x = 0.$

<u>Solution</u>. Writing the equations of Γ in parametric form:

$$\Gamma: x = 0, \quad y = 1+\cos t, \quad z = \sin t$$



we have

$$S: \frac{x-h}{h-0} = \frac{y}{0-(1+\cos t)} = \frac{z-0}{0-\sin t}$$

or

$$\frac{x-h}{h} = \frac{y}{-(1+\cos t)} = \frac{z}{-\sin t}$$

$$\Rightarrow \quad 1+\cos t = -\frac{hy}{x-h}\,, \qquad \sin t = -\frac{hz}{x-h}$$

$$\Rightarrow \quad \left(-\frac{hy}{x-h} - 1\right)^2 + \left(-\frac{hz}{x-h}\right)^2 = 1$$

$$\Rightarrow \quad (x-h+hy)^2 + h^2z^2 = (x-h)^2$$

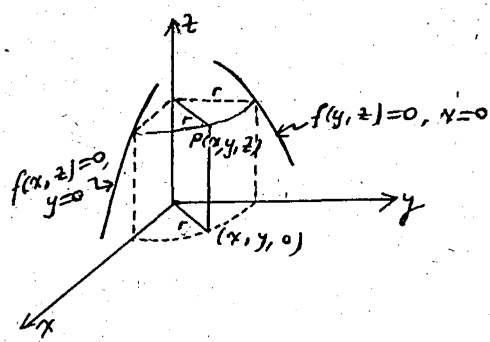
$$\Rightarrow \quad (x+hy-h)^2 - (x-h)^2 + h^2z^2 = 0$$

3. Surfaces of revolution.

If the generatrix Γ lies on yz-plane (or on xz-plane) with equation $f(y, z) = 0$, $x=0$ (or $f(x, z)$, $y=0$) and Oz is the axis of revolution Δ, then as Γ revolves about z-axis, any point P_0 on Γ describes a circle

$$x^2+y^2 = r^2, \quad z = z$$

with center on z-axis and radius r, where r satifies $f(r, z) = 0$.



Then eliminating r between $x^2+y^2 = r^2$ and $f(r, z) = 0$, one gets the required equation

$$S: f(\pm\sqrt{x^2+y^2}, z) = 0$$

of the surface of revolution S,

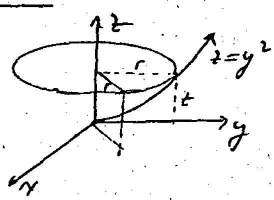
The other similar cases, where Δ is on Ox or Oy, are treated similarly.

Example. Given the curve Γ: $z = y^2$ on yz-plane, obtain the equation of the surfaces of revolution when Γ is revolved about

a) z-axis b) y-axis

Solution.

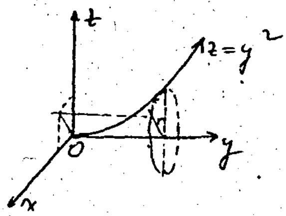
a)



$x^2+y^2= r^2, \quad z=r^2$

$\Rightarrow \quad x^2+y^2 = z$

b)



$x^2+z^2 = r^2, \quad y^2=r$

$\Rightarrow \quad x^2+z^2 = y^4$

D. QUADRICS (In Standard forms)

The following standard equations of second degree represent surfaces, called quadrics with certain orientations, where $a, b, c > 0$.

a) Ellipsoids (and spheres)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad \text{(Sphere if } a=b=c)$$

b) Hyperboloids

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \qquad \text{(Hyperboloid of one sheet)}$$

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad \text{(Hyperboloid of two sheets)}$$

c) Paraboloids:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \lambda z \qquad \text{(Elliptic paraboloid)}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \lambda z \qquad \text{(Hyperbolic paraboloid)}$$

Sketching:

a) Since in $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ the terms are non negative, it follows that $|x/a|\leqslant 1$, $|y/b|\leqslant 1$, $|z/c|\leqslant 1$ or $-a\leqslant x\leqslant a$, $-b\leqslant y\leqslant b$, $-c\leqslant z\leqslant c$ and the surface is bounded.

Setting $z = k$ $(|k|\leqslant c)$, one has

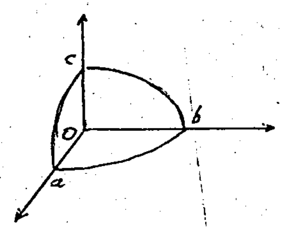
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{k^2}{c^2} \qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{c^2-k^2}{c^2}$$

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1 \quad \text{with } \alpha=\frac{a}{c}\sqrt{c^2-k^2},\ \beta=\frac{b}{c}\sqrt{c^2-k^2}$$

showing that cross sections parallel to xy-planes are ellipses. Similarly cross sections parallel to xz- and yz-planes are ellipses. Then the surface is called an ellipsoid.

a, b, c are the semi axes and 0 the center of the ellipsoid.

If a=b=c, then the cross sections are all circles, and the surface is a sphere.



Ellipsoid in the I. octant

If two of a,b,c are equal the ellipsoid is of revolution and called a spheroid, which is oblate (prolate) if the third semi axis is smaller (larger) then the other.

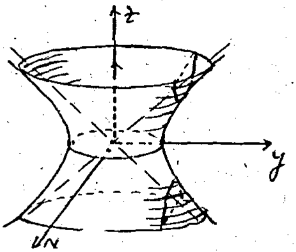
The equation

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}+\frac{(z-\ell)^2}{c^2}=1$$

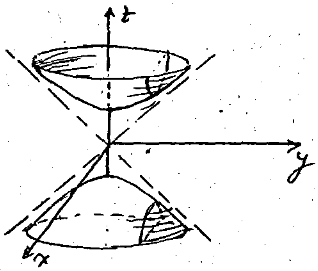
represents also an ellipsoid centered at (h, k, ℓ) with the same orientation as previous one.

b) $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ (H_1), $\quad -\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ (H_2)

The cross sections of H_1 and H_2 are hepherbolas for $x=k$ or for $y=k$ and ellipses for $z=k$. Since cross sections are hyperbolas in two ways, they are called <u>hyperboloid</u>. The cross sections in H_1 are ellipses for $z=k$ for any k and the surface consists of a single piece (sheet), while in H_2 the sections are ellipses only for $|k| > c$. Hence the surface consists of two disjoint pieces (sheets). Accordingly H_1 is called a <u>hyperboloid of one sheet</u>, H_2 a <u>hyperboloid of two sheets</u>.



A Hyperboloid of one sheet



A hyperboloid of two sheets

a, b, c are semi-axes and 0 the center of the surface. They are of revolution when $a=b$.

Similar surfaces are obtained for other combinations of signs.

The equations

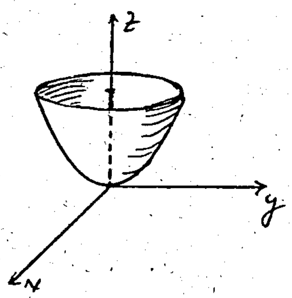
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} - \frac{(z-\ell)^2}{c^2} = 1, \quad -\frac{(x-b)^2}{a^2} - \frac{(y-k)^2}{b^2} + \frac{(z-\ell)^2}{c^2} = 1$$

represent clearly hyperboloids with the same orientation of axes but center at (h, k, ℓ).

c) Consider the equations

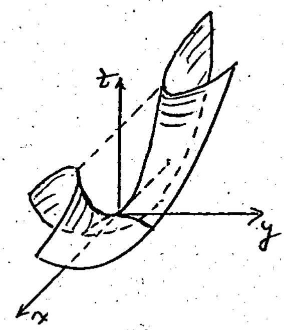
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \lambda z \quad (EP), \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = \lambda z \quad (HP)$$

The cross sections of EP and HP for $x=k$ or $y=k$ are parabolas and ellipses (hyperbolas) for $z=k$ in EP (HP). Since sections are parabolas in two ways are called paraboloids. The third cross sections in EP (HP) being ellipses (hyperbolas) they are respectively called elliptic parabaloid (EP), hyperbolic paraboloid (HP).



$\lambda > 0$

An elliptic paraboloid



$\lambda > 0$

A hyperbolic paraboloid

The origin in H_1 and H_2 is called the vertex. The hyperbolic paraboloid is of saddle shape in the neighborhood of the origin and the origin is called the saddle point of the surface, and the surface H_2 is sometimes called a saddle shape surface.

Similar results are obtained when x or y are linear instead of z.

The equations

$$\frac{(x-h)^2}{a^2} \pm \frac{(y-k)^2}{b^2} = \lambda(z-\ell)$$

represent clearly paraboloids having vertex at (h, k, ℓ).

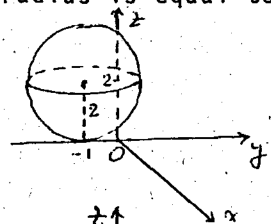
Example 1. Sketch

a) the sphere $(x+1)^2 + y^2 + (z-2)^2 = 4$

b) the ellipsoid $4x^2+y^2+z^2 = 16$

<u>Solution</u>.

a) The center is at $(-1, 0, 2)$ and radius is equal to 2.



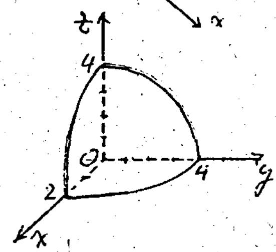
b) Writing it in the form

$$\frac{x^2}{2^2} + \frac{y^2}{4^2} + \frac{z^2}{4^2} = 1$$

one has

$$a = 2, \quad b = 4, \quad c = 4$$

and that part in the I. octant is shown in the figure.

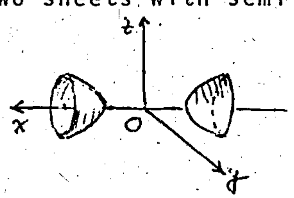


<u>Example 2</u>. Sketch the quadrics:

a) $\frac{x^2}{4} - \frac{y^2}{9} - \frac{z^2}{4} = 1$ b) $\frac{x^2}{4} - z^2 = 2y$

<u>Solution</u>.

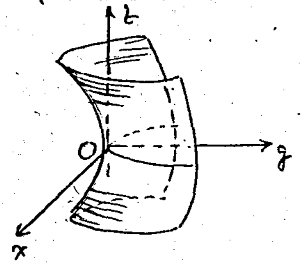
a) The surface is a hyperboloid of two sheets with semi axes $a = 2$, $b = 3$, $c = 2$ admitting Ox as axis. Cross sections // yz-plane cease to exist when $-2 < x < 2$.



b) The surface is a hyperbolic paraboloid (or a saddle shaped surface).

$$x = 0 \implies y = -\frac{z^2}{2} \quad \text{(a parabola)}$$

$$z = 0 \implies y = x^2/8 \quad \text{(a parabola)}$$



E. <u>SECOND DEGREE SURFACES</u>.

The quadrics being second degree surfaces, their equations are included in the general equation

$$A_1x^2 + A_2y^2 + A_3z^2 + B_1yz + B_2zx + B_3xy$$
$$+ C_1x + C_2y + C_3z + D = 0 \qquad (1)$$

of second degree where the terms with coefficients A_i, B_i, C_i. and D are called quadratic terms, cross terms, linear terms and constant term respectively.

If (1) does not involve a cross term, by completing the squares, it can be transformed into the standard form:

$$A_1(x-x_0)^2 + A_2(y-y_0)^2 + A_3(z-z_0)^2 + D' = 0$$

If there is only one cross term, say B_3xy, a rotation of xy-plane about z-axis, (by a proper angle) eliminates the cross term:

$$A'_1x'^2+A'_2y'^2+A_3z^2+C'_1x'+C'_2y'+C_3z+D' = 0$$

If more than one cross term are present, the elimination of them can oly be performed by a rotation of the system Oxyz about a line through the origin.

The full discussion of reduction of (1) into a standard form (including degenerate cases) is given in LINEAR ALGEBRA by writing (1) in matrix form as follows:

(1) is rendered homogeneous by introducing a variable of homogeneity t:

$$A_1x^2+A_2y^2+A_3z^2+B_1yz+B_2zx+B_3xy$$
$$+C_1xt+C_2yt+C_3zt+Dt^2 = 0 \qquad (1')$$

Next one denotes x, y z and t by x_1, x_2, x_3 and x_4 and sets

$$A_1 = a_{11}, \qquad A_2 = a_{22}, \qquad A_3 = a_{33}$$

$$B_1 = 2a_{23}, \qquad B_2 = 2a_{31}, \qquad B_3 = 2a_{13}$$

$$C_1 = 2a_{14}, \qquad C_2 = 2a_{24}, \qquad C_3 = 2a_{34}, \quad D = a_{44}$$

with $a_{ij} = a_{ji}$. Then (1') becomes

$$\begin{aligned} & a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 \\ & + 2a_{23}\, x_2x_3 + 2a_{31}\, x_3x_1 + 2a_{12}\, x_1x_2 \\ & + 2a_{14}\, x_1x_4 + 2a_{24}\, x_2x_4 + 2a_{34}\, x_3x_4 + a_{44}\, x_4^2 = 0 \end{aligned}$$

or

$$\begin{aligned} & a_{11}\, x_1^2 + a_{12}\, x_1x_2 + a_{13}\, x_1x_3 + a_{14}\, x_1x_4 \\ & + a_{21}\, x_2x_1 + a_{22}\, x_2^2 + a_{23}\, x_2x_3 + a_{24}\, x_2x_4 \\ & + a_{31}\, x_3x_1 + a_{32}\, x_3x_2 + a_{33}\, x_3^2 + a_{34}\, x_3x_4 \\ & + a_{41}\, x_4x_1 + a_{42}\, x_4x_2 + a_{43}\, x_4x_3 + a_{44}\, x_4^2 = 0 \end{aligned} \tag{1"}$$

which is seen to be the same as

$$[x_1 \ x_2 \ x_3 \ x_4] \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0, \ (x_4 = 1) \quad (2)$$

where the matrix $[a_{ij}]$ is symmetric.

