MIDDLE EAST TECHNICAL UNIVERSITY

FRESHMAN

CALCULUS

BOOK ONE
part two

B. SÜER & H. DEMİR

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Sayın
Haluk Bingöl'e
sevgi ile başarı
dilekleriyle

FRESHMAN CALCULUS

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DETERMINANTS AND SISTEMS OF LINEAR EQUATIONS

3. I. DETERMINANTS

A. DEFINITIONS

A square array of the form

$$D = |a_{ij}|_n = \begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix} \quad \begin{matrix} (i=1\ , \ldots , n; \\ j=1\ , \ldots , n) \end{matrix}$$

of n^2 elements (entries) is called a *determinant* of *order* n. A determinant consisting of n rows and n columns is said to be of *size* $n \times n$. An element a_{ij} lies in the ith row and jth column, i.e., occupies the place ij.

The elements $a_{ii}(i=1\ , \ldots , n)$ are said to lie on the *main (leading, principal) diagonal* or the *axis* and are called the *diagonal elements*, in general, while the elements a_{n1}, $a_{n-1\ 2}$, ... , a_{1n}, where $i + j = n + 1$, are the elements of the *secondary diagonal*.

In a determinant D, if $a_{ij} \in R$, then D is called a *real determinant* which is equal to a real number. The evaluation of the determinant D will be defined and discussed soon.

Example 1. Given the determinant

$$\begin{vmatrix} -1 & 2 & 5 \\ 4 & 3 & 1 \\ 0 & 7 & 2 \end{vmatrix}$$

a) What are the size and order?

b) What are the elements of 2nd row?

c) What is the place of the element 7?

d) What is the element of the place 1 2?

e) What is the sum of the diagonal elements?

Answer.

a) 3 x 3, 3   b) 4, 3, 1   c) 3 2,   d) 2,   e) 4.

A determinant is *symmetric* if $a_{ij} = a_{ji}$, and *skew symmetric* if $a_{ij} = -a_{ji}$ for all places. Certainly the (main) diagonal elements in a skew symmetric determinant are zero $(a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0)$ and such a determinant is *zero axial*.

Example 2. Complete the real determinant

$$\begin{vmatrix} 0 & 3 & \cdot & \cdot \\ \cdot & t & -2 & \cdot \\ 1 & \cdot & 0 & 4 \\ -5 & 0 & \cdot & 2t \end{vmatrix}$$

a) if it is symmetric,   b) if skew symmetric

Answer.

a) $\begin{vmatrix} 0 & 3 & 1 & -5 \\ 3 & t & -2 & 0 \\ 1 & -2 & 0 & 4 \\ -5 & 0 & 4 & 2t \end{vmatrix}$ b) $\begin{vmatrix} 0 & 3 & -1 & 5 \\ -3 & 0 & -2 & 0 \\ 1 & 2 & 0 & 4 \\ -5 & 0 & -4 & 0 \end{vmatrix}$ $(t = 0)$

Minor and cofactor:

In a determinant D of order n, by the *minor* M_{ij} of the place $i\ j$ (or of the element a_{ij}) is meant the determinant of order $n-1$ obtained by removing from D the ith row and jth column. The *cofactor* C_{ij} of the same place is $(-1)^{i+j} M_{ij}$.

Example 3. Find the minors and cofactors of the elements 5 and 4 in the determinant of Example 1.

<u>Answer</u>.

$$M_{13} = \begin{vmatrix} 4 & 3 \\ 0 & 7 \end{vmatrix}, \quad C_{13} = (-1)^{1+3} M_{13} = M_{13};$$

$$M_{21} = \begin{vmatrix} 2 & 5 \\ 7 & 2 \end{vmatrix}, \quad C_{21} = (-1)^{2+1} M_{21} = -M_{21}$$

<u>Transpose of a Determinant:</u>

By the *transpose* of a determinant

$$D = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = |a_{ij}|_n$$

is meant the determinant

$$\begin{vmatrix} a_{11} & \cdots & a_{n1} \\ & & \cdot \\ a_{1n} & \cdots & a_{nn} \end{vmatrix} = |a_{ji}|_n$$

obtained from D by replacing each row by respective column. It is denoted by D^T (read: D transpose.)

Thus the transpose of

$$D = \begin{vmatrix} 1 & 2 & 3 \\ 5 & 5 & 5 \\ 0 & 7 & 4 \end{vmatrix} \quad \text{is} \quad D^T = \begin{vmatrix} 1 & 5 & 0 \\ 2 & 5 & 7 \\ 3 & 5 & 4 \end{vmatrix}$$

Why the transpose of a symmetric determinant is identical with itself, and that of a skew symmetric one is skew symmetric?

B. Evaluation of a Determinant.

The real determinant $|a_{11}|$ of order 1 is by definition the real number a_{11} itself. Thus, $|-5| = -5$, $|\sqrt{2}| = \sqrt{2}$.

If the order is greater than 1, we define it by cofactors as follows:

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11}C_{11} + \ldots + a_{1j}C_{1j} + \ldots + a_{1n}C_{1n} = \sum_{j=1}^{n} a_{1j}C_{1j} \qquad (1)$$

where $C_{1j} = (-1)^{1+j} M_{1j}$ $(j=1, \ldots, n)$ are cofactors so that the evaluation of D is reduced to the evaluation of determinants of order n-1, which in turn are reduced to the evaluation of determinants of order n-2, and so on. Thus

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12} = a_{11}a_{22} - a_{12}a_{21}.$$

The value given by (1) is the *Laplace expansion* of D with respect to the first row.

The same determinant D has Laplace expansions with respect to any other row or any column. It is proved in Linear Algebra that all these expansions have the same value, hence each one can be used for the evaluation of D.

Thus we have

$$D = a_{i1}\, C_{i1} + a_{i2}\, C_{i2} + \ldots + a_{in}\, C_{in} = \sum_{j=1}^{n} a_{ij}\, C_{ij} \qquad (2)$$

as Laplace expansion with respect to the ith row, and

$$D = a_{1j}\, C_{1j} + a_{2j}\, C_{2j} + \ldots + a_{nj}\, C_{nj} = \sum_{i=1}^{n} a_{ij}\, C_{ij} \qquad (3)$$

as Laplace expansion with respect to the jth column.

<u>Example 4</u>. Evaluate

$$D = \begin{vmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{vmatrix}$$

by expanding it with respect to the 3rd column, and 2nd row.

Solution.

$$D = 6.(-1)^{1+3}\begin{vmatrix}3 & 5\\ 4 & 9\end{vmatrix} + 7.(-1)^{2+3}\begin{vmatrix}8 & 1\\ 4 & 9\end{vmatrix} + 2.(-1)^{3+3}\begin{vmatrix}8 & 1\\ 3 & 5\end{vmatrix}$$

$$= 6(27 - 20) - 7(72 - 4) + 2(40 - 3)$$

$$= 42 - 476 + 74 = 66 - 476 = 116 - 476 = -360.$$

$$D = 3.(-1)^{2+1}\begin{vmatrix}1 & 6\\ 9 & 2\end{vmatrix} + 5.(-1)^{2+2}\begin{vmatrix}8 & 6\\ 4 & 2\end{vmatrix} + 7.(-1)^{2+3}\begin{vmatrix}8 & 1\\ 4 & 9\end{vmatrix}$$

$$= -3(2 - 54) + 5(16 - 24) - 7(72 - 4)$$

$$= 156 - 40 - 476 = -360$$

Any determinant can be evaluated this way by expanding it with respect to any row (column), but as the order gets heigher, calculations become laborious. The following theorems on determinants are helpful in simplifying the computations.

C. THEOREMS ON DETERMINANTS.

Theorem 1. If D is a determinant and D^T is its transpose, then $D^T = D$.

This is a consequence of evaluation of determinant by (2) and (3).

Theorem 2. If two rows (columns) of a determinant are interchanged, the determinant is changed in sign only.