When (2) is written in the original notation, we have (for $x_4 = 1$)

$$[x \quad y \quad z \quad 1] \begin{bmatrix} A_1 & \frac{1}{2}B_3 & \frac{1}{2}B_2 & \frac{1}{2}C_1 \\ \frac{1}{2}B_3 & A_2 & \frac{1}{2}B_1 & \frac{1}{2}C_2 \\ \frac{1}{2}B_2 & \frac{1}{2}B_1 & A_3 & \frac{1}{2}C_3 \\ \frac{1}{2}C_1 & \frac{1}{2}C_2 & \frac{1}{2}C_3 & D \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0 \quad (2')$$

It is proved that the surface represented by (1) or by (2) is a quadric if $\det(a_{ij}) \neq 0$, and degenerate if $\det(a_{ij}) = 0$, or (1) is a quadric or degenerate according as

$$T = \begin{vmatrix} 2A_1 & B_3 & B_2 & C_1 \\ B_3 & 2A_2 & B_1 & C_2 \\ B_2 & B_1 & 2A_3 & C_3 \\ C_1 & C_2 & C_3 & 2D \end{vmatrix}$$

is not zero or zero.

Quadrics are non degenerate surfaces, while cylinders, cones and sets of two planes are degenerate ones

Example 1. Transform the following into standard forms and test them for degeneracy

a) $9x^2 - 4y^2 + z^2 + 18x - 4z + 13 = 0$

b) $x^2 - y^2 + z^2 - 2xz + 2x - 4y - 2z - 3 = 0$

Solution.

a) Since there are no cross terms, standard equations are obtained by completing squares:

$$(9x^2+18x)-4y^2+(z^2-4z)+13 = 0$$

$$\Rightarrow 9(x^2+2x)-4y^2+(z-2)^2-4+13 = 0$$

$$\Rightarrow 9(x+1)^2-9-4y^2+(z-2)^2+9 = 0$$

$$\Rightarrow 9(x+1)^2-4y^2+(z-2)^2 = 0 \quad \text{(cone, vertex at } (-1, 0, 2))$$

Since

$$T = \begin{vmatrix} 18 & 0 & 0 & 18 \\ 0 & -8 & 0 & 0 \\ 0 & 0 & 2 & -4 \\ 18 & 0 & -4 & 26 \end{vmatrix} = 0$$

the cone is degenerate,

b) Since there is only one cross term, namely $-2xz$, the standard equation is obtained by rotating $0xz$ about 0 by a proper angle θ:

$$(x^2-2xz+z^2)-y^2+2x-4y-2z-3 = 0 \qquad (1)$$

$$\tan 2\theta = \frac{-2}{1-1} = \infty \quad \Rightarrow \quad \theta = \pi/4 \quad \Rightarrow \cos\theta = \sin\theta = \sqrt{2}/2$$

$$x = \frac{\sqrt{2}}{2}(x'-z'), \quad z = \frac{\sqrt{2}}{2}(x'+z'), \quad y = y'.$$

Setting these values in (1), we have

$$\frac{1}{2}(x'-z')^2 - 2.\frac{2}{4}(x'^2-z'^2) + \frac{1}{2}(x'+z')^2-y'^2$$

$$+ \sqrt{2}(x'-z')-4y'-\sqrt{2}(x'+z')-3 = 0$$

$$4z'^2-2y'^2-8y'-4\sqrt{2}z'-6 = 0$$

$$2(z' - \frac{\sqrt{2}}{2})^2 - (y'+2)^2 = 0 \quad \text{(two intersecting planes)}$$

as in (a) there is degeneracy.

Example .2.

a) Find the locus of points equidistant from the point $F(0, 0, 2)$ and the plane $\pi: \quad z = -2$.

b) Sketch the obtained locus

Solution.

a) Let $P(x, y, z)$ be equidistant from $F(0, 0, 2)$ and $\pi: \quad z = -2$. Then

$$d(P, F) = \sqrt{x^2+y^2+(z-2)^2}, \qquad d(P, \pi) = |z+2|$$

$$\Rightarrow \quad x^2+y^2+(z-2)^2 = (z+2)^2 \Rightarrow x^2+y^2 = 8z. \quad \text{(EP)}$$

b) Domain: $\mathbb{R}^2$

Traces: xy-trace: $z=0 \Rightarrow x^2+y^2 = 0$ (origin)

xz-trace: $y=0 \Rightarrow x^2+8z =$ (parabola)

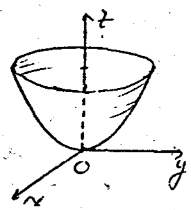
yz-trace: $x=0 \Rightarrow y^2=8z$ (parabola)

cross sections:

// xy-plane : $z=k$ $\quad x^2+y^2 = k$ (circles for k 0)

// xz-plane : $y=k$ $\quad x^2+k^2 = 8z$ (parabolas)

// yz-plane : $x=k$ $\quad y^2+k^2 = 8z$ (parabolas)



EXERCISES (3, 3)

96. Plot the following points in a cylindrical coordinate system:
$A(\pi/4, 3, 4)$, $B(\pi/2, 4, 1)$, $C(0, -2, 1)$, $D(0, 0, 2)$.

97. Plot the following points in a spherical coordinate system:
$A(\pi/4, \pi/2, 2)$, $B(\pi/2, \pi/4, 3)$, $C(0, \pi/2, -2)$, $D(\pi/4, \pi/4, 1)$

98. Sketch each of the surfaces given by equations in spherical coordinates:

a) $\rho = 5$, b) $\varphi = 2\pi/3$, c) $\theta = \pi/2$, d) $\rho = 2\cos\varphi$.

99. Sketch the surfaces:

a) $x^2+y^2 = 36$ b) $y^2+4z^2 = 0$, c) $x^2-z^2 = 16$

Is there any degenerate one?

100. Same question for:

a) $x^2+16y^2-4x = 0$ b) $x^2+z^2-4y = 0$, c) $x^2-y^2 = z^2$

101. Find the projections of the following curves on the coordinate planes:

a) $z = x^2+y^2$, $z = 4y$ b) $x^2+z^2 = 9$, $y^2+z^2 = 4$

c) $x^2+y^2+z^2 = 4$, $x+y+z = 2$.

102. Find the equation of the cylinder with directrix Γ and direction Δ given below:

a) Γ : $z^2+2x = 8$, $y=0$, Δ // y-axis

b) Γ : $x = 2+t$, $y = t^2$, $z = t^3+1$, Δ// x-axis

103. Find the equation of the cone with given vertex V and directrix Γ :

a) $V(0, 0, 0)$, Γ: $x^2+y^2 = 16$, $z = 2$

b) $V(3, 1, 2)$, Γ: $x = 2+t$, $y = t^2$, $z = t^3+1$.

104. Find the equations of the surfaces of revolution with given generatrix Γ and axis Δ:

a) Γ: $y^2 = 4x-16$, Δ: x-axis

b) Γ: $y^2 = 2pz$, Δ: z-axis

c) $\Gamma: \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad \Delta:$ y-axis

105. Find the locus of points equidistant from

a) $(0, 0, 0)$ and $\pi: z = -4$

b) $(3, 1, -5)$ and $\pi: x+2y-2z = 1$

106. Construct and discuss the surfaces:

a) $z^2 = 16$ b) $4y^2-25 = 0$ c) $3y^2+7z^2 = 0$

d) $4x^2+z^2 = 16$ e) $y^2-9z^2 = 0$ f) $16x^2-4y^2-z^2 = 0$

107. Same question for:

a) $4x^2+9y^2+16z^2 = 144$ b) $9x^2-y^2+9z^2 = 36$

c) $4x^2-9$[illegible]$^2 = 144$ d) $x^2+y^2-z^2 = 25$

108. Construct th[illegible]olid, in the I. octant, bounded by

a) $x^2+y^2 = a^2, \quad z = 2mx \quad (a>0, \ m>0)$

b) $x^2+y^2 = az, \quad x^2+y^2 = 2ax, \quad z=0 \quad (a>0)$

109. Determine the relative positions of the line and quadric Q:

a) $\ell: \frac{x+6}{2} = \frac{y-6}{-2} = \frac{z-3}{-1}, \quad Q: x^2+y^2+4x^2 = 16$

b) $\ell: x = 3+4t, \quad y = \frac{5}{2}t, \quad z = -2t, \quad Q: x^2-z^2 = 2y.$

c) $\ell: \frac{x-2}{2} = \frac{y+2}{2} = z-5, \quad Q: x^2+y^2+z^2 = 36$

d) $\ell: x = 6t, \quad y = 9+3t, \quad z = 1-2t, \quad Q: y^2+4z^2 = 8x.$

110. Write the following in standard form:

a) $x^2+y^2+z^2-4x+2y+6z-9 = 0$ b) $4x^2-4y^2+16x+8z = 0$

111. Same question for:

a) $x^2+y^2+z^2+2xy = 0$ b) $y^2-z^2-4xz = 0$

112. Write $x^2+3y^2+2z^2+2xy+3x+2y+1 = 0$ in matrix form.

113. Show that the following are degenerate second degree surfaces:

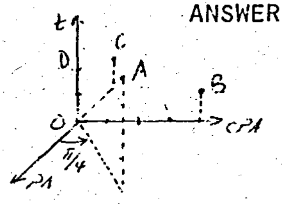
a) $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ b) $z^2 = ax+by$

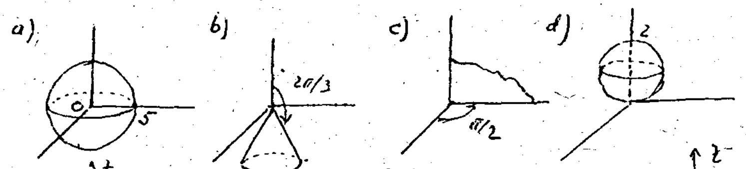
c) $Ax_1^2+Bx_1x_2+Cx_2^2+Dx_1+Ex_2+F = 0$

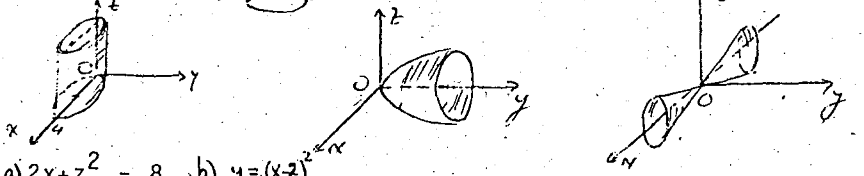
114. Find the surface through the curve of intersection of $z = x^2+2y^2$, $3x+4y = 0$ and passing through the point (0, 1, 4).

115. Write the equation of second degree surface passing through the nine points: (0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1), (0, 1, 1), (1, 0, 1), (1, 1, 0), (1, 1, 1), (1, 2, 3)

ANSWERS TO EVEN NUMBERED EXERCISES

96. 

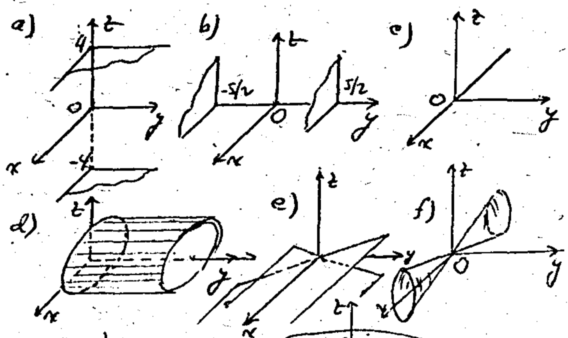
98. 

100. 

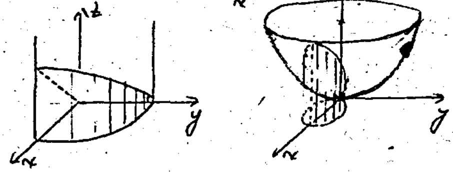
102. a) $2x+z^2 = 8$, b) $y=(x-2)^2$

104. $y^2+z^2 = 4x-16$, b) $x^2+y^2 = 2pz$, c) $\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{a^2} = 1$

106.



108.



110. a) $(x-2)^2+(y+1)^2+(z+3)^2 = 23$

b) $(x+2)^2 - y^2 = -2z-4$

112.

$$\begin{bmatrix} x & y & z & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 3/2 \\ 1 & 3 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 3/2 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0,$$ non deg.

114. $2x^2+4y^2+3x+4y-2z = 0.$

3.4. SPACE CURVES

A. <u>VECTOR FUNCTIONS</u> (of one variable).

If I is an interval on a t-axis, then the function

$$\vec{r}: I \to R^3$$

is called a <u>vector function</u> of one variable from reals to vectors in 3-space.

The value of $\vec{r}$ at $t \varepsilon I$ is $\vec{r}(t)$. If the position vector $\vec{r}(t)$ has scalar components $x(t)$, $y(t)$, $z(t)$, then

$$\vec{r}(t) = x(t)i + y(t)j + z(t)k$$

or

$$\vec{r}(t) = (x(t), y(t), z(t))$$

For the function $\vec{r}(t)$, limit, derivative, differential, continuity and integrals are defined as follows:

$$\lim_{t \to t_0} \vec{r}(t) = (\lim_{t \to t_0} x(t), \lim_{t \to t_0} y(t), \lim_{t \to t_0} z(t))$$

$$\frac{d}{dt} \vec{r}(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t \; \Delta t) - \vec{r}(t)}{\Delta t}$$

$$= (\frac{d}{dt} x(t), \frac{d}{dt} y(t), \frac{d}{dt} z(t))$$

$$d\vec{r}(t) = (d\,x(t), d\,y(t), d\,z(t))$$

$$= (\dot{x}\,dt, \dot{y}\,dt, \dot{z}\,dt)$$

$$= (\dot{x}, \dot{y}, \dot{z})dt$$

The notations for the first derivative are

$$\frac{d}{dt}\vec{r}(t), \quad \frac{d\vec{r}}{dt}, \quad D_t\vec{r}(t), \quad \vec{r}(t)$$

$\vec{r}(t)$ is said to be continuous at t_0 if

$$\lim_{t \to t_0} \vec{r}(t) = \vec{r}(t_0) = (x(t_0),\ y(t_0),\ z(t_0))$$

or if $x(t)$, $y(t)$, $z(t)$ are continuous at t_0.

If $\vec{r}(t)$ is continuous, then the integrals of $x(t)$, $y(t)$, $z(t)$ exist, and

$$\int \vec{r}(t)dt = (\int x dt,\ \int y dt,\ \int z dt)$$

$$\int_\alpha^\beta \vec{r}(t)dt = (\int_\alpha^\beta x dt,\ \int_\alpha^\beta y dt,\ \int_\alpha^\beta z dt).$$

From now on, the arrows of vector function will be omitted.

Theorem. If the vector functions $F(t)$, $G(t)$, $H(t)$ and the scalar function $u(t)$ are differentiable on $[\alpha, \beta]$, then

1. $(F \pm G)^\cdot = \dot{F} \pm \dot{G}$
2. $(uF)^\cdot = \dot{u}F + u\dot{F}$
3. $(F \cdot G)^\cdot = \dot{F} \cdot G + F \cdot \dot{G}$
4. $(F\times G)^\cdot = \dot{F}\times G + F\times\dot{G}$
5. $(FGH)^\cdot = (\dot{F}\ G\ H) + (F\ \dot{G}\ H) + (F\ G\ \dot{H})$
6. $\left((F\times G)\times H\right)^\cdot = (\dot{F}\times G)\times H + (F\times\dot{G})\times H + (F\times G)\times\dot{H}.$

where "dot" means derivative with respect to the parameter t.

Proof. Let $F = (F_1, F_2, F_3)$, $G = (G_1, G_2, G_3)$, $H = (H_1, H_2, H_3)$.

1. $$\begin{aligned}(F \pm G)^\cdot &= (F_1 \pm G_1,\ F_2 \pm G_2,\ F_3 \pm G_3)^\cdot \\ &= ((F_1 \pm G_1)^\cdot,\ (F_2 \pm G_2)^\cdot,\ (F_3 \pm G_3)^\cdot) \\ &= (\dot{F}_1 \pm \dot{G}_1,\ \dot{F}_2 \pm \dot{G}_2,\ \dot{F}_3 \pm \dot{G}_3) \\ &= (\dot{F}_1, \dot{F}_2, \dot{F}_3) \pm (\dot{G}_1, \dot{G}_2, \dot{G}_3) = \dot{F} \pm \dot{G}\end{aligned}$$

2. $(uF)^{\cdot} = ((uF_1)^{\cdot}, (uF_2)^{\cdot}, (uF_3)^{\cdot})$

$= (\dot{u}F_1 + u\dot{F}_1, \dot{u}F_2 + u\dot{F}_2, \dot{u}F_3 + u\dot{F}_3)$

$= (\dot{u}F_1, \dot{u}F_2, \dot{u}F_3) + (u\dot{F}_1, u\dot{F}_2, u\dot{F}_3)$

$= \dot{u}\,F + u\,\dot{F}$

3. $(F.G)^{\cdot} = (F_1G_1 + F_2G_2 + F_3G_3)^{\cdot}$

$= (\dot{F}_1G_1 + \dot{F}_2G_2 + \dot{F}_3G_3) + (F_1\dot{G}_1 + F_2\dot{G}_2 + F_3\dot{G}_3)$

$= \dot{F}.G + F.\dot{G}$

4. $(F\times G)^{\cdot} = ((F_2G_3 - F_3G_2)^{\cdot}, (F_3G_1 - F_1G_3)^{\cdot}, (F_1G_2 - F_2G_1)^{\cdot})$

$= (\dot{F}_2G_3 - \dot{F}_3G_2, \dot{F}_3G_1 - \dot{F}_1G_3, \dot{F}_1G_2 - \dot{F}_2G_1)$

$+ (F_2\dot{G}_3 - F_3\dot{G}_2, F_3\dot{G}_1 - F_1\dot{G}_3, F_1\dot{G}_2 - F_2\dot{G}_1)$

$= \dot{F} \times G + F \times \dot{G}$

5. Since the mixed product $(F\,G\,H)$ is equal to $F\times G.H$ (or to $F.G\times H$) the derivative is obtained by the application of the properties (3) and (4).

6. The proof is obtained by applying twice the property (4).

The property (5) gives a rule for differentiating .. ter-minant of order 3 where elements are functions of the sam variable:

$$(FGH) = \begin{vmatrix} F_1 & F_2 & F_3 \\ G_1 & G_2 & G_3 \\ H_1 & H_2 & H_3 \end{vmatrix}$$

$\Rightarrow$

$$\frac{d}{dt}\begin{vmatrix} F_1 & F_2 & F_3 \\ G_1 & G_2 & G_3 \\ H_1 & H_2 & H_3 \end{vmatrix} = \begin{vmatrix} \dot F_1 & \dot F_2 & \dot F_3 \\ G_1 & G_2 & G_3 \\ H_1 & H_2 & H_3 \end{vmatrix} + \begin{vmatrix} F_1 & F_2 & F_3 \\ \dot G_1 & \dot G_2 & \dot G_3 \\ H_1 & H_2 & H_3 \end{vmatrix} + \begin{vmatrix} F_1 & F_2 & F_3 \\ G_1 & G_2 & G_3 \\ \dot H_1 & \dot H_2 & \dot H_3 \end{vmatrix}$$

This rule of differentiation holds for determinants of any order.

<u>Example</u>. Given the vector function

$$\vec{r}(t) = \text{Ch } t \; i - \text{Sh } 2t \; j + e^{-t} \; k$$

evaluate the following

a) $D\,\vec{r}(\ell n\ 3) = D\,\vec{r}(t)\Big|_{t=\ell n\ 3}$ b) $\int_0^{\ell n\ 2} \vec{r}(t)dt$

<u>Solution</u>.

a) $$D\,\vec{r}(t) = (D \text{ Ch } t,\ -D \text{ Sh } 2t,\ D\, e^{-t})$$
$$= (\text{Sh } t,\ -2 \text{ Ch } 2t,\ -e^{-t})$$
$$D\,\vec{r}(\ell n\ 3) = (\text{Sh } \ell n\ 3,\ -2 \text{ Ch } 2\ell n\ 3,\ -e^{-\ell n\ 3})$$
$$= \left(\frac{e^{\ell n 3} - e^{-\ell n 3}}{2},\ -2\,\frac{e^{2\ell n 3} + e^{-2\ell n 3}}{2},\ -\frac{1}{3}\right)$$
$$= \left(\frac{3 - \frac{1}{3}}{2},\ \frac{-1}{2}(9 + \frac{1}{9}),\ -\frac{1}{3}\right)$$
$$= \left(\frac{4}{3},\ -\frac{41}{9},\ -\frac{1}{3}\right)$$

b) $$\int_0^{\ell n\ 2} \vec{r}(t)dt = \left(\int_0^{\ell n\ 2} \text{Ch } t\ dt,\ -\int_0^{\ell n\ 2} \text{Sh } 2t\ dt,\ \int_0^{\ell n\ 2} e^{-t}\ dt\right)$$
$$= \left(\text{Sh } t\Big|_0^{\ell n\ 2},\ -\frac{1}{2} \text{ Ch } 2t\Big|_0^{\ell n\ 2},\ -e^{-t}\Big|_0^{\ell n\ 2}\right)$$
$$= \left(\frac{2 - \frac{1}{2}}{2},\ -\frac{1}{2}\left[\text{Ch } 2\ \ell n\ 2 - 1\right],\ -(\frac{1}{2} - 1)\right)$$
$$= \left(\frac{3}{4},\ -\frac{1}{2}\left(\frac{4 + \frac{1}{4}}{2} - 1\right),\ \frac{1}{2}\right)$$
$$= \left(\frac{3}{4},\ -\frac{9}{16},\ \frac{1}{2}\right).$$

B. SPACE CURVES

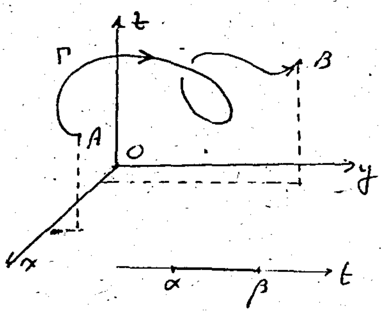
Definitions:

A vector function

$$r(t) = x(t)i + y(t)j + z(t)k, \qquad t\in[\alpha, \beta]$$

considered as a variable position vector $r(t) = P(t)$ defines a curve Γ (or an arc of curve) as the locus of $P(x(t), y(t), z(t))$ when t varies in the interval $[\alpha, \beta]$, of which the parametric equations are

$$\Gamma: \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}, \qquad t\in[\alpha, \beta].$$



The initial point of Γ is $A(t = \alpha)$ and the end point is $B(t = \beta)$. Γ is said to be oriented from A to B and the sense is indicated by an errow put on the curve.

Γ is a plane curve or a skew curve according as it lies or does not lie on a plane. A plane curve or a skew curve is called a space curve.

Elemination of t between two coordinates gives a relation betweer these coordinates showing that Γ lies on a cylinder. Such a cylinder is called a projecting cylinder. Γ defines three projecting cylinders.

The space curve Γ is closed if $r(\alpha) = r(\beta)$ (or if $A \cong B$), and called simple if Γ does not intersect itself. Γ is said to be smooth if admits tangent line of every point of it.

Lines are simple examples of space curves. Below we give an interesting curve which is skew.

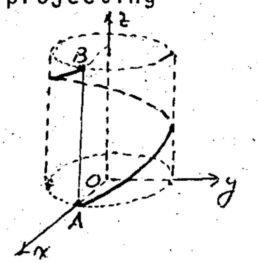
<u>Example.</u> (Circular helix). Consider the curve

$$r(\theta) = a\cos\theta\ i + a\sin\theta\ j + b\ \theta\ k$$

or

$$\Gamma : \begin{cases} x = a\cos\theta \\ y = a\sin\theta \qquad (a>0,\quad b>0),\ \ \theta\in\mathbb{R}. \\ z = b\ \theta \end{cases}$$

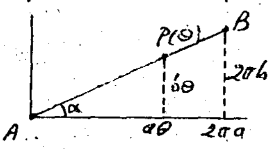
Elimination of θ between x, y gives the relation $x^2+y^2 = a^2$ which is a rihgt circular cylinder (a projecting cylinder). Hence Γ lies on this cylinder. As θ varies, $P(a\cos\theta, a\sin\theta, b\theta)$ lying on this cylinder describe a curve called a <u>circular helix.</u>



That arc corresponding to the interval $(0, 2\pi)$ is shown on the figure, starting at $A(a, 0, 0)$ and ending at $B(a, 0, 2\pi b)$.

The whole curve is obtained by translating this arc in the direction of z-axis by multiplies of $2\pi b$.