When the given determinant is

$$D = \begin{vmatrix}\cdots a_{1k} \cdots a_{1r} \cdots\\ \\ \cdots a_{nk} \cdots a_{nr} \cdots\end{vmatrix}, \text{then } D' = \begin{vmatrix}\cdots a_{1r} \cdots a_{1k} \cdots\\ \\ \cdots a_{nr} \cdots a_{nk} \cdots\end{vmatrix} = -D.$$

This can be proved by induction.

Corollary. If two rows (columns) of a determinant are identical, the determinant is zero:

$$D = -D \Rightarrow D = 0,$$

from interchanging two identical rows (columns).

Theorem 3. If every element in any row (column) of a determinant is multiplied by the same factor, the whole determinant is multiplied by that factor.

If the given determinant is

$$D = \begin{vmatrix} \cdots & a_{1j} & \cdots \\ & \cdot & \\ \cdots & a_{nj} & \cdots \end{vmatrix}, \text{ then } \quad D' = \begin{vmatrix} \cdots & ca_{1j} & \cdots \\ & \cdot & \\ \cdots & ca_{nj} & \cdots \end{vmatrix} = cD$$

The expansions of D' and D with respect to the jth column prove the assertion.

Corollary 1. To multiply a determinant by a factor, one may multiply every element in any row (column) by that factor.

Corollary 2. If every element in any row (column) of a determinant is zero, the determinant is zero.

Corollary 3. If the corresponding elements in two rows (columns) of a determinant are proportional, the determinant is zero.

$$\begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{k1} & \cdot & \cdot & \cdot & a_{kn} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ ca_{k1} & \cdot & \cdot & \cdot & ca_{kn} \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = c \begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{k1} & \cdot & \cdot & \cdot & a_{kn} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{k1} & \cdot & \cdot & \cdot & a_{kn} \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0$$

Theorem 4. If every element in any row (column) can be expressed as the sum of two quantities, then the given determinant can be expressed as the sum of two determinants of the same order:

$$D = \begin{vmatrix} a_{11} \cdots a_{1k} + b_{1k} \cdots a_{1n} \\ \vdots \qquad\quad \vdots \qquad\quad \vdots \\ a_{n1} \cdots a_{nk} + b_{nk} \cdots a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} \cdots a_{1k} \cdots a_{1n} \\ \vdots \qquad \vdots \qquad \vdots \\ a_{n1} \cdots a_{nk} \cdots a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} \cdots b_{1k} \cdots a_{1n} \\ \vdots \qquad \vdots \qquad \vdots \\ a_{n1} \cdots b_{nk} \cdots a_{nn} \end{vmatrix}.$$

It can be shown by expanding the three determinants with respect to the kth column.

Corollary. A determinant is unaltered if to each element of any row (column) is added the corresponding element of any other row (column) multiplied by a factor:

$$\begin{vmatrix} \cdots a_{1k} + ca_{1r} \cdots a_{1r} \cdots \\ \vdots \qquad\quad \vdots \\ \cdots a_{nk} + ca_{nr} \cdots a_{nr} \cdots \end{vmatrix} = \begin{vmatrix} \cdots a_{1k} \cdots a_{1r} \cdots \\ \vdots \qquad \vdots \\ \cdots a_{nk} \cdots a_{nr} \cdots \end{vmatrix} + \begin{vmatrix} \cdots ca_{1r} \cdots a_{1r} \cdots \\ \vdots \qquad \vdots \\ \cdots ca_{nr} \cdots a_{nr} \cdots \end{vmatrix}$$

$$= \begin{vmatrix} \cdots a_{1k} \cdots a_{1r} \cdots \\ \\ \cdots a_{nk} \cdots a_{nr} \cdots \end{vmatrix} + 0$$

which is the original determinant.

Theorem 5. If the elements of a determinant D are polynomials in a variable x and the determinant vanishes for $x=c$, then $x-c$ is a factor of D.

Since the element of D are polynomials in x, the expansion of D will be a polynomial $D(x)$ and $D(c)=0$ implies $x - c \mid D(x)$.

Theorem 6. If the elements of a row (column) of a determinant are multiplied by the cofactors of respective elements of any other row (column), the sum D' of the products thus obtained is zero.

Proof. The expansion of D with respect to the jth column is

$$D = a_{1j} C_{1j} + \dots + a_{nj} C_{nj},$$

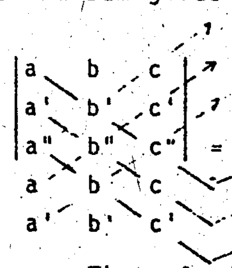
and

$$D' = a_{1k} C_{1j} + \dots + a_{nk} C_{nj} \Rightarrow D' = \begin{vmatrix} \dots a_{1k} \dots a_{1k} \dots \\ \vdots \quad\quad \vdots \\ \dots a_{nk} \dots a_{nk} \dots \end{vmatrix} = 0$$

by the Corollary of Theorem 2.

Rule of Sarrus:

For determinants of order 3 and *only for these,* there is a rule for evaluation commonly used in practice. This rule of SARRUS consists of rewriting the first two rows below the third one, and then multiplying the three elements on the main diagonal, multiplying those just below these elements and multiplying three others below the latter, and then obtaining the sum of these three products; next doing the same for the elements of the secondary diagonal and related ones, obtaining a second sum of three products. Then the difference between the first and second sum gives the value of the determinant:



$$= (ab'c'' + a'b''c + a''bc') - (a''b'c + ab''c' + a'bc'')$$

The rule is applied also by rewriting the first two columns after the third one.

Examples

1. Evaluate the determinant

$$D = \begin{vmatrix} -1 & 2 & 4 \\ 5 & -7 & 3 \\ 0 & 6 & 2 \end{vmatrix}$$

a) by the use of SARRUS' rule, b) by Laplace expansion

Solution.

a) $$D = \begin{vmatrix} -1 & 2 & 4 \\ 5 & -7 & 3 \\ 0 & 6 & 2 \end{vmatrix} \begin{matrix} -1 & 2 \\ 5 & -7 \\ 0 & 6 \end{matrix}$$

$$= (-1)(-7)(2) + (2)(3)(0) + (4)(5)(6)$$

$$- (0)(-7)(4) - (6)(3)(-1) - (2)(5)(2) = 132.$$

b) Since the first column contains a zero element,

$$D = -1.(-1)^{1+1} \begin{vmatrix} -7 & 3 \\ 6 & 2 \end{vmatrix} + 5.(-1)^{2+1} \begin{vmatrix} 2 & 4 \\ 6 & 2 \end{vmatrix}$$

$$= -(-14 - 18) - 5(4 - 24) = 32 + 100 = 132$$

2. Why the following determinants are zero?

$$A = \begin{vmatrix} 15 & -7 & 0 & 13 \\ 8 & 12 & 0 & 9 \\ -11 & 6 & 0 & 0 \\ 7 & 5 & 0 & 2 \end{vmatrix}, \quad B = \begin{vmatrix} -2 & 8 & -10 & 18 \\ 7 & 7 & 7 & 7 \\ -1 & 4 & -5 & 9 \\ 17 & -7 & 9 & 0 \end{vmatrix}$$

$$C = \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix} \qquad D = \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$$

Answer:

$A = 0$, since every element in the third column is zero.

$B = 0$, since two rows are proportional (which ones?).

C = 0, since two columns are proportional.

D = 0, since it will have two proportional columns after adding the second and the third column.

3. Why $\begin{vmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \\ 0 & 8 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 2 & 3 \\ 8 & -5 & 6 \\ 0 & 4 & 1 \end{vmatrix}$?

Answer.

$$2 \begin{vmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \\ 0 & 4 & 1 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \\ 0 & 4 & 1 \end{vmatrix}$$

4. Write the sum

$$D = \begin{vmatrix} 1 & 4 & 2 \\ -2 & 4 & 3 \\ 3 & 7 & 6 \end{vmatrix} + \begin{vmatrix} 1 & 6 & 2 \\ -2 & 7 & 3 \\ 3 & 8 & 6 \end{vmatrix} + \begin{vmatrix} 0 & 10 & 2 \\ 3 & 11 & 3 \\ 8 & 15 & 6 \end{vmatrix}$$

as a single determinant.