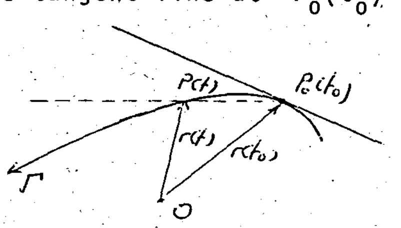
The circular helix has the property that when the cylinder is cut along the line AB (or the line $x=a$, $y=0$) and developed on a plane the curve becomes a line with slope $2\pi b/2\pi a = b/a$.



<u>Geometric interpretation of $\dot{r}(t)$</u>. Let

$$\Gamma : r(t) = (x(t), y(t), z(t)), \quad t\in(\alpha, \beta)$$

be a space curve. Let the equation of the tangent line at $P_0(t_0)$ be desired. Consider a nearby point $P(t)$ on Γ. The tangent line at P_0 is the limiting position of the line P_0P as $P \to P_0$. The direction numbers



of the line P_0P being $x-x_0$, $y-y_0$, $z-z_0$ or

$$\frac{x-x_0}{t-t_0}, \quad \frac{y-y_0}{t-t_0}, \quad \frac{z-z_0}{t-t_0}$$

their limits, as $t \to t_0$, are $\dot{x}(t_0)$, $\dot{y}(t_0)$, $\dot{z}(t_0)$ and are components of a tangent vector at P_0. Hence

$$\dot{r}(t_0) = (\dot{x}(t_0), y(t_0), \dot{z}(t_0))$$

is a <u>tangent vector</u> to Γ at $P(t_0)$, and

$$\dot{r}(t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t))$$

is a tangent vector at $P(t)$. The differential

$$dr = (x, \dot{y}, \dot{z})dt$$

is also a tangent vector.

The vector of equation of the tangent line at P_0 is then

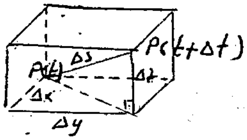
$$P = P_0 + \lambda \; \dot{r}(t_0)$$

<u>Arclength.</u>

Consider two nearby points

$P(t)$ and $P(t+\Delta t)$

on Γ. THe length Δs of the chord joing these points is an approximation to the are from $P(t)$ to $P(t+\Delta t)$. Now, from the figure



$$(\Delta s)^2 = [x(t+\Delta t)-x(t)]^2 + (y(t+\Delta t) - y(t))^2 + (z(t+\Delta t) - z(t))^2$$
$$= \Delta x^2 + \Delta y^2 + \Delta z^2$$

$$\Rightarrow \left(\frac{\Delta s}{\Delta t}\right)^2 = \left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2$$

When $x(t)$, $y(t)$, $z(t) \varepsilon D(\alpha, \beta)$, one has

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$$

$$\Rightarrow \quad ds^2 = dx^2 + dy^2 + dz^2$$

$$s = \int_\alpha^\beta \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt$$

Physical interpretation of $\dot{r}(t)$:

If one considers the parameter t as time, then the curve

$$\Gamma: r(t) = (x(t), y(t), z(t))$$

becomes the path (trajectory) of a moving particle.

The tangent vector $\dot{r}(t)$ is the velocity vector $v(t)$ of the moving particle, since $\dot{x}(t)$, $\dot{y}(t)$, $\dot{z}(t)$ are velocity components in the directions of coordinate axes:

$$v(t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t))$$

or

$$v(t) = (v_x, v_y, v_y)$$

The magnitude of the velocity vector is the speed of the particle:

$$v = |v(t)|$$

Example 1. Find the arc length of the circular helix

$$r(\theta) = a \cos\theta\ i + a \sin\theta\ j + b\ \theta\ k \qquad a>0, \quad b>0$$

between the points $A(\theta = 0)$, $P(\theta)$.

Solution.

$$x = a \cos\theta, \qquad y = a \sin\theta, \qquad z = b\theta$$

$$\Rightarrow \quad \dot{x} = -a \sin\theta, \qquad \dot{y} = a \cos\theta, \qquad \dot{z} = b$$

$$\Rightarrow \quad \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = a^2 + b^2$$

$$s = \int_0^\theta \sqrt{a^2+b^2}\, d\theta = \sqrt{a^2+b^2}\ \theta$$

Example 2. Let

$$P = (t^3,\ 2t,\ t^2), \qquad Q = (t^2+4t-4,\ \ t^2+1,\ \ 2t^3-1)$$

be the paths of two particles

a) Do the particles collide?

b) If they collide, find velocity vectors at the time of collision.

Solution.

a) For collision, the coordinates of two particles are to be the same at the same time. Equating the first coordinates we have:

$$t^3 = t^2+4t-4 \quad \Rightarrow \quad t^3-t^2-4t+4 = 0 \quad \Rightarrow \quad t=1,\ t=2,\ t=-2$$

of which only $t=1$ satisfies the other equations $2t = t^2+1$ and $t^2 = 2t^3-1$. Hence the particles collide when $t = 1$.

b) $v_1(t) = (3t^2, 2, 2t)$, $v_2(t) = (2t+4, 2t, 6t^2)$

$$\Rightarrow \quad v_1(1) = (3, 2, 2), \quad v_2(1) = (6, 2, 6).$$

FRENET Frame.

Given a space curve Γ, to any point P of Γ there is associated a system of three mutually orthogonal unit, vectors T, N, B as described below. Such a system is called the FRENET Frame.

When the arc length s is used as parameter (instrinsic parameter), t becomes a function of s, and by chain rule, we have

$$r'(t) = \frac{dr}{ds} = \frac{dr(t)}{dt}\ \frac{dt}{ds} = \dot{r}(t)\ \frac{dt}{ds}$$

which is then a vector tangent to Γ at any point $P(t)$

where

$$\dot{r}(t) = \frac{dr}{dt}\ , \qquad r'(t) = \frac{dr}{ds}$$

These notations will be used throughout this Section.

Now we show that the tangent vector $r'(t)$ is a unit vector:

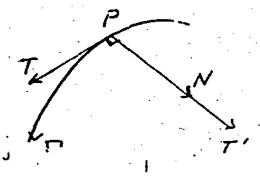
$$(r'(t))^2 = r'(t).r'(t) = \dot{r}^2\left(\frac{dt}{ds}\right)^2$$

$$= (\dot{x}^2+\dot{y}^2+\dot{z}^2)dt^2/ds^2 = ds^2/ds^2 = 1.$$

This <u>unit tangent vector</u> r' having the same sense as Γ is denoted by T:

$$T = r'(t) = \dot{r}(t)\,\frac{dt}{ds}$$

Since $T.T = 1$, by differentiation, we get $T'.T = 0$ showing that T' is perpendicular to T.



Defining a unit vector N in the same direction and sense as T' we write

$$T' = \kappa N$$

where N is called the <u>principal unit normal vector</u> and the scalar κ the <u>curvature</u>, and its reciprocal $\rho = 1/k$ the <u>radius of curvature</u>(1) of Γ at $P(t)$.

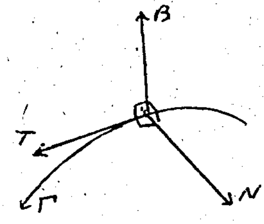
Having defined two unit orthogonal vectors T and N, we define now a third one B by

$$B = T \times N$$

which is orthogonal to both T and N, called the <u>binormal unit vector</u> of Γ at $P(t)$.

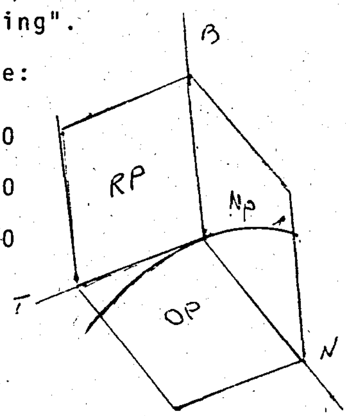
These three mutually orthogonal unit vectors T, N, B form, in this order, a positive system called the <u>FRENET system</u> of Γ at $P(t)$.

(1) The reason for calling the "radius curvature" is that when evaluated for a circle it gives the radius of that circle.



The lines through P along T, N, B are the tangent line, principal normal and binormal of Γ. These lines determine three planes: The one determined by N, B is the normal plane (NP) of Γ at P. The other two are tangent to Γ at P. The one formed by T, N is called the osculating plane (OP) and the other the rectifying plane (RP) of Γ at P.

The osculating plane can be shown to be that tangent plane that fits the curve best. The name "rectifying" may be explained as follows: When P describes Γ, the RP becomes tangent to a certain surface S containing the curve Γ. This surface can be developed onto a plane. When developed its curve Γ is transformed into a "straight line". Hence the name "rectifying".



The equations of these planes at P_0 are:

Normal plane: $(P-P_0).T = 0$

Rectifying plane: $(P-P_0).N = 0$

Osculating plane: $(P-P_0).B = 0$

Example. Given the circular helix

$$r(t) = a\cos t\, i + a\sin t\, j + bt\, k$$

a) find T, N, B, κ

b) find the equation of the osculating plane at $P(t)$

Solution.

a) $T = \frac{dr}{ds} = \frac{dr}{dt}\frac{dt}{ds} = (-a\sin t, a\cos t, b)\frac{1}{\sqrt{a^2+b^2}}$.

$$\frac{dT}{ds} = \kappa N \Rightarrow \frac{dT}{ds} = \frac{dT}{dt}\frac{dt}{ds} = (-a\cos t, -a\sin t, 0)\frac{1}{a^2+b^2}$$

$$(-\cos t, \ -\sin t, \ 0)\frac{a}{a^2+b^2} = kN$$

$$\Rightarrow \quad K = \frac{a}{a^2+b^2}, \quad N = (-\cos t, -\sin t, 0)$$

$$B = T \times N = \begin{vmatrix} i & j & k \\ -a\sin t & a\cos t & b \\ -\cos t & -\sin t & 0 \end{vmatrix} \frac{1}{\sqrt{a^2+b^2}} = \frac{(b\sin t, -b\cos t, a)}{\sqrt{a^2+b^2}}$$

∴ Principal normal vector is parallel to xy-plane and curvature is constant.

b) $(x-a\cos t)b\sin t - (y-a\sin t)b\cos t + (z-bt)a = 0$

$$(b\sin t)x - (b\cos t)y \ az = abt$$

$$\frac{x}{a}\sin t - \frac{y}{a}\cos t + \frac{z}{b} = t.$$

<u>SERRET-FRENET formulas</u>.

These are expressions for T', N', B' as linear combinations of T, N, B.

T' was already obtained as kN.

B' is derivable from $B.B = 1$, $B.T = 0$ by differentiation:

$$B.B = 1, \quad B.T = 0 \quad \Rightarrow \quad B'.B = 0, \quad B'.T + B.T' = 0$$

$$\Rightarrow \quad B'.B = 0, \quad B'.T + B.kN = 0$$

$$\Rightarrow \quad B'.B = 0, \quad B'.T = 0 \quad \Rightarrow \quad B' \perp B, \quad B' \perp T \Rightarrow B' // N.$$

Hence there is a scalar τ (tau) such that

$$B' = -\tau N$$

where τ is called the <u>torsion</u> of Γ at $P(t)$.

Now,

$$N' = (B\times T)' = B'\times T + B\times T'$$
$$= -\tau N\times T + B\times \kappa N = -\kappa T + \tau B$$

Thus we have obtained

$$\begin{aligned} T' &= \kappa N \\ N' &= -\kappa T + \tau B \\ B' &= -\tau N \end{aligned} \quad \text{or} \quad \begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

which are called the SERRET-FRENET formulas, discovered by SERRET in 1851, and by FRENET in 1852, where the square matrix is skew symmetric.

The reciprocals ρ, σ of κ (Kappa), τ (tau) are called the radius of curvature and radius of torsion of the space curve Γ at $P(t)$.

For a straight line, T is a constant vector, and $T' \equiv 0$ implying that $\kappa = 0$. κ is a measure of deviation (departure) of the curve from the tangent line.

For a plane curve, B is a constant vector and $B' \equiv 0$ implying that $\tau = 0$. τ is a measure of deviation (departure) of the curve from the osculating plane.

Converses are also true, that is,

$$k \equiv 0 \Rightarrow \Gamma \text{ is a straight line,}$$
$$\tau \equiv 0 \Rightarrow \Gamma \text{ is a plane curve.}$$

From SERRET-FRENET formulas, we have

$$\kappa^2 = T'.T' = r''.r'', \quad \tau^2 = B'.B'$$

Evaluation of T'', N'', B'':

Setting

$$F = \begin{bmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}, \qquad U = \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

the SERRET-FRENET formulas are written as

$$U' = F\,U$$

which, when differentiated, gives

$$\begin{aligned} U'' &= F'U + FU' \\ &= F'U + F(FU) = (F' + F^2)U \end{aligned}$$

where

$$F' = \begin{vmatrix} 0 & k' & 0 \\ -k' & 0 & \tau' \\ 0 & -\tau' & 0 \end{vmatrix}, \qquad F^2 = \begin{vmatrix} -k^2 & 0 & k\tau \\ 0 & -k^2-\tau^2 & 0 \\ k\tau & 0 & -\tau^2 \end{vmatrix}$$

Then

$$\begin{bmatrix} T'' \\ N'' \\ B'' \end{bmatrix} = \begin{bmatrix} -k^2 & k' & k\tau \\ -k' & -k^2-\tau^2 & \tau' \\ k\tau & -\tau' & -\tau^2 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

or

$$\begin{aligned} T'' &= -k^2T + k'N + k\tau B \\ N'' &= -k'T - (k^2+\tau^2)N + \tau'B \\ B'' &= k\tau T \quad -\tau' N - \tau^2 B \end{aligned}$$

Now we evaluate the mixed products $(T\ T'\ T'')$, $(N,\ N',\ N'')$ and $(B,\ B',\ B'')$:

$$\begin{aligned} (T\ T'\ T'') &= (T\ \ kN\ \ -k^2T + k'N + k\tau B) \\ &= (T\ \ kN\ \ k\tau B) = k^2\tau(T\ N\ B) = k^2\tau \\ (N\ N'\ N'') &= (N\ \ -kT+\tau B\ \ -k'T-(k^2+\tau^2)N+\tau'B) \\ &= (N\ \ -kT\ \ \tau'B) + (N\ \ \tau B\ \ -k'T) \\ &= -k\tau'\ (N\ T\ B) - k'\tau\ (N\ B\ T) \\ &= k\tau' - k'\tau \end{aligned}$$

$$(B\ B'\ B'') = (B \quad -\tau N \quad k\tau T - \tau' N - \tau^2 B)$$
$$= (B \quad -\tau N \quad k\tau T) = k\tau^2$$

Thus we have

$$(T\ T'\ T'') = (r'\ r''\ r''') = k^2\tau$$
$$(N\ N'\ N'') = k\tau' - k'\tau$$
$$(B\ B'\ B'') = k\tau^2$$

showing that the sign of K is that of $(B'\ B\ B'')$, and the sign of τ is that of $(T\ T'\ T'')$.

Physical interpretation of r(t)

If a particle moves on the trajectory

$$\Gamma : r(t) = x(t)i + y(t)j + z(t)k,$$

where the parameter t is time, then $r(t) = \dot{v}(t)$ is the acceleration vector $a(t)$ of the particle at time t.

We show that $a(t)$ lies on the osculating plane of Γ at $P(t)$:

$$a(t) = \frac{d}{dt}\, v(t) = \frac{d}{dt}\left(\frac{ds}{dt}\, T\right)$$
$$= \frac{d^2s}{dt^2}\, T + \frac{ds}{dt}\, \frac{d}{dt}\, T$$
$$= \ddot{s}\, T + \frac{ds}{dt}\left(\frac{ds}{dt}\, \frac{dT}{ds}\right)$$
$$= \ddot{s}\, T + v^2\, k\, N$$
$$= \ddot{s}\, T + \frac{v^2}{\rho}\, N$$

showing also that the tangential and normal components are

$$a_T = \ddot{s}\ ; \qquad a_N = \frac{v^2}{\rho}$$

If the motion is uniform $(ds/dt = \text{constant})$ along Γ, then $\ddot{s} = 0$, and the acceleration vector lies along the principal normal of Γ

Example. A particle is moving on the path

$$\Gamma : r(t) = \ln t\ i + \frac{1}{2} t^2\ j + \sqrt{2}\ t\ k.$$

a) find velocity and acceleration vectors,

b) find the components a_T, a_N of $a(t)$ at $t = 1$.

Solution.

a) $v(t) = \dot{r}(t) = \frac{1}{t}\ i + tj + \sqrt{2}\ k$

$a(t) = \ddot{r}(t) = -\frac{1}{t^2}\ i + j$

b) $a_T = \ddot{s}$ where

$$ds = \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\ dt = \frac{t^2+1}{t}\ dt$$

$$\dot{s} = \frac{ds}{dt} = \frac{t^2+1}{t} \Rightarrow \ddot{s} = \frac{t^2-1}{t^2} \Rightarrow a_T(1) = \ddot{s}(1) = 0.$$

$$a_N = \frac{v^2}{\rho} = v^2 k \quad \text{where} \quad v^2 = \left(\frac{t^2+1}{t}\right)^2 \Rightarrow v^2(1) = 4$$

$$K = |r''| = \left|\frac{dr'}{ds}\right| ,$$

$$r' = \dot{r}(t)\ \frac{dt}{ds} = \left(\frac{1}{t}\ i + tj + \sqrt{2}k\right) \frac{t}{t^2+1}$$

$$r'' = \left[\left(-\frac{1}{t^2}\ i+j\right) \frac{t}{t^2+1} + \left(\frac{1}{t}\ i+tj+\sqrt{2}k\right) \frac{1-t^2}{(t^2+1)^2}\right]\frac{t}{t^2+1}$$

$$\Rightarrow \quad r''(1) = \left[(-i+j)\tfrac{1}{2}\right]\tfrac{1}{2} = \frac{1}{4}\ (-i+j)$$

$$\Rightarrow \quad k(1) = |r''(1)| = \sqrt{2}/4. \quad \text{Then}$$

$$a_N(1) = v^2(1)\ K(1) = \sqrt{2}.$$

Curvature of plane curves.

Let

$$\Gamma : \ r(x) = x\boldsymbol{i} + y\boldsymbol{j} , \quad y = f(x)$$

be a plane curve.

$$r'(x) = (i + y'j)\frac{dx}{ds} \left(\text{with } ds = \sqrt{1+y'^2}\,dx\right)$$

$$= (i + y'j)\frac{1}{\sqrt{1+y'^2}}$$

$$r''(x) = \left[(y''j)\frac{1}{\sqrt{1+y'^2}} - (i+y'j)\frac{y'y''}{(1+y'^2)^{3/2}}\right]\frac{1}{\sqrt{1+y'^2}}$$

$$= \left[-\frac{y'}{1+y'^2}\,i + \left(1 - \frac{y'^2}{1+y'^2}\right)j\right]\frac{y''}{1+y'^2}$$

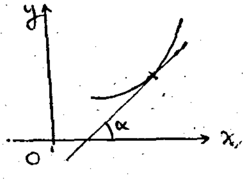
$$= (-y'i + j)\frac{y''}{(1+y'^2)^2}$$

$$K^2 = (1+y'^2)\frac{y''^2}{(1+y'^2)^4} = \frac{y''^2}{(1+y'^2)^3}$$

$$K = \pm\frac{y''}{(1+y'^2)^{3/2}} \Rightarrow \rho = \frac{(1+y'^2)^{3/2}}{|y''|}$$

For a plane curve the curvature has the equivalent definition, namely

$$K = \frac{d\alpha}{ds}$$



Indeed,

$$\tan\alpha = y' \Rightarrow \alpha = \arctan y' \Rightarrow \frac{d\alpha}{ds} = \frac{y''}{1+y'^2}\frac{dx}{ds}$$

$$K = \frac{d\alpha}{ds} = \frac{y''}{1+y'^2}\frac{1}{\sqrt{1+y'^2}} = \frac{y''}{(1+y'^2)^{3/2}} .$$

We have obtained K when the plane curve Γ is given in vector or cartesian form. Now, for a <u>parametric curve</u>

$$\Gamma : \quad x = x(t), \quad y = y(t),$$

setting

$$y' = \dot{y}/\dot{x}, \qquad y'' = \frac{d}{dx}\,\frac{\dot{y}}{\dot{x}} = \frac{\ddot{y}\,\dot{x} - \dot{y}\,\ddot{x}}{\dot{x}^2}\;\frac{1}{\dot{x}},$$

in $K = y''/(1+y'^2)^{3/2}$, we have

$$K = \frac{\dot{x}\,\ddot{y} - \ddot{x}\,\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$$

When Γ is given in <u>polar form</u>:

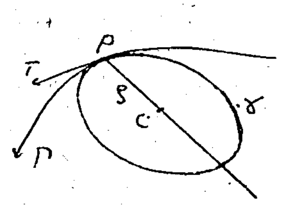
$$\Gamma: \ r = f(\theta) \quad \text{or} \quad x = r\cos\theta, \quad y = r\sin\theta,$$

computing $\dot{x}, \dot{y}, \ddot{x}, \ddot{y}$ and setting in above formulas one gets

$$K = \frac{r^2 + 2r'^2 - rr''}{(r^2+r'^2)^{3/2}}$$

which can also be obtained from $K = dx/ds$ where $\alpha = \psi + \theta$ and $\psi = \arctan \frac{r}{r'}$.

<u>Circle of curvature</u>:



The circle of curvature of a curve Γ at a point P of it, is the limiting circle of the circle passing through P and two nearby points Q, R when $Q \to P$, $R \to P$.

It can be shown that the circle center at $C = P + \rho N$ and radius ρ is the circle of curvature γ at P (in 3- or 2-space).

γ lies in the osculating plane at P since it is tangent to Γ. (or to tangent vector T) and center is on principal normal.

The coordinates of the center C are, for a plane curve,

$$\left|\begin{array}{l} \xi = x - y' \dfrac{1+y'^2}{y''} \\[2em] \eta = y + \dfrac{1+y'^2}{y''}, \end{array}\right. \qquad \left|\begin{array}{l} \xi = x(t) - \dot{x} \dfrac{\dot{x}^2 + \dot{y}^2}{\begin{vmatrix} \dot{x} & \dot{y} \\ \ddot{x} & \ddot{y} \end{vmatrix}} \\[2em] \eta = y(t) - \dot{y} \dfrac{\dot{x}^2 + \dot{y}^2}{\begin{vmatrix} \dot{x} & \dot{y} \\ \ddot{x} & \ddot{y} \end{vmatrix}} \end{array}\right.$$

and the equation of γ is

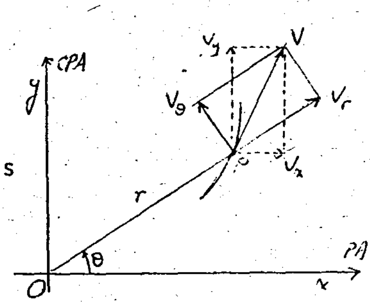
$$(x - \xi)^2 + (y - \eta)^2 = \rho^2$$

Example. Given the cardioid $r = 1+\cos\theta$, find extrema of the curvature.

Solution. $r' = -\sin\theta$, $r'' = -\cos\theta$

$$K = \frac{r^2 + 2r'^2 - rr''}{(r^2+r'^2)^{3/2}} \Rightarrow \qquad K = \frac{3}{\cos^3 \frac{\theta}{2}} \frac{1}{\sqrt{8}}$$

$K = 3/\sqrt{8}$ is minimum when $\theta = 0$,



Curvilinear motion (in plane).

Let Γ be the trajectory of a particle. In the cartesian system, the components of the velocity vector are:

$$V_x = \frac{dx}{dt}, \quad V_y = \frac{dy}{dt} \quad \left(\frac{V_y}{V_x} = y' = \tan\alpha\right)$$

and the components of the acceleration vector become

$$a_x = \frac{d^2x}{dt^2}, \quad a_y = \frac{d^2y}{dt^2}.$$

In the polar system the components are taken in the direction of the radius vector and perpendicular to that direction:

$$\left| \begin{array}{l} V_r = \dfrac{dr}{dt} = \dot{r} \\[2ex] V_\theta = \dfrac{V_\theta}{V_r}\, v_r = (\tan\psi)v_r = \dfrac{r}{dr/d\theta}\cdot\dfrac{dr}{dt} = \omega r \end{array} \right.$$

$$\left| \begin{array}{l} V_x = \dfrac{d}{dt}\,(r\cos\theta) = \dot{r}\cos\theta - (r\sin\theta)\dot{\theta} = \dot{r}\cos\theta - \omega r\sin\theta \\[2ex] V_y = \dfrac{d}{dt}\,(r\sin\theta) = \dot{r}\sin\theta + (r\cos\theta)\dot{\theta} = \dot{r}\sin\theta + \omega r\cos\theta \end{array} \right.$$

$$a_x = \frac{d}{dt}\,(\dot{r}\cos\theta - \omega r\sin\theta)$$

$$= \ddot{r}\cos\theta - (\dot{r}\sin\theta)\omega - \dot{\omega}r\sin\theta - (\omega\dot{r}\sin\theta) - (\omega r\cos\theta)\omega$$

$$= \ddot{r}\cos\theta - 2\omega\dot{r}\sin\theta - \dot{\omega}r\sin\theta - \omega^2 r\cos\theta$$

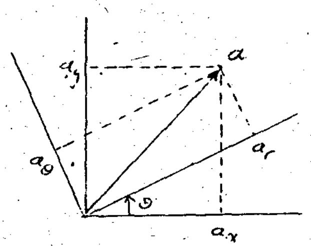
$$\ddot{r}\sin\theta + 2\omega\dot{r}\cos\theta + \dot{\omega}r\cos\theta - \omega^2 r\sin\theta$$

or

$$a_x = (\ddot{r} - \omega^2 r)\cos\theta - (2\omega\dot{r} + \dot{\omega}r)\sin\theta$$

$$a_y = (\ddot{r} - \omega^2 r)\sin\theta + (2\omega\dot{r} + \dot{\omega}r)\cos\theta$$

$$\Rightarrow \quad \begin{array}{l} a_r = \ddot{r} - \omega^2 r \\ a_\theta = 2\omega\dot{r} + \dot{\omega}r \end{array} \quad \text{(by a rotation)}$$



where a_r, a_θ are the radial and transverse components of the acceleration.