Solution. The first two, having two identical corresponding columns, are written as a single determinant, which by the same reasoning can be added to the third determinant:

$$D = \begin{vmatrix} 1 & 4+6 & 2 \\ -2 & 4+7 & 3 \\ 3 & 7+8 & 6 \end{vmatrix} + \begin{vmatrix} 0 & 10 & 2 \\ 3 & 11 & 3 \\ 8 & 15 & 6 \end{vmatrix} + \begin{vmatrix} 1+0 & 10 & 2 \\ -2+3 & 11 & 3 \\ 3+8 & 15 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 10 & 2 \\ 1 & 11 & 3 \\ 11 & 15 & 6 \end{vmatrix}$$

5. Evaluate the determinant

$$D = \begin{vmatrix} 2 & 3 & -4 & 5 \\ 4 & 4 & 2 & 1 \\ 3 & 0 & 6 & 4 \\ 3 & -2 & 4 & 1 \end{vmatrix}$$

Solution. We expand D with respect to that row (column) having more zeros and more simple elements. Then by the use of Corollary of Theorem 4 we get more zeros on that row (column).

Selecting the second column for expansion, multiplying the last row by 2 and adding to the second one, we get another zero element on that column:

$$D = \begin{vmatrix} 2 & 3 & -4 & 5 \\ 4 & 4 & 2 & 1 \\ 3 & 0 & 6 & 4 \\ 3 & -2 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 3 & -4 & 5 \\ 10 & 0 & 10 & 3 \\ 3 & 0 & 6 & 4 \\ 3 & -2 & 4 & 1 \end{vmatrix}$$

$$= -3 \begin{vmatrix} 10 & 10 & 3 \\ 3 & 6 & 4 \\ 3 & 4 & 1 \end{vmatrix} - (-2) \begin{vmatrix} 2 & -4 & 5 \\ 10 & 10 & 3 \\ 3 & 4 & 1 \end{vmatrix}$$

$$= -3 \begin{vmatrix} 10 & 0 & 3 \\ 3 & 3 & 4 \\ 3 & 1 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & -6 & 5 \\ 10 & 0 & 3 \\ 3 & 1 & 1 \end{vmatrix} = 84 - 100 = -16$$

6. Show that $x-5$ and $x+6$ are factors of

$$P(x) = \begin{vmatrix} x & 2 & -3 \\ 3 & 4 & -x \\ 5 & 2 & -3 \end{vmatrix}$$

<u>Solution</u>. $P(5) = 0$, since two rows are identical; $P(6) = 0$, since two columns are proportional.

7. Show that a skew symmetric determinant of order 3 is zero and that of order 4 is a perfect square of a polynomial.

<u>Solution</u>.

$$\begin{vmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{vmatrix} = -a_1 \begin{vmatrix} -a_1 & a_3 \\ -a_2 & 0 \end{vmatrix} + a_2 \begin{vmatrix} -a_1 & 0 \\ -a_2 & -a_3 \end{vmatrix} = 0$$

The property is true for all odd ordered skew symmetric determinants. The proof will be given on Matrices in Book II.

For $n = 4$ we have

$$\begin{vmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & a_4 & a_5 \\ -a_2 & -a_4 & 0 & a_6 \\ -a_3 & -a_5 & -a_6 & 0 \end{vmatrix} = (a_1a_6 - a_2a_5 + a_3a_4)^2$$

This property is true for all even ordered skew symmetric determinants, but the proof will not be given.

8. Prove by induction:

$$P_n(x) = \begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ -1 & x & 0 & \dots & 0 \\ 0 & -1 & x & & \vdots \\ \vdots & & & & 0 \\ 0 & \dots & 0 & -1 & x \end{vmatrix}_{n+1} = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n \quad (1)$$

Proof. The equality is certainly true for $n = 0$. Suppose the property is true for $n = k$, then we show that the determinant

$$P_{k+1} = \begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_k & a_{k+1} \\ -1 & x & 0 & \dots & 0 & 0 \\ 0 & & & & & \vdots \\ \vdots & & & & x & 0 \\ 0 & \dots & & 0 & -1 & x \end{vmatrix}_{k+2}$$

of order $k + 2$ is equal to $a_0x^{k+1} + a_1x^k + \dots + a_{k+1}$.

In the expansion of this determinant with respect to the last column, the cofactor of x is $P_k(x)$:

$$P_{k+1}(x) = x\,P_k(x) + a_{k+1}(-1)^{1+(k+2)} \begin{vmatrix} -1 & x & 0 & \dots & 0 \\ 0 & & & & \\ \vdots & & & & 0 \\ \vdots & & & & x \\ 0 & \dots & 0 & & -1 \end{vmatrix}_{k+1}$$

where

$$P_k(x) = a_o x^k + \ldots + a_k$$

from the induction hypothesis, and the determinant here is equal to $(-1)^{k+1}$. Hence

$$P_{k+1}(x) = (a_o x^{k+1} + \ldots + a_k x) + a_{k+1} \cdot (-1)^{1+(k+2)+(k+1)}$$
$$= a_o x^{k+1} + \ldots + a_k x + a_{k+1}.$$

EXERCISES (3. I)

1. Let a_{ij} be a determinant of order n. Find a relation between the indices i and j, for a_{ij} to be

   a) below (above) the main diagonal,

   b) below (above) the secondary diagonal,

   c) above the main and above secondary diagonal,

   d) below the main and above the secondary diagonal.

2. Compute the following determinants:

   a) $\begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix}$ b) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 7 \\ 3 & 6 & 8 & 9 \\ 4 & 7 & 9 & 10 \end{vmatrix}$

3. Expand and write the result as a polynomial in decreasing powers of x:

$$\begin{vmatrix} a_o & a_1 & a_2 & a_3 \\ -1 & x & 0 & 0 \\ 0 & -1 & x & 0 \\ 0 & 0 & -1 & x \end{vmatrix}$$

4. Evaluate the following determinants:

a) $\begin{vmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{vmatrix}$ b) $\begin{vmatrix} 10 & 17 & 4 \\ 5 & 11 & 7 \\ 3 & 19 & 6 \end{vmatrix}$

5. Evaluate

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$

6. Show that

$$\begin{vmatrix} 1 & 2a & a^2 \\ 1 & a+b & ab \\ 1 & 2b & b^2 \end{vmatrix} = -(a-b)^2$$

7. Evaluate

$$\begin{vmatrix} 2 & -2 & 1 & -4 \\ 1 & 3 & -2 & -3 \\ -1 & 2 & -3 & 2 \\ 4 & 5 & 2 & -2 \end{vmatrix} \qquad \begin{vmatrix} 5 & 4 & 2 & 1 \\ 2 & 3 & 1 & -2 \\ -5 & -7 & -3 & 9 \\ 1 & -2 & -1 & 4 \end{vmatrix}$$

8. Evaluate

$$\begin{vmatrix} 0 & 1 & -2 & 3 & 4 \\ -1 & 0 & 5 & -6 & 7 \\ 2 & -5 & 0 & 8 & -9 \\ -3 & 6 & -8 & 0 & 10 \\ -4 & -7 & 9 & -10 & 0 \end{vmatrix}$$

9. Show that $-(a+b+c)$ is a root of the equation:

$$\begin{vmatrix} x+a & b & c \\ b & x+c & a \\ c & a & x+b \end{vmatrix} = 0$$

10. Find all linear factors of

$$\begin{vmatrix} x & a & b & x \\ a & x & x & b \\ b & x & x & a \\ x & b & a & x \end{vmatrix}$$

11. Same question for:

$$\begin{vmatrix} 0 & a & b & c \\ a & 0 & c & b \\ b & c & 0 & a \\ c & b & a & 0 \end{vmatrix}$$

12. Prove the equality

$$\begin{vmatrix} 0 & a^2 & b^2 & c^2 \\ a^2 & 0 & f^2 & e^2 \\ b^2 & f^2 & 0 & d^2 \\ c^2 & e^2 & d^2 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c^2f^2 & b^2e^2 \\ 1 & c^2f^2 & 0 & a^2d^2 \\ 1 & b^2c^2 & a^2d^2 & 0 \end{vmatrix}$$

13. Prove

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 \\ 1 & c^2 & 0 & a^2 \\ 1 & b^2 & a^2 & 0 \end{vmatrix} = \begin{vmatrix} 0 & a & b & c \\ a & 0 & c & b \\ b & c & 0 & a \\ c & b & a & 0 \end{vmatrix} = -16\Delta^2,$$

where $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ is the area of the triangle with sides a, b, c $(2s = a + b + c)$

14. If

$$\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} = 0$$

show that at least two of the numbers a, b, c, d must be equal to each other.