The motion under a central force (as in planetary motion) the transverse component of the accelation is zero. We have

$$a_\theta = 2\omega\dot{r} + \dot{\omega}r = \frac{1}{r}\,(2\omega r\dot{r} + \dot{\omega}r^2) = \frac{1}{r}\,\frac{d}{dt}\,(\omega r^2)$$

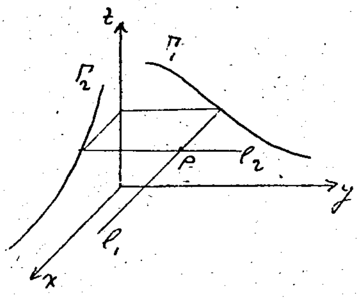
$$= \frac{1}{r}\,\frac{d}{dt}\,\frac{r^2 d\theta}{dt} = \frac{1}{r}\,\frac{d}{dt}\,\left(\frac{d\,R_{\theta r}}{dt}\right) = 0 \quad \text{(for central force)}$$

$$\Rightarrow \quad \frac{d\,R_{\theta r}}{dt} = \text{const.}$$ (<u>KEPLER's second law</u>: The area swept by the radius vector is proportional to time).

Curve Sketching:

Sketching of a space curve Γ is usually done by the use of two of its projecting cylinders.

Let directrices of its two projecting cylinders be Γ_1, Γ_2, say, on yz- and xz-planes. Since a plane parallel to xy-plane intersect the cylinders along two generatrices ℓ_1, ℓ_2 the intersection P of ℓ_1, ℓ_2 belong to the curve Γ. Then Γ is plotted by taking some number of planes //xy-plane.



A projecting cylinder is obtained by eliminating the parameter between two coordinates if Γ is given in parametric form, and by eliminating a variable if Γ is given as $F(x, y, z) = 0$, $G(x, y, z) = 0$, since the equation of a cylinder (generatrices // are axis) involves only two variables.

Example 1. Find all three projecting cylinders of the following curves:

a) Γ : $x = t$, $y = t^2$, $z = 1+\sqrt{t}$

b) Γ : $x^2+4z^2 = 16$, $x^2-y^2+z^2 = 0$

Solution.

a) xy-proj. cyl.: $y = x^2$

xz-proj. cyl.: $(z-1)^2 = x$

yz-proj. cyl.: $(z-1)^2 = \sqrt{y}$

b) xz-proj. cyl.: $x^2+4z^2 = 16$ (given)

xy-proj. cyl.: $4y^2-3x^2 = 16$

yz-proj. cyl.: $y^2+3z^2 = 16$

Example 2. Sketch the curve

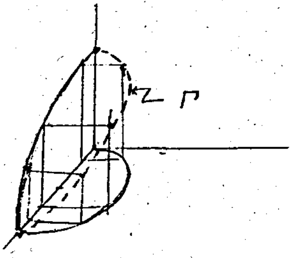
$$\Gamma: \quad x^2+y^2+z^2 = 16, \quad x^2+y^2-4x = 0 \quad \text{(A VIVIANI curve)}$$

in the I. octant.

Solution.

xy-proj. cyl. : $(x-2)^2+y^2 = 4$

xz-proj. cyl. : $z^2+4x = 16$



EXERCISES (3, 4)

116. At what points does the curve

$$r(t) = 2t^2 i + (1-t)j + (3+t^2)k$$

intersect the plane $3x-14y+z = 10$?

117. Find the unit tangent vector T for:

a) $r(t) = (t^3,\ 1-t,\ 2t+1)$

b) $r(t) = e^{-2t}i + e^{2t}j + (1+t^2)k$

118. Find cosine of the angle of intersection of the curves

$r(t) = (1+t^4,\ 2\cos\pi t,\ t^3)$, $r(t) = (t+t^2,\ t-3t^2,\ te^{1-t})$

at $(2, -2, 1)$.

119. Find the are length in the given interval:

a) $r(t) = (t \operatorname{Sin}t,\ t\cos t,\ t)$, $[0, \pi/2]$

b) $r(t) = ti + \frac{3}{2}t^2 j + \frac{3}{2}t^3 k$, $[0, 2]$

c) $x = t$, $y = \ln(\sec t + \tan t)$, $z = \ln \sec t$, $[0, \pi/4]$

120. Show that the curve

$$r(t) = e^t \sin t\ i + e^t \cos t\ j + e^t k$$

lies on a cone $ax^2+by^2+cz^2 = 0$, and $T(t)$ makes a constant angle with z-axis.

121. Obtain the equation of the surface generated by the tangent lines to the curves:

a) $r(t) = a \cos\theta \, i + a \sin\theta \, j + b\theta \, k$

b) $x = t, \quad y = t^2, \quad z = t^3$

c) $x^2+y^2+z^2 = 16, \quad x^2+y^2-4x = 0$

122. Find the condition on a, b, c so that the curve

$$r(t) = at^4 i + bt^3 j + ct^6 k$$

lies on the surface $z^2 = x^3+2y^4$

123. Find the principal normal vector N of the curve

$$r(t) = e^t i + e^t \cos t \, j + e^t \sin t \, k$$

at $(1, 1, 0)$

124. If $r(t)$ is a vector function with $r = |r(t)|$ prove

a) $\vec{r} \cdot d\vec{r} = 0 \Rightarrow r$ is constant

b) $\vec{r} \times d\vec{r} = 0 \Rightarrow r$ is constant.

125. For the curve

$$x = 6 \sin 2t, \quad y = 6 \cos 2t, \quad z = 5t$$

a) find T

b) find s for $t \in (0, \pi)$

c) find N, K, B

d) find the osculating plane at $t = \pi/6$.

126. Find the equation of the osculating plane of

Γ: $r(t) = (e^{3t}, e^{-3t}, 3\sqrt{2}t)$ at $t = 1$.

127. Find the equation of the normal plane:

a) $r(t) = \sin 3t \, i + \cos 3t \, j + 2t^{3/2} k$ at $(0, 1, 0)$

b) $r(t) = e^t \, i + tj + t^2 k$ at $t = 1$.

128. For the given trajectories find the velocity vector $v(t)$ and the accelerations vector $a(t)$ with its components a_r, a_N

a) $r(t) = ti + \frac{3}{2}t^2j + \frac{3}{2}t^3k$ at $A(2, 6, 12)$

b) $r(t) = \frac{1}{3}t^3i + 2t\,j + \frac{2}{t}k$ at $B(\frac{1}{3}, 2, 2)$

129. Find the velocity vector of motion

$$r(t) = \sin 2t\, i + \ln(1\ t)j + tk$$

130. At what point of the parabola $y = x^2/4$ is the radius of curvature a minimum?

131. Find the curvature and equation of the circle of curvature of the curve

$x = \ln \cos y$ at $(0, 0)$

132. Find K, T, N for the circular helix:

$$r(t) = \cos t\, i + \sin t\, j + 2t\, k$$

133. For the curve

$$\Gamma: r(t) = a(3t-t^3)i + 3at^2j + a(3t+t3)k$$

prove

$$K = \tau = \frac{1}{3a(1+t^2)^2}$$

134. Find the path of a particle which starts from the origin at $t = 0$ and moves with the velocity $v(t) = i\cos t + j\sin t + k$

135. Find the curvature and torsion of

a) $x = a(t-\sin t)$, $y = a(1-\cos t)$, $z = bt$

b) $x = a\cos t$, $y = a\sin t$, $z = a\cos 2t$

136. Show that the xy-projection of the curve

Γ: $x^2+y^2+z^2-2ax-a^2 = 0$, $x^2+y^2-z^2-2ax+a^2 = 0$

is a circle.

137. Write the equations of the projecting cylinders of the following curves:

a) $r(t) = (t^2, t^4, t^6)$, b) $r(t) = (\mathrm{Sh}\, t, \mathrm{Ch}\, t, e^t)$

138. Find the projecting cylinders:

a) $x = \cos t - \sin t$ b) $x = e^t$

$y = \cos t + \sin t$ $y = e^{-t}$

$z = 2 \sin 2t$ $z = \mathrm{Ch}\, t$

139. Sketch the curves:

a) $z = xy, \quad x + z = y^2$ b) $x^2 - 4x + y^2 = 0, \quad x^2 + y^2 = z^2$

140. Sketch the lines

a) $\frac{x-1}{-2} = \frac{9}{1} = \frac{z-1}{-3}$ b) $x + 2y + 5z = 10, \quad -x + 2y + z = 2$

by the use of projecting cylinders.

ANSWERS TO EVEN NUMBERED EXERCISES

116. $(2, 0, 4)$, $(18, 4, 12)$

11[illegible] $12/5\sqrt{34}$

122. $a^3 + 2b^4 - c^2 = 0$

126. $x - e^6 y - \sqrt{2} e^3 z + 6e^3 = 0$

128. a) $v = i + 6j + 18k$,

$a = 3j + 18k$, $a_T = 18$, $a_N = 3$

b) $v = i + 2j - 2k$

$a = 2i + 4k$, $a_T = -2$, $a_N = 4$

130. at $(0, 0)$

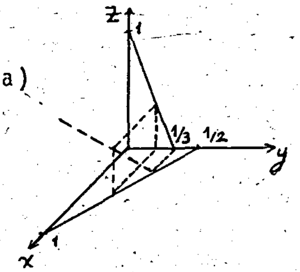
132. $1/5$, $\frac{1}{\sqrt{5}}(-\sin t\, i + \cos t\, j + 2k)$, $-\cos t\, i - \sin t\, j$

134. $r(t) = \sin t\, i + (1 - \cos t) j + tk$

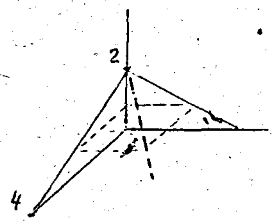
138. a) $x^2 + y^2 = 2$, $z = 2y^2 - 2$, $z = 2 - 2x^2$

b) $xy = 1$, $\quad x+y-2z = 0$, $\quad x^2+1 = 2xz$

140. a)



b)



A SUMMARY

3. 1. Dot product: $A.B = |A||B| \cos\theta = a_1b_1+a_2b_2+a_3b_3 = B.A$

$A.(B+C) = A.B+A.C$, $\quad A \perp B \Leftrightarrow A.B = 0$

Cross product: $AxB = \vec{n}|A||B|\sin\theta$ where $|\vec{n}| = 1$; A, B, n is a positive system.

$$AxB = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = -BxA, \quad Ax(B+C) = AxB + AxC$$

$|AxB| = |OARB|_2$ where $\vec{OR} = \vec{OA}+\vec{OB}$

Mixed product: $(ABC) = AxB.C = A.BxC = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Triple vector product:

$$Ax(BxC) = (A.C)B - (A.B)C, \quad (AxB)xC = (A.C)B - (B.C)A$$

Inner product in $\mathbb{R}^n$ $\qquad$ in $c[a, b]$

$\langle A, B\rangle = a_1b_1 + \dots + a_nb_n$ $\qquad$ $\langle f, g\rangle = \int_a^b f(t)\, g(t)dt$

$||A|| = (a_1^2 + \dots + a_n^2)^{1/2}$ $\qquad$ $||f|| = (\int_a^b f^2(t)dt)^{1/2}$

3. 2. PLANES:

Plane through $P_0(x_0, y_0, z_0)$ and $\perp$ $N = (A, B, C)$:

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$$

General equation: $Ax + By + Cz + D = 0$

Normal equations:

$x\cos\alpha + y\cos\beta + z\cos\gamma - p = 0$, $ax+by+cz+d=0$ $(a^2+b^2+c^2=1)$

$(Ax+By+Cz+D)/\sqrt{A^2+B^2+C^2} = 0$

Plane through P_1, P_2, P_3: Intercept form: $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0,$$

Vectoral form: $P = P_0 + sU + tV$

LINES.

Line through P_0 and $//$ $\vec{D} = (a, b, c)$:

$P = P_0 + tD$ (vectoral equation)

$x = x_0+ta$, $y = y_0+tb$, $z = t_0+tc$ (param.cartes.equ)

$\frac{x-x_0}{a} = \frac{y-y_0}{c} = \frac{z-z_0}{c}$ (symmetric equations)

$Ax+By+Cz+D = 0$, $A'x+B'y+C'z+D' = 0$ (simultaneo equ)

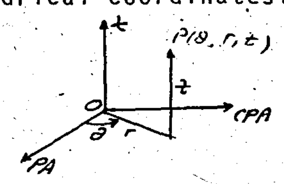
3. 3. SURFACES.