15. Show that the determinant $|a_{ij} + x|$ of order n is of the form $E + Fx$ where E, F are independent of x.

16. Show that one root of the equation

$$\begin{vmatrix} 11-x & -6 & 2 \\ -6 & 10-x & -4 \\ 2 & -4 & 6-x \end{vmatrix} = 0$$

is 6, and find the other two roots.

is 6, and find the other two roots.

17. If $n \in N$, show that $(b-c)(c-a)(a-b)$ is a factor of

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^{n+2} & b^{n+2} & c^{n+2} \end{vmatrix}$$

18. If $\alpha + \beta + \gamma = 2s$, prove

$$\begin{vmatrix} 1 & \cos\gamma & \cos\beta \\ \cos\gamma & 1 & \cos\alpha \\ \cos\beta & \cos\alpha & 1 \end{vmatrix} = 4 \sin s \sin(s-\alpha) \sin(s-\beta) \sin(s-\gamma)$$

19. Show that

$$\begin{vmatrix} 1 & \cos x - \sin x & \cos x + \sin x \\ 1 & \cos y - \sin y & \cos y + \sin y \\ 1 & \cos z - \sin z & \cos z + \sin z \end{vmatrix} = 2 \begin{vmatrix} 1 & \cos x & \sin x \\ 1 & \cos y & \sin y \\ 1 & \cos z & \sin z \end{vmatrix}$$

20. Evaluate the following determinants:

a) $\begin{vmatrix} 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 5 & 6 \\ -1 & -4 & 0 & 7 & 8 \\ -2 & -5 & -7 & 0 & 9 \\ -3 & -6 & -8 & -9 & 0 \end{vmatrix}$ b) $\begin{vmatrix} 0 & 0 & a & b \\ 0 & 0 & c & d \\ -a & -c & 0 & 0 \\ -b & -d & 0 & 0 \end{vmatrix}$

(See Example 7 and 8.)

ANSWERS TO EVEN NUMBERED EXERCISES

2. a) $adf + 2bce - ae^2 - dc^2 - fb$ b) -2

4. a) $(a^3 - 1)^2$, b) -575

8. 0

10. $(a-b)^2 (2x-a-b)(a+b-2x)$

16. 3; 18

20. 0; $(bc - cd)^2$

3. 2. Systems of Linear Equations

A. DEFINITIONS.

A relation of the form

$$a_1x_1 + \ldots + a_nx_n = b \qquad (1)$$

between n unknowns $x_1, \ldots, x_n$ is called a *linear equation* where $a_1, \ldots, a_n$ are *coefficients* and b is the *constant term*.

(1) is called *homogeneous linear equation* if $b = 0$, and *non homogeneous* otherwise.

Some number of linear equations is called a *system of linear equations:*

$$\begin{array}{ccccc} a_{11} x_1 & + \ldots + & a_{1n} x_n & = & b_1 \\ \vdots & & \vdots & & \vdots \\ a_{m1} x_1 & + \ldots + & a_{mn} x_n & = & b_m \end{array} \qquad (2)$$

(2) is a system of linear equations involving n unknowns and m equations.

If all b's are zero, the system is said to be a *homogeneous* linear system of equations (HLS), and *non homogeneous* linear system of equations (NHLS) otherwise.

The system (2) is *rectangular* and includes the case $m=n$ corresponding to a *square system*. Every square system is rectangular, but not every rectangular system is a square one.

A *solution* of a single linear equation (1) is a point

$$(s_1, s_2, \ldots, s_n)$$

in $\mathbb{R}^n$, whose coordinates satisfy the equation. Therefore

$$a_1s_1 + a_2s_2 + \dots + a_ns_n = b.$$

If a point is a solution of every equation of the system (2) it is a solution point of the system.

As we shall see, some systems have no solution, some have a unique solution and some others have infinitely many solutions. The system having no solution is said to be *inconsistent*, otherwise *consistent*.

Every HLS has the zero solution point, called the *trivial solution*.

B. Solution by Determinants(*)

Square NHLS:

Let

$$\begin{matrix} a_{11}x_1 + \dots + a_{1j}x_j + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{n1}x_1 + \dots + a_{nj}x_j + \dots + a_{nn}x_n = b_n \end{matrix} \qquad (m = n) \qquad (1)$$

be a square NHLS. The determinant of the coefficients is

$$D = D_{coeff} = \begin{vmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix}$$

Theorem. (CRAMER's Rule) The square system (1) has the unique solution point

$$\left(x_1 = \frac{D_1}{D}, \dots, x_j = \frac{D_j}{D}, \dots, x_n = \frac{D_n}{D}\right)$$

if $D \neq 0$, and has no solution or infinitely many solution points if $D = 0$, where D_j is the determinant obtained from D replacing its jth column by the column of constants.

(*) Solution by matrices will be given in Book II.

Proof. To find the unknown x_j, one multiplies the equations (first, second, ...) by the cofactors A_{1j}, A_{2j}, ... of the jth column respectively, and then adds them side by side:

$$\underbrace{(a_{11}A_{1j} + \ldots + a_{n1}A_{nj})}_{0}x_1 + \ldots + \underbrace{(a_{1j}A_{1j} + \ldots + a_{nj}A_{nj})}_{D}x_j + \ldots + \underbrace{(a_{1n}A_{1j} + \ldots + a_{nn}A_{nj})}_{0}x_n =$$

$$= b_1A_{1j} + \ldots + b_nA_{nj} = D_j \quad \text{(by the Theorem 6)}$$

$$Dx_j = D_j \Rightarrow x_j = \frac{D_j}{D} \quad (D \neq 0). \text{ (unique sol.)}$$

Examples. Solve the NH systems of linear equations:

a) $2x-y = 3$, $x+y = 3$

b) $x+2y-z = 2$, $x+y+z = 6$, $2x-y+z = 3$

c) $x+2y-z = 2$, $x+y+z = 6$, $2x+3y = 5$

Solution.

a) $D = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 3, \quad D_x = \begin{vmatrix} 3 & -1 \\ 3 & 1 \end{vmatrix} = 6, \quad D_y = \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} = 3$

$x = \frac{6}{3} = 2, \quad y = \frac{3}{3} = 1 \qquad (2, 1)$

b) $D = \begin{vmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = 7, \quad D_x = \begin{vmatrix} 2 & 2 & -1 \\ 6 & 1 & 1 \\ 3 & -1 & 1 \end{vmatrix} = 7,$

$D_y = \begin{vmatrix} 1 & 2 & -1 \\ 1 & 6 & 1 \\ 2 & 3 & 1 \end{vmatrix} = 14, \quad D_z = \begin{vmatrix} 1 & 2 & 2 \\ 1 & 1 & 6 \\ 2 & -1 & 3 \end{vmatrix} = 21$

$(1 \quad 2 \quad 3)$

c) $D = \begin{vmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & 3 & 0 \end{vmatrix} = 0, \quad D_1 = \begin{vmatrix} 2 & 2 & -1 \\ 6 & 1 & 1 \\ 5 & 3 & 0 \end{vmatrix} = -9 \neq 0$

(continue to evaluate if $D_1 = 0$)

The system is inconsistent (no solution).

An example for the case with infinitely many solutions is

$$\begin{aligned} x + y - z &= 1 \\ 2x + 2y - 2z &= 2 \qquad (D = 0,\ D_1 = 0,\ D_2 = 0,\ D_3 = 0). \\ 3x - y + z &= 3 \end{aligned}$$

It is clear that any scalar t satisfies the equation $D \cdot x_i = D_j (0 \cdot x_j = 0)$. This conclusion leads us to a solution point involving a *parameter t*. This means that one of the unknowns can be taken as parameter (one degree of freedom), say $x = t$.

In this example, the first two equations being identical, taking the first and the last one, we have

$$\left.\begin{aligned} x + y &= 1 + t \\ 3x - y &= 3 - t \end{aligned}\right\} \quad D = -4,\ D_1 = -4,\ D_2 = -4t$$

Then the solution point is

$$\{1, t, t\}, \quad \text{for all } t \in R.$$

Note: The solution shows that, x cannot be taken as parameter, since it has a unique value.

If for a square NHLS,

$$D = 0; \quad D_1 = 0, \ \ldots, \ D_n = 0,$$

we have, in general, one degree of freedom as given in the above example. After taking one of the unknown as parameter t_1, we obtain a system of n equations with n-1 unknowns. Considering any n-1 of these n equations we have a square system (the first reduced system). If this system has a solution point involving t_1, this may be a solution point of the original system if it satisfies the non considered equation (one degree of freedom).

If it does not satisfy, there is no solution of the original system.

If, for the first reduced system,

$$D = 0; \quad D_1 = 0 \ , \ \dots \ , \ D_{n-1} = 0,$$

we have one degree of freedom for the reduced system and two degree of freedom for the original system. One of the $n-1$ unknowns can be taken as parameter t_2. Then considering $n-2$ of the n equations of the original system (or $n-2$ of the $n-1$ equations of the first reduced system) one may find a solution point involving t_1, t_2 $(x_j = x_j\ (t_1, t_2))$. If this solution satisfies the two non considered equations it is the solution of the original system, otherwise it is not, and so on.

If any one of the reduced system has no solution, the system has no solution.

<u>Square homogeneous linear system (SHLS):</u>

Let

$$\begin{matrix} a_{11}x_1 & + \dots + & a_{1n}x_n & = & 0 \\ \vdots & & \vdots & & \vdots \\ a_{n1}x_1 & + \dots + & a_{nn}x_n & = & 0 \end{matrix}$$

be a SHLS, where all constant terms are zero.

From CRAMER's Rule, we have $D \cdot x_j = 0$, since in the determinant D_j, the elements of jth column are zero.

If $D \neq 0$, we have the unique solution point $(0, 0, \dots, 0)$ which is the trivial solutions of the HLS.

<u>Theorem</u>. The necessary condition for a square HLS to have a non trivial solution is $D = 0$.

<u>Proof</u>. Suppose on the contrary that $D \neq 0$. Then by the

CRAMER's Rule the system has unique trivial solution, contradicting that the system has a non trivial solution.

This means that the system has one degree of freedom, since $D_j = 0$ for all j. To find a non trivial solution, if any, continue in the way done in NH case.

<u>Example</u>. Solve the HS:

$$\begin{aligned} x_1 + 2x_2 + 4x_4 &= 0 \\ 3x_1 - x_2 + 5x_3 - 2x_4 &= 0 \\ 4x_1 + x_2 + 5x_3 + 2x_4 &= 0 \\ -2x_1 + 3x_2 - 5x_3 + 6x_4 &= 0 \end{aligned}$$

<u>Solution</u>. Since $D = 0$, the system may have a non trivial solution, and we have one degree of freedom. Taking $x_4 = t$, we obtain

$$\begin{aligned} x_1 + 2x_2 &= -4t \\ 3x_1 - x_2 + 5x_3 &= 2t \\ 4x_1 + x_2 + 5x_3 &= -2t \\ -2x_1 + 3x_2 - 5x_3 &= -6t \end{aligned} \qquad (1)$$

Leaving the last equation out of consideration we have a reduced system:

$$\begin{aligned} x_1 + 2x_2 &= -4t \\ 3x_1 - x_2 + 5x_3 &= 2t \\ 4x_1 + x_2 + 5x_3 &= -2t \end{aligned}$$

of which

$$D = 0, \quad D_1(t) = 0, \quad D_2(t) = 0, \quad D_3(t) = 0.$$

Then we have one more degree of freedom. Taking $x_3 = s$ in the first reduced form, we have

$$x_1 + 2x_2 = -4t$$
$$3x_1 - x_2 = 2t - 5s \qquad (2)$$
$$4x_1 + x_2 = -2t - 5s$$

Discarding one of these equations, say the last one, we have

$$x_1 + 2x_2 = -4t$$
$$3x_1 - x_2 = 2t - 5s$$

whose solution point is

$$\left(x_1 = -\frac{10}{7}s, \quad x_2 = \frac{5}{7}s - 2t\right).$$

If this is a solution of the reduced form (2), it must satisfy the discarded equation:

$$4.(-\frac{10}{7}s) + (\frac{5}{7}s - 2t) = -2t-5s \quad \text{(satisfied!)}$$

Then

$$\left(-\frac{10}{7}s, \ \frac{5}{7}s - 2t, \ s\right)$$

is a solution of (2). For this to be a solution of (1) it must satisfy the discarded last equation:

$$-2(-\frac{10}{7}s) + 3(\frac{5}{7}s - 2t) - 5s = -6t \quad \text{(satisfied!)}$$

Then the point

$$\left(-\frac{10}{7}s, \ \frac{5}{7}s - 2t, \ s, \ t\right)$$

is the solution point of the given system for all $s, t \in R$.

Non square case:

We distinquish two cases as $m > n$ and $m < n$:

1) $m > n$. The number of equations is greater than the number of unknowns:

$$\begin{aligned} a_{11}x_1 + \dots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{n1}x_1 + \dots + a_{nn}x_n &= b_n \\ &\vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n &= b_m \end{aligned} \qquad (1)$$

Consider the system (1') of as many equations as there are unknowns (n), say the first n equations. Solve this square system by the previous method. Obviously if (1') has no solution, the given system (1) has no solution.

If (1') has a solution point S, then substitute the coordinates of S in the remaining equation(s) successively. If all satisfied, then S is the solution of (1), otherwise (1) has no solution.

Example. Solve

a) $2x - y = 7$
$x + y = 2$
$x + 2y = 2$

b) $2x - y = 7$
$x + y = 2$
$x + 2y = 1$
$x - 3y = 6$

Solution.

a) The number of unknowns is two. Consider the system of two equations, say of the first two. It has the solution (3, -1) which when substituted in the remaining third equation, we have

$$3 + 2(-1) = 2 \qquad \text{(not satisfied).}$$

Hence the system has no solution.

b) The number of unknowns is again 2. Considering again the system (1') of the first two equations, we get (3, -1) as solution, and the remaining equations are seen to be satisfied:

$$3 + 2(-1) = 1, \quad 3 - 3(-1) = 6$$

Hence the system b) has (3, -1) as solution:

2) $m < n$. The number of equations is less than the number of unknowns:

$$\begin{matrix} a_{11}x_1 + \dots + a_{1m}x_m + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \dots + a_{mm}x_m + \dots + a_{mn}x_n = b_m \end{matrix} \qquad (1)$$

Take the unknowns $x_{m+1}, \dots, x_n$ as parameters $t_1, \dots, t_{n-m}$ and transpose the related terms to the second side obtaing a square system

$$\begin{matrix} a_{11}x_1 + \dots + a_{1m}x_m = b'_1 \\ \vdots \\ a_{m1}x_1 + \dots + a_{mm}x_m = b'_{mm} \end{matrix} \qquad (1')$$

where $b'_1, \dots, b'_m$ are functions of parameters $t_1, \dots, t_{n-m}$

This square system is solved by the method given earlier. If it has solution, the system has solution point involving $n-m$ parameters or more.