A surface is the graph of an equality relation $F(x, y, z)=0$.

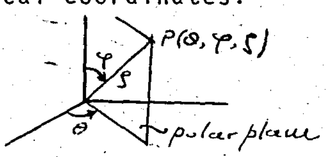
$x = f(u, v)$, $y = g(u, v)$, $z = h(u, v)$ (parametric equ)

$\lambda F(x, y, z) + \mu G(x, y, z) = 0$ (linear family of planes)

Cylindrical coordinates:



Spherical coordinates:



Cylinders: S: $\frac{x-f(t)}{a} = \frac{y-g(t)}{b} = \frac{z-h(t)}{c}$

with direction $\Delta = (a, b, c)$ and directrix. $x = f(t)$, $y = g(t)$, $z = h(t)$.

Cones: S: $\frac{x-x_0}{x_0-f(t)} = \frac{y-y_0}{y_0-g(t)} = \frac{z-z_0}{z_0-h(t)}$

with vertex (x_0, y_0, z_0) and directrix $x = f(t)$, $g = g(t)$, $z = h(t)$.

Surface of revolution with axis on z-axis:

$$S: x^2+y^2 = r^2, \quad z = h(t), \quad r^2 = f^2(t)+g^2(t)$$

$$S: x^2+y^2 = f^2(t)+g^2(t), \quad z = h(t)$$

where $x = f(t)$, $y = g(t)$, $z = h(t)$ is the generatrix.

QUADRICS:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} + \frac{(z-\ell)^2}{c^2} = 1 \quad \text{(Ellipsoid)}$$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} - \frac{(z-\ell)^2}{c^2} = 1 \quad \text{(Hyperboloid of one sheet)}$$

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} + \frac{(z-\ell)^2}{c^2} = 1 \quad \text{(Hyperboloid of two sheets)}$$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = \lambda(z-\ell) \quad \text{(Elliptic paraboloid)}$$

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = \lambda(z-\ell) \quad \text{(Hyperbolic paraboloid)}$$

SOME DEGENERATE QUADRICS:

$f(x, y) = 0$ or $g(x, z) = 0$ or $h(y, z) = 0$ (cylinders)

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} - \frac{(z-\ell)^2}{c^2} = 0 \quad \text{(cone)}$$

3. 4. SPACE CURVES:

Vector function: $r(t) = x(t)i + y(t)j + z(t)k = (x(t), y(t), z(t))$

tangent vector: $\dot{r}(t) = \dot{x}(t)i + \dot{y}(t)j + \dot{z}(t)k$, $(\dot{r} = dr/dt)$

unit tangent vector $T = r'(t) = dr/ds$

unit principal normal vector $N = K^{-1} T'$

unit binormal vector $B = T \times N$

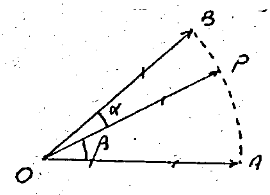
FRENET formulas:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & K & 0 \\ -K & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

MISCELLANEOUS EXERCISES

1. Show that the midpoints of the sides of a quadrangle are the v... of a parallelogram, (VARIGNON, 1654-1722).

42. Pro...



$$\sin(\alpha+\beta)\overrightarrow{OP} = \sin\alpha\,\overrightarrow{OA} + \sin\beta\,\overrightarrow{OB}$$

Show t...

If $\alpha A + \beta$ ... $\gamma C + \delta D = 0$, $\alpha+\beta+\gamma+\delta = 0$ $(\alpha, \beta, \gamma, \delta \neq 0)$, then A, B, C, D are coplanar.

144. O is the circumcenter and H the orthocenter of $\triangle ABC$. Prove: $\vec{OA} + \vec{OB} + \vec{OC} = \vec{OH}$

145. Use dot ... to prov... an angle inscribed in a semi-circle is a ...

146. Show that A(0, 0, 0, ... 3, 4), C(4, -1, 2, -3) are the vertices of a rig... in $\mathbb{R}^4$.

147. Show that

a) In an isoceles trapezoid the di...als are congruent,

b.) In a right triangle the midpoint of the hypothesis is equindistant from the vertices.

148. By projecting the sides a regular pentagon onto a line which makes an angle θ with one of its sides, p... that

$$\cos\theta + \cos\left(\theta + \frac{2\pi}{5}\right) + \cos\left(\theta + \frac{4\pi}{5}\right) + \cos\left(\theta + \frac{6\pi}{5}\right) + \cos\left(\theta + \frac{8\pi}{5}\right) = 0$$

149. If $a = |\vec{a}|$ and $b = |\vec{b}|$ show that

$$\vec{r} = \frac{a\,\vec{b} + b\,\vec{a}}{a + b}$$

bisects the angle between $\vec{a}$ and $\vec{b}$.

150. Show that

a) in a parallelogram diagonals bisects each other,

b) in quadrilateral the bimedians (segments joining the midpoints of opposite sides) bisect each other.

151. Show that in a parallelogram ABCD the line poining D to the midpoint I of [BC] trisects the diagonal [AC].

152. Let $A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$ and $C(c_1, c_2, c_3)$ be vertices of the triangle ABC, prove

a) the law of cosine b) the law of sine

153. Show that, if in a tetrahedron two pairs of opposite edges are othogonal, then the third pair is orthogonal.

154. Find the conditions for nonzero vectors $\vec{a}, \vec{b}, \vec{c}$ for which $(\vec{a}x\vec{b})x\vec{c} = \vec{a}x(\vec{b}x\vec{c})$ holds.

155. Show that for three vectors,

a) $(bxc,\ cxa,\ axb) = (abc)^2$ b) $\sum ax(bxc) = 0$

156. Prove that

a) $[(AxB)xC]\ X\ A + Ax[(BxC)xA] = 0$

b) $\{[(A_1 x A_2) x A_3] x A_4\} x A_5$ is equal to

$$\begin{vmatrix} A_1 & A_2 & A_4 \\ A_1 . A_3 & A_2 . A_3 & 0 \\ A_1 . A_5 & A_2 . A_5 & A_4 . A_5 \end{vmatrix}$$

157. Given $V_1 = \frac{3}{5}\, i + \frac{4}{5}\, j$, $V_2 = -\frac{4}{5}\, i + \frac{3}{5}\, j$

show that V_1, V_2 in an orthonormal system in $\mathbb{R}^2$.

158. Which vectors of the bases $\{(1\ \ 2\ \ -1\ \ 3)^T,\ (2\ \ 1\ \ 1\ \ 1)^T,$

$(3\ \ -2\ \ 1\ \ 4)^T$, $(1\ \ 1\ \ 1\ \ 1)^T$ for the vectors of R^4, can be replaced by the vector $(2\ \ -3\ \ 0\ \ 3)^T$?

159. Let $\vec{F}$ be a sliding force along a line d and Δ be a line. Let $\vec{G}$ be the moment of $\vec{F}$ with respect to a point 0 on Δ. Show that the projection vector of $\vec{G}$ on Δ is independent of 0 taken on Δ.

160. Prove that the centroid of equal masses of the vertices of a quadrangle is the center of the VARIGNON parallelogram. (See Exercises 141).

161. Prove that the perpendicular bisectors of the edges in a tetrahedron are concurrent.

162. Find the equation of the planes for which

a) $\alpha = \pi/4$, $\beta = \pi/3$, $\gamma = \pi/3$, $p = 6$

b) $\frac{\cos\alpha}{-2} = \frac{\cos\beta}{-1} = \frac{\cos\gamma}{-2}$, $p = 2$

163. Show that the plane $x + 2y - 2z - 9 = 0$ passes through the point of intersection of the planes $x+y+z = 1$, $x-y-z = 1$ and $2x+3y = 8$.

164. Find the point of intersection of the planes (if any): $x+2y+z = 0$, $x-2y-8 = 0$ and $x+y+z-3 = 0$

165. Show that the four planes $x+y+2z = 2$, $x+y-2z = -2$, $x-y+8 = 0$, $3x-y-2z+18 = 0$ passes through some point.

166. Find the angles between the following pairs of planes:

a) $x+y-4z = 0$
$3y-3z+2 = 0$

b) $2x-y+z = 7$
$x+y+2z = 11$

167. For what values of α, β, γ and p, the plane

$x\cos\alpha + y\cos\beta + z\cos\gamma - p = 0$

a) is // to xz-plane   b) passes through y-axis

168. Find the intercepts on the axes and the traces of:

a) $2x+3y+4z = 24$ b) $3x+5z+45 = 0$

169. Find the traces of the following lines:

a) $\left|\begin{array}{l} 2x+y-z = 2 \\ x-y+2z = 4 \end{array}\right.$ b) $\left|\begin{array}{l} x+2y = 8 \\ 2x-4y = 7 \end{array}\right.$

170. Find the angle between the following lines:

a) $\left|\begin{array}{l} x+y-z = 0 \\ y+z = 0 \end{array}\right.$ and $\left|\begin{array}{l} x-y = 1 \\ x-3y+z = 0 \end{array}\right.$

b) $\left|\begin{array}{l} x-2y+z = 2 \\ 2y-z = 1 \end{array}\right.$ and $\left|\begin{array}{l} x-2y+z = 2 \\ x-2y+2z = 4 \end{array}\right.$

171. Show that the following pairs of lines intersect and perpendicular:

a) $\left|\begin{array}{l} x+2y = 1 \\ 2y-z = 1 \end{array}\right.$ and $\left|\begin{array}{l} x-y = 1 \\ x-2z = 3 \end{array}\right.$ b) $\left|\begin{array}{l} 3x+y-z = 1 \\ 2x-z = 2 \end{array}\right.$ and $\left|\begin{array}{l} 2x-y+2z = 4 \\ x-y+2z = 3 \end{array}\right.$

172. Find the equation of the projecting planes of the following lines:

a) $\left|\begin{array}{l} 2x+y-z = 0, \\ x-y+2z = 3 \end{array}\right.$ b) $\left|\begin{array}{l} x+z = 1 \\ x-z = 3 \end{array}\right.$

173. Find the distance measured along the line $x = 2 - \frac{3}{13}t$, $y = 4 + \frac{12}{13}t$, $z = -3 + \frac{4}{13}t$ from $(2, 4, -3)$ to the intersection of the line with plane $4x-y-2z = 6$.

174. Reduce the equation of the following lines to the symmetric form:

a) $4x-5y+3z = 3,$ b) $x = mz+a,$

$4x-5y+z+9 = 0$ $y = nz+b$

175. Find the equation of the plane determined by the line $\frac{x-z}{2} = \frac{y+3}{-2} = \frac{z-1}{1}$ and the point $(0, 3, -4)$.

176. Find the equation of the plane determined by the parallel lines

$$\frac{x+1}{3} = \frac{y-2}{2} = \frac{z}{1} \quad \text{and} \quad \frac{x-3}{3} = \frac{y+4}{2} = \frac{z-1}{1}$$

177. Find the distance between the parallel planes $2x-y+z = 4$ and $4x-2y+2z = 10$

178. Find the distance between the given plane and the point:

a) $2x-y+3z = 4,\ (1, 2, -3)$ b) $x+2y-z = 5,\ (2, 3, -1)$

179. Find the distance between two lines:

$$\frac{x-1}{2} = \frac{y}{3} = \frac{z+1}{1} \quad \text{and} \quad \frac{x}{3} = \frac{y-1}{2} = \frac{z}{-1}$$

180. Find the distance between the line $\frac{x}{3} = \frac{y+1}{2} = \frac{z-2}{3}$ and the point $P(0, 1, 3)$.

181. Find the distance between the parallel lines

$$\frac{x+1}{3} = \frac{y+2}{2} = \frac{z}{1} \quad \text{and} \quad \frac{x-3}{3} = \frac{y+4}{2} = \frac{z+1}{1}$$

182. Identify the surfaces

a) $x = a\,u\cos v$ b) $x = a\,u\,\mathrm{Ch}\,v$

$y = b\,u\sin v$ $y = b\,u\,\mathrm{Sh}\,v$

$z = u^2$ $z = c\,u^2$

c) $x = a\,\mathrm{Sh}\,u\cos v$ d) $x = a\,\mathrm{Sh}\,u\,\mathrm{Sh}\,v$

$y = a\,\mathrm{Sh}\,u\sin v$ $y = a\,\mathrm{Sh}\,u\,\mathrm{Ch}\,v$

$z = c\,\mathrm{Ch}\,u$ $z = d\,\mathrm{Ch}\,u$

e) $x = a\,\frac{u-v}{u+v}$, $y = b\,\frac{uv+1}{u+v}$, $z = c\,\frac{uv-1}{u+v}$

183. Identify the surfaces:

a) $x^2 + \frac{y^2}{4} + \frac{z^2}{5} = 1$ b) $\frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{4} = 9$ c) $x^2 - \frac{y^2}{9} = z^2+1$

d) $z^2 = x^2 + \frac{y^2}{9} + 4$ e) $x = z^2 - y^2$ f) $x = z^2-x^2$

g) $x = a(u-v)$, $y = b(u-v)$, $z = uv$

h) $\varrho = z$, i) $\phi = \pi/4$, k) $\frac{x-u}{1} = \frac{y-u^2}{2} = \frac{z-(u-1)}{2}$ $(= v)$

l) $r(u,v) = (u-v+1)i + (2u+v)j + (u-2v)k$

184. Discuss and construct the loci

a) $y^2+z^2 = 4x$ b) $y^2-z^2 = 4x$

185. Construct the following surfaces and shade that part of the first cut off by the second: (in the I. octant)

a) $x^2+4y^2+9z^2 = 36$, $x^2+y^2+z^2 = 16$

b) $x^2+y^2+z^2 = 64$, $x^2+y^2-8x = 0$

c) $4x^2+y^2-4z = 0$, $x^2+4y^2-z^2 = 0$

186. Given $x^2+2yz-4x = 0$, after a suitable transformation obtain its standard equation and identity.

187. Find the cartesian equation of the cylinder

a) Δ: $x = y$, $z = 0$; Γ: $x = \sin t$, $y = \cos t$, $z = \sin t \cos t$

b) Δ: $y = z$, $x = 0$; Γ: $y = 0$, $x+y = 2ax+2ay$

c) axis: $x^2-2y^2+z+a = 0$, $x+2y+3z+a = 0$ and passing through $(0, 0, 0)$.