If (1') has no solution(*), try another square system in a similar manner, and so on. If no one has a solution the system has no solution.

Example. Solve

a)
$$x - 3y + z = 2$$
$$x - 2y + 2z = 6$$

b)
$$2x + z = 4$$
$$-2y + 3v = 3$$
$$4x - 2y + 2z + 3v = 2$$

Solution.

a) Setting $z = t$ we have

$$x - 3y = 2 - t$$
$$x - 3y = 6 - 2t$$

(*) There are two reasons: 1) Some of the unknowns taken as parameters may have a fixed value, 2) the system may have no solution.

which has no unique solution since $D = 0$. This gives $z = t = 4$ which, being a fixed value, cannot be taken as parameter.

Let $x = t$. Then

$$\begin{aligned} -3y + z &= 2 - t \\ -3y + 2z &= 6 - t \end{aligned} \quad \Rightarrow \quad \begin{aligned} y &= \frac{t}{3} + \frac{2}{3} \\ z &= 4 \end{aligned} \quad \Rightarrow \left(t, \frac{t}{3} + \frac{2}{3}, 4\right).$$

b) Let $v = t$. Then

$$\begin{aligned} 2x + z &= 4 \\ -2y &= 3-t \\ 4x - 2y + 2z &= 2-3t \end{aligned}$$

where $D = 0$, $D_1 \neq 0$. No solution with this parameter.

Let $z = t$:

$$\begin{aligned} 2x &= 4 - t \\ -2y + 3v &= 3 \\ 4x - 2y + 3v &= 2 - 2t \end{aligned}$$

where $D = 0$, $D_1 = 0$, $D_2 \neq 0$. Again no solution with this parameter. See that there is no solution when $y = t$ or $x = t$.

This system is inconsistent, because the left hand side of the last equation is equal to twice the first plus the second equation, but the right side is not.

EXERCISES (3, 2)

21. Solve the following linear equations:

a) $2x - 3y = 5$ b) $x + 3y - 5z = 2$

22. Solve the system by CRAMER's Rule:

$$\begin{aligned} 2x - y - z &= 4 \\ -x + 2y - z &= -5 \\ x - y + 2z &= 1 \end{aligned}$$

23. Show that the system

$$x + 2y - z = 0, \quad x + y - 2z = -1, \quad -x + 4y + 7z = 6$$

is consistent, and solve it.

24. Examine the consistency of the systems

$$\begin{aligned} x + y + z &= a \\ ax + by + z &= b \\ ax + b^2y + z &= 1 \end{aligned}$$

25. Solve the system of linear equations:

$$\begin{aligned} 3x_1 - 2x_2 + 2x_3 &= 10 \\ x_1 + 2x_2 - 3x_3 &= -1 \\ 4x_1 + x_2 + 2x_3 &= 3 \end{aligned}$$

26. Solve

$$\begin{aligned} (c^2 - a^2)y - bc\, z + ba\, x &= 0 \\ (a^2 - b^2)z - ca\, x + cb\, y &= 0 \end{aligned}$$

27. Solve and discuss

$$\begin{aligned} \lambda x + y + z + u &= a \\ x + \lambda y + z + u &= b \\ x + y + \lambda z + u &= c \\ x + y + z + \lambda u &= d \end{aligned}$$

28. If a, b, c are not all zero, show that the system

$$bz - cy = a', \quad cx - az = b', \quad ay - bx = c'$$

is consistent or inconsistent according as $aa' + bb' + cc'$ is zero or non zero.

29. For what values of λ the system

$$\begin{aligned} x + 2y + (\lambda + 2)z &= 10 \\ 2x + 3y + (\lambda + 3)z &= 16 \\ 3x + (6\lambda - 1)y + 7z &= 26 \end{aligned}$$

a) has no solution b) has one solution

c) has infinitely many solutions.

30. Check the consistency of the system, and solve (if possible):

a) $3x_1 + 2x_2 + x_3 = 7$
$2x_1 + x_2 - 2x_3 = 4$
$4x_1 + 3x_2 + 4x_3 = 20$

b) $2x_1 + 7x_3 = 4$
$x_1 + x_2 + 2x_3 = 1$
$2x_1 + 2x_2 + 4x_3 = 2$

31. Solve

a) $x + 2y = 3$
$2x + y = 0$
$3x - y = 5$

b) $x + 2y = 3$
$2x + y = 0$
$x + y = 2$

32. Solve

a) $x^2 + 5y^2 - z^2 = 0$
$2x^2 + 3y^2 - z^2 = 2$
$x^2 - y^2 + 2z^2 = 21$

b) $\frac{2}{x} + \frac{1}{y} + \frac{3}{z} = 1$
$\frac{6}{x} - \frac{2}{y} + \frac{6}{z} = 3$
$\frac{1}{x} + \frac{1}{2y} - \frac{3}{z} = 0$

33. Solve

a) $3x - y - z = 2$
$4x + y + 2z = 7$
$x + 2y + 3z = 4$

b) $s + t = 2$
$t - u = 3$
$u - z = 2$
$s - z = 7$

34. If the following systems have the same solution, what is the relation between a, b, c?

$2x - 3y + z = 5$
$ax + y - bz = 2$
$x - y + 2z = 0$

$5x - y + 2z = 4$
$x + cy - z = 0$

35. Discuss the solution of

$$x + a^2y + a^4z = a$$
$$x + b^2y + b^4z = b$$
$$x + c^2y + c^4z = c$$

ANSWERS TO EVEN NUMBERED EXERCISES

22. (0, -3, -1)

24. $a \neq 1$, $b \neq 1$, $a \neq b$ consistency

26. $(\lambda a, \lambda b, \lambda c)$

30. a) No solution, b) $(2 - 7t/2, -1 + 3t/2, t)$

32. a) $(\pm 2, \pm 1, \pm 3)$ (all combinations of signs) b) (2, -1, 3)

34. $5a + 6b = 17$, $c = 11/7$

A SUMMARY
(CHAPTER 3)

Expansion of a determinant of order n:

$$D = |a_{ij}|_n = \sum_{i=1}^{n} a_{ij} C_{ij} \quad \text{by the jth column}$$

$$= \sum_{j=1}^{n} a_{ij} C_{ij}, \quad \text{by the ith row.}$$

CRAMER's Rule: For a SNHLS,

$$x_j = \frac{D_j}{D} \quad \text{if } D \neq 0, \text{ where}$$

D is the determinant of coefficients, D_j is the determinant obtained from D replacing its jth column by the column of constants.

For SHLS,

when $D = 0$ system may have non trivial solution.

MISCELLANEOUS EXERCISES

36. Show that $x + 1$ is a factor of

$$\begin{vmatrix} x+1 & 2 & 3 \\ 1 & x+1 & 3 \\ 3 & -6 & x+1 \end{vmatrix}$$

and factorize it completely.

37. If a, b, c are distinc real numbers, show

$$\begin{vmatrix} a & a^2 & a^3-1 \\ b & b^2 & b^3-1 \\ c & c^2 & c^3-1 \end{vmatrix} = 0 \qquad abc = 1$$

38. Prove that

$$\begin{vmatrix} x^3 & x^2 & x & 1 \\ 3x^2 & 2x & 1 & 0 \\ y^3 & y^2 & y & 1 \\ 3y^2 & 2y & 1 & 0 \end{vmatrix} = (x-y)^4$$

39. Show that

$$\begin{vmatrix} 1 & A & B & AB \\ 1 & a & B & aB \\ 1 & A & b & Ab \\ 1 & a & b & ab \end{vmatrix} = (A-a)^2\ (B-b)^2$$

40. Show that

$$\begin{vmatrix} \cos(x+y) & \sin(x+y) & -\cos(x+y) \\ \sin(x-y) & \cos(x-y) & \sin(x-y) \\ \sin 2x & 0 & \sin 2y \end{vmatrix} = \sin 2(x+y)$$

41. Prove $D_n = a_n D_{n-1} + D_{n-2}$, where

$$D_n = \begin{vmatrix} a_1 & 1 & 0 & \cdots & 0 \\ -1 & a_2 & 1 & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & -1 & a_n \end{vmatrix}$$

(<u>Hint</u>: Expand by the last row or column)

42. If D_n denotes the determinant

$$\begin{vmatrix} a & 1 & 0 & \cdots & 0 \\ 1 & a & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & a \end{vmatrix}_n$$

then show

$$D_{n+1} - a\, D_n + D_{n-1} = 0,$$

and prove that

$$D_n = (p^{n+1} - q^{n+1})/(p-q)$$

where p and q are the roots of $x^2 - ax + 1 = 0$.

43. Evaluate

$$D(x) = \begin{vmatrix} 1+x & 1 & \cdots & 1 \\ 1 & 1+x & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 1+x \end{vmatrix}$$

44. Show

$$\begin{vmatrix} a^2-b^2-c^2+d^2 & 2(ab+cd) & -2(ac-bd) \\ -2(ab-cd) & a^2-b^2+c^2-d^2 & 2(ad+bc) \\ 2(ac+bd) & 2(ad-bc) & a^2+b^2-c^2-d^2 \end{vmatrix} = (a^2+b^2+c^2+d^2)^3$$

45. The roots of $x^4 - 3px - q = 0$ are x_1, x_2, x_3, x_n, and

$$A_n = \begin{vmatrix} 1 & x_1^n & x_1^{n+1} & x_1^{n+2} \\ 1 & x_2^n & x_2^{n+1} & x_2^{n+2} \\ 1 & x_3^n & x_3^{n+1} & x_3^{n+2} \\ 1 & x_4^n & x_4^{n+1} & x_4^{n+2} \end{vmatrix}$$

Find the values of A_4/A_1 and A_5/A_1 in terms of p, q.

46. Characterize quadrilateral with consecutive sides a, b, c, d such that

$$D = \begin{vmatrix} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{vmatrix} = 0$$

47. Prove that the system

$$x - y + z = 0$$
$$2x + y - z = 0$$
$$x + 5y - 5z = 0$$

is consistent and solve it.

48. Solve

$$3x + 5y - 7z = 13$$
$$4x + y - 12z = 6$$
$$2x + 9y - 3z = 20$$

49. Solve

$$x_1 + 2x_2 + 3x_3 + 4x_4 = 5$$
$$2x_1 + x_2 + 4x_3 + x_4 = 2$$
$$3x_1 + 4x_2 + x_3 + 5x_4 = 6$$
$$2x_1 + 3x_2 + 5x_3 + 2x_4 = 3$$

50. Solve

$$x_1 + x_2 + x_3 + x_4 + x_5 = 1$$
$$x_1 - x_2 + x_3 + x_4 + x_5 = 1$$
$$2x_1 + 3x_2 + x_3 - x_4 + 2x_5 = -4$$
$$x_1 + 3x_3 + x_5 = 6$$
$$x_2 + 2x_3 + 3x_4 - x_5 = 10$$

ANSWERS TO EVEN NUMBERED EXERCISES

36. $(x + 1)(x^2 + 2x + 8)$

46. $D = (a + b + c + d)(a - b + c - d)\left[(a-c)^2 + (b-d)^2\right] = 0$

$a + c = b + d$ (circumscibed); $a = c$, $b = d$ (parallelogram

48. $(1, 2, 0)$

50. $(-2, 0, 3, 1, -1)$

CHAPTER 4

ANALYTIC GEOMETRY IN $\mathbb{R}^2$

4. 1. A Review of Line, Circle and Conics (in standard forms)

A 2-space (two dimentional space) $\mathbb{R}^2$ is the set

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y): x, y \in \mathbb{R}\}$$

of ordered pairs of real numbers x, y where x is the *abscissa* (the first coordinate) and y is the *ordinate*(the second coordinate) of a point in $\mathbb{R}^2$. There is one-to-one correspondance between the points and the ordered pairs.

Let $A(x_1, y_1)$, $B(x_2, y_2)$ be two given points. Then the distance $d(A, B) = |AB|$ between A and B is:

$$d(A, B) = |AB| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$$

The coordinates x, y of the point C dividing the segment $[AB]$ in the division ratio $r = \overline{CA}/\overline{CB}$ (inner division when $r < 0$, outer division when $r > 0$) are

$$x = \frac{x_1 - rx_2}{1 - r}, \qquad y = \frac{y_1 - ry_2}{1 - r}$$

Then the midpoint I of $[AB]$ is

$$x = \frac{x_1 + x_2}{2}, \qquad y = \frac{y_1 + y_2}{2} \quad (r = -1)$$

A. Lines. We have the following forms of equations of a line:

1. $y - y_1 = m(x - x_1)$ Point-slope form
2. $y = mx + n$ Slope-intercept form

3. a) $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1}$ Symmetric form — Two-point forms

b) $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$ Determinant form — Two-point forms

In (a) $x = x_1$ if $x_2 - x_1 = 0$ (or $y = y_1$ if $y_2 - y_1 = 0$)

4. $\dfrac{x}{u} + \dfrac{y}{v} = 1$ Intercept form

5. $Ax + By + C = 0$ $(A^2 + B^2 \neq 0)$ General form

(General linear equation)

6. a) $x\cos\omega + y\sin\omega - p = 0$ (Hess equation; Euler's equation) — Normal forms

b) $ax + by + c = 0$ $(a^2 + b^2 = 1)$ — Normal forms

In (a) p is the distance of the origin from the line and is the oriented angle from positive x-axis to the ray through the origin intersecting the line perpendicularly.

<u>Derivation of normal form:</u>

Intercepts of the line ℓ in terms of p and ω being

$$u = p/\cos\omega, \qquad v = p/\sin\omega,$$

the intercept form (4) gives

$$x\cos\omega + y\sin\omega - p = 0$$

when $\omega \neq k\pi$ or $\omega \neq (2k+1)\pi/2$ $(k = 0, 1)$

$x - p = 0$ is a vertical line when $\omega = k\pi$,

$y - p = 0$ is a horizontal line when $\omega = (2k+1)\pi/2$

<u>Example</u>. Write the equation of the family of lines

a) having the constant distance 3 units from the origin,

b) having the constant slope angle α.

Solution.

a) By the use of normal form: $x \cos\omega + y \sin\omega - 3 = 0$

b) By the use of slope-intercept form: $y = x \tan\alpha + n$ or

by normal form $x \cos(\alpha - \frac{\pi}{2}) + y \sin(\alpha - \frac{\pi}{2}) - p = 0$,

since $\omega = \alpha - \frac{\pi}{2}$.

The equation of a line in any form can obviously be written as the general form but the converse may not be true.

Example. Transform the general equation $\sqrt{3}x + y - 4 = 0$ into other forms.