188. Find the equation of the cone with directrix

Γ: $x = a+a\cos t$, $y = a\sin t$, $z = 0$

and vertex at $(0, 0, h)$

189. Find the equation of the cone:

a) vertex at 0, directrix $x = \frac{1}{x-a}$, $y = \frac{1}{y-b}$, $z = \frac{1}{z-c}$

b) vertex at (1, 1, 1), directrix $x = 3t^2-2at$, $y = 2t^3-t$, $z = t^2$

c) c is $x = y = z$, semiangle at vertex is $\pi/6$.

190. Find the equation of the surface of revolution generated when Γ: $y = z$, $x^2+y^2-2ax-2ay = 0$ is revolved about z-axis.

191. Find the cartesian equation of the surface of revolution

a) axis: az, generated by Γ: $x=t$, $y = t^2$, $z = t^3$

b) axis: Oz, generated by Γ: $y=z$, $(x-y)^2-2a(x+y) = 0$

c) axis: Ox, generated by Γ: $r=a \cos\theta + b$.

192. Find the lo f the points equidistant from the y-axis and xz-plane.

193. Find the locus of the point that are equidistant from

a) two skew lines b) a line and two points

194. Find the locus of the point equidistant from the z-axis and the plane z -1.

195. Find the locus of center of spheres of radius "r" that are tangent to two intersecting lines having angle 2α between them.

196. Find and sketch the locus of the points equidistant from the x-axis and the point A(0, 3, 0).

197. Find the locus of the mid-points of the chords of a space curve Γ: $x = f(t)$, $y = g(t)$, $z = h(t)$

198. Find the locus of the point equidistant from the x-axis and the plane $y = z$.

199. Find the locus of the point ratio of its distance from the point $A(2, 0, 0)$ and the plane $\pi: x = 1$ is 2.

200. Find the locus of the points, ratio of its distances from the point $A(0, 0, 2)$ and $B(0, 0, -2)$ is 3.

201. Find the equation of the planes whose intersection with the ellipsoid $9x^2+25y^2+169z^2 = 1$ are circles.

202. Find the condition that (x_1, y_1, z_1) should be the middle point of the chord of the hyperholoid $x^2-y^2+4z^2 = 16$ formed by

$$\frac{x-x_1}{2} = \frac{y-y_1}{-1} = \frac{z-z_1}{-2}$$

203. Find the condition that the line $\frac{x-2}{\cos\alpha} = \frac{y-1}{\cos\beta} = \frac{z+1}{\cos\gamma}$ should be tangent to the paraboloid $x^2-y^2+3z = 0$.

204. Let $\vec{P} = (\cos\lambda t)A + (\sin\ t)B$ where λ, A, B are constant. Show that

a) $P \times \frac{dP}{dt} = Const.$ b) $\frac{d^2P}{dt^2} + \lambda^2 P = 0$

205. If $P(t)$, $Q(t)$ are vector functions, show that

a) $\frac{d}{dt}\left(P \times \frac{dP}{dt}\right) = P \times \frac{d^2P}{dt^2}$ b) $P \cdot \frac{dP}{dt} = |P| \frac{d}{dt}|P|$

c) $\left|\frac{dP}{dt}\right| = \frac{d}{dt}|P|$ d) $\frac{d}{dt}\left(P \times \frac{dQ}{dt} - \frac{dP}{dt} \times Q\right) = P \times$ [illegible] $- \frac{d^2P}{dt^2} \times Q$

206. Let P, Q be two fixed points of a solid in motion, and let $\vec{V}(P)$, $\vec{V}(Q)$ be the velocity vectors at P and Q. Show that the projections of these velocitions on the line PQ are equal to each other.

207. Find the arc length of the VIVIANI curve $\Gamma: x^2+y^2+z^2 = 4a^2$, $x^2+y^2-2ax = 0$ in the first octant.

208. Find T, N, B, κ at the given point of the given curve:

a) $r(t) = e^t\cos t\ i + e^t\sin t\ j + e^t k$, $A(1, 0, 1)$

b) $r(t) = (1+t)i + (3-t)j + (2t+4)k$, $B(4, 0, 10)$

c) $r(t) = 2\ \mathrm{Ch}\frac{t}{2}\ i + 2\ \mathrm{Sh}\frac{t}{2}\ j + 2tk$, $C(2, 0, 0)$

209. Find the equation of the FRENET planes of the curve

$\vec{r} = \sin 3t\ i + \cos t\ j + 2\ t^{3/2}k$ at $(0, 1, 0)$

210. Consider the space curve

$$\Gamma:\ x = \frac{(t-a)^3(t-b)}{t}, \quad y = \frac{(t-a)(t-b)^3}{t}, \quad z = t^3$$

show that if the osculating plane at a point P passes through Q on Γ, the osculating plane at Q passes through P.

211. Prove that.

a) $(T', T'', T''') = \kappa^3(\kappa\tau'-\kappa'\tau) = \kappa^5 \frac{d}{ds}\left(\frac{\tau}{\kappa}\right)$

b) $(B', B'', B''') = \tau^3(\kappa'\tau-\kappa\tau') = \tau^5 \frac{d}{ds}\left(\frac{\kappa}{\tau}\right)$

212. Prove that

$B = \dot{r}\times\ddot{r}/\kappa\dot{s}^3$, $N = (\dot{s}\ddot{r}-\ddot{s}\dot{r})/\kappa\dot{s}^2$, $\kappa^2 = (\ddot{r}^2-\ddot{s}^2)/\dot{s}^4$. and

$\tau = (\dot{r}\ \ddot{r}\ \dddot{r})/\kappa^2\dot{s}^6$.

213. Prove:

a) $r'.r'' = 0$, $r'.r''' = -\kappa^2$, $r'.r'''' = -3\kappa\kappa'$

b) $r'''.r''' = \kappa'\kappa'' + 2\kappa^3\kappa' + \kappa^2\tau\tau' + \kappa\kappa'\tau^2$

c) $T'.B' = -\kappa\tau$.

214. Squaring $r''' = -\kappa^2 T + \kappa' N + \kappa\tau B$ obtain

a) $\tau^3 = \frac{1}{\kappa^2} r'''^2 - \kappa^2 - \left(\frac{\kappa'}{\kappa}\right)^2$

b) $r'''' = -3\kappa\kappa' T + (\kappa''-\kappa^3-\kappa\tau^2)N + (2\kappa'\tau + \kappa\tau')B$

215. Given $r^{(n)} = a_n T + b_n N + C_n B$ show that

$$a_{n+1} = a'_n - Kb_n, \quad b_{n+1} = K\,a_n + b'_n - Tc_n, \quad c_{n+1} = T\,b_n + c'_n.$$

216. Show that $\rho = r\dfrac{dr}{dp}$ where p is the distance of the pole from the tangent line at $P(\theta, r)$.

217. Evaluate radius of curvature:

a) $y^2 = \dfrac{x^3}{a-x}$, b) $3ay^2 = x^3$, c) $r^m = a^m \operatorname{Cos} m\theta$.

d) d) $y = \ln \operatorname{Cos}x$, e) $r = \dfrac{\sin\theta - \cos\theta}{2+2\cos^3\theta}$ at the pole.

f) $x = \operatorname{Cos}^2 t + \ln \sin t$, $y = \operatorname{Cos}t \sin t$,

g) $x = (1+\operatorname{Cos}^2 t)\sin t$, $y = \sin^2 t \cos t$

h) $x = 3t$, $y = 3t^2$, $z = 2t^3$, i) $x = a\cos t$, $y = b\sin t$, $z = ct$.

218. Show that

$$\Gamma:\left.\begin{aligned} x &= -f'(\theta)\operatorname{Cos}\theta - f''(\theta)\operatorname{Sin}\theta \\ y &= f'(\theta)\operatorname{Cos}\theta + f''(\theta)\operatorname{Cos}\theta \\ z &= a\left(f'(\theta)+f''(\theta)\right) \end{aligned}\right\}$$

is an helix traced on a cylinder with direction // to z-axis.

219. Find the condition for a point on the curve of $y = f(x)$ to have a maximum ρ.

220. Evaluate the radius of curvature of

a) $y = \sin x$ at $x = \frac{\pi}{2}$, b) $y = x \ln x$ at $x = 1$

c) $x = t+t^2$, $y = (1+t)e^{1/t}$ at the origin.

ANSWERS TO EVEN NUMBERED EXERCISES

154. $b \parallel a \times c$ or $a \parallel c$

158. $(3 \quad -2 \quad 1 \quad 4)^T$.

162. a) $\sqrt{2}x+y+z-12 = 0$, b) $2x+y+2z+6 = 0$

164. $(2, -3, 4)$

166. a) arc cos(5/6) b) $\pi/3$

168. a) x-int: 12 xy-trace: $2x+3y = 24$

y-int: 8 xz-trace: $2x+4z = 24$

z-int: 6 yz-trace: $3y+4z = 24$

b) x-int: -15 xy-trace: $x+15 = 0$

y-int: non xz-trace: $3x+5z+45 = 0$

z-int: -9 yz-trace: $z+9 = 0$

170. a) $\pi/3$ b) arc cos (1/5)

172. a) $5x+y = 3$ $3x+z = 3$, $3y-5z+6 = 0$

b) $x = 2$, $z = -1$

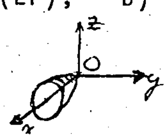
174. a) $\frac{x}{5} = \frac{y-3}{4}$, $z = 6$, b) $\frac{x-a}{m} = \frac{y-b}{n} = \frac{z}{T}$

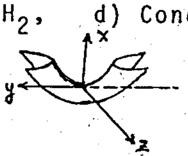
176. $8x+y-26z+6 = 0$

178. a) $13/\sqrt{14}$, b) $4/\sqrt{6}$

180. $\sqrt{61}/\sqrt{22}$

182. a) (EP), b) (HP), c) H_2, d) Cone e) H_1

184. a) 

b) 

186. By rotation about x-axis by an angle $\pi/4$: $(x-2)^2+y'^2-z'^2 = 4$, hyperboloid of one skeet.

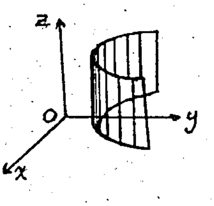
188. $h(x^2+y^2) + 2ax(x-h) = 0$

190. $(x-a)^2 + (y-a)^2 = 2a^2$.

192. $x^2+z^2-y^2 = 0$ cone.

194. $x^2+y^2 = (z+1)^2$, cone vertex at (0, 0, -1)

196. $6y = x^2+9$, parabolic cylinder;



198. $z^2+4y = 4$ parabolic cylinder

200. $x^2+y^2+(z-5/2)^2 = \frac{9}{4}$

202. $2x_1+y_1-8z_1 = 0$

208. a) $T = \frac{1}{\sqrt{3}}(i+j+k)$, $N = -\frac{1}{\sqrt{2}}(i-j)$ $B = -i-j+2k$, $K = \sqrt{2}/3$

b) $T = \frac{1}{\sqrt{6}}(i-j+2k)$, $N = 0$, $B = 0$, $K = 0$

c) $T = \frac{1}{\sqrt{5}}(i+2k)$, $N = i$, $B = \frac{1}{\sqrt{5}}(2i-k)$, $K = 1/10$

220. a) 1 , b) $2\sqrt{2}$, c) ∞ b) $1/2\sqrt{2}$ c) 0

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