Solution. Slope: $m = -\sqrt{3}$

$$x_1 = \sqrt{3} \Rightarrow y_1 = 1 \Rightarrow A(\sqrt{3}, 1)$$
$$x_2 = -\sqrt{3} \Rightarrow y_2 = 7 \Rightarrow B(-\sqrt{3}, 7)$$

Point-slope form: $y - 1 = -3(x-3)$

Two-point form: $\frac{x - \sqrt{3}}{-2\sqrt{3}} = \frac{y - 1}{6}$

Now

$$x = 0 \Rightarrow y = v = 4$$
$$y = 0 \Rightarrow x = u = 4/\sqrt{3},$$

Intercept form: $\frac{x}{4/\sqrt{3}} + \frac{y}{4} = 1$

Slope-intercept form: $y = -\sqrt{3}\,x + 4.$

Since $A^2 + B^2 = 4$, $\sqrt{A^2 + B^2} = 2$ and $C = -4 < 0$, we have

Normal form: $\frac{\sqrt{3}}{2} x + \frac{1}{2} y - 2 = 0$ or

$$x \cos\frac{\pi}{6} + y \sin\frac{\pi}{6} - 2 = 0$$

Linear family of lines:

Let

$$Ax + By + C = 0 \quad \text{and} \quad A'x + B'y + C' = 0$$

be the equations of two lines. Then for $\lambda, \lambda' \in R$, the equation

$$\lambda(Ax + By + C) + \lambda'(A'x + B'y + C') = 0 \qquad (1)$$

represents linear family of lines (passing through the point of intersection of the original lines when they are intersecting, since the coefficients of λ and λ' vanish for the coordinates of the common point).

If the lines are parallel the same equation represents a family of lines parallel to these. Why?

For what values of λ, λ' the equation (1) represents the first or second line?

Division by one of the parameters, say λ, gives

$$(Ax + By + C) + \mu(A'x + B'y + C') = 0$$

involving a single parameter μ, representing the same family except $A'x + B'y + C' = 0$.

<u>Distance of a point from a line:</u> $d(P_0, \ell)$

Let $P_0(x_0, y_0)$ be a point and ℓ: $Ax + By + C = 0$ be a line. Then the distance of the point P_0 from ℓ is

$$d(P_0, \ell) = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

In the case of normal form:

$$d(P_0, \ell) = \left| x_0 \cos\omega + y_0 \sin\omega - p \right|$$

and

$$x_0 \cos\omega + y_0 \sin\omega - p \begin{cases} < 0 & \text{when } P_0 \text{ and origin are on the same side of } \ell, \\ > 0 & \text{other wise} \end{cases}$$

The distance between two lines is given by

$$d(\ell, \ell') = \begin{cases} 0 & \text{when } \ell, \ell' \text{ intersect,} \\ d(P, \ell') & \text{when } P\in\ell \text{ and } \ell // \ell' \end{cases}$$

Example. Given the points $A(t, 3)$, $B(4, 5)$ and $C(4,8)$

a) find t if A, B, C are collinear (Points of the same line)

b) Setting $t=-2$ find the equations of the lines AB and BC

c) Find the equation of the line through B and $P(1, -1)$

d) Find $d(P, BC)$, $d(P, AB)$

Solution.

a) Setting the coordinates of the point in $Ax + By + C = 0$ we have the HLS

$$\begin{aligned} tA + 3B + C &= 0 \\ 4A + 5B + C &= 0 \\ 4A + 8B + C &= 0 \end{aligned}$$

To have a non trivial solution we get

$$\begin{vmatrix} t & 3 & 1 \\ 4 & 5 & 1 \\ 4 & 8 & 1 \end{vmatrix} = 0 \quad \text{giving} \quad t = 4.$$

This is one of the ways of solution. The others are:

1) By the use of distance: For the collinearity, one of $|AB|$, $|BC|$, $|CA|$ must be the sum of the other two.

2) By the use of slope: For the collinearity, the slopes of AB and BC must be equal.

3) By the use of equation: For the collinearity, the coordinate of one of the points, say of A, must satisfy the equation of the line through the other two points B, C. (This mean that $d(A, BC) = 0$)

b) $$\frac{x - x_1}{x_2 - x_1} \quad \frac{y - y_1}{y_2 - y_1} \quad \Rightarrow$$

$$AB: \quad \frac{x + 2}{y + 2} \quad \frac{y - 3}{y - 3} \Rightarrow x - 3y + 11 = 0,$$

$$BC: \quad x_1 = x_2 = 4 \quad \Rightarrow x = 4$$

c) Since B is the common point of the line AB and BC, the lines

$$\lambda(x - 4) + \lambda'(x - 3y + 11) = 0$$

through B, and P must satisfy

$$\lambda(1 - 4) + \lambda'(1 + 3 + 11) = 0 \quad \lambda = \lambda'$$

$$\Rightarrow \lambda'\big(5(x - 4) + (x - 3y + 11)\big) = 0 \quad (\lambda' = 0).$$

$$\Rightarrow \quad 2x - y - 3 = 0.$$

d) Since BC is a vertical line, $d(P, BC) = 3$,

$$d(P, AB) = \frac{|1 + 3 + 11|}{\sqrt{1 + 9}} = 15/\sqrt{10}.$$

Angle between two lines:

The angle Θ between two lines with slopes m_1, m_2 is given by

$$\tan\Theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

where Θ is the positively oriented angle from the first to the second line.

It follows that the lines are parallel iff (if and only if) $m_1 = m_2$ and perpendicular iff $m_1 m_2 = -1$.

Example. Given the line

$$\ell: \ 2x + 3y - 5 = 0,$$

a) Find the equation of the line $\ell' // \ell$ and through $A(-2,1)$.

b) Find the equation of the line $\ell'' \perp \ell$ and through $A(-2,1)$.

c) Show that the sum of the products of the corresponding coefficients in the equations of ℓ and ℓ'' is zero. Generalize.

Solution.

a) Any line $// \ell$ can be written as

$$2x + 3y + C = 0,$$
$$\Rightarrow \quad 2.(-2) + 3.1 + C \quad 0 \Rightarrow C = 1$$
$$\Rightarrow \quad \ell': 2x + 3y + 1 = 0.$$

b) The slope of ℓ being $m = -2/3$ that of ℓ'' will be $m'' = -1/m = 3/2$. Hence by the point slope form we have

$$y - 1 = \frac{3}{2}(x + 2),$$
$$\ell'' : 3x - 2y + 8 = 0$$

c) For the mentioned product we have indeed,

$$2 . 3 + 3(-2) = 0$$

To generalize, let the lines

$$\ell : Ax + By + C = 0,$$
$$\ell': A'x + B'y + C' = 0$$

be perpendicular. Having

$$m = -A/B, \quad m' = -A'/B, \quad \text{and} \quad mm' = -1,$$

it follows that $(-A/B)(-A'/B') = -1$ or $AA' + BB' = 0$

B. Conics

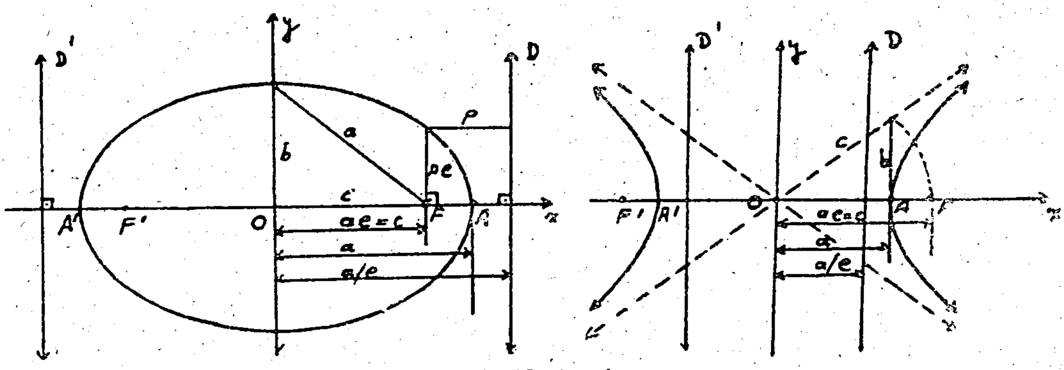
Conics are defined in various ways. Here it is preferred to define them by a fixed line D, a fixed point F with $d(F, D) = p \neq 0$, a fixed ratio e, called respectively *directrix*, *focus* and *eccentricity*:

Definition. The set

$$\{P: d(P, F)/d(P, D) = e\}$$

of points is called

$$\begin{cases} \text{an } \textit{ellipse} & \text{when } e < 1, \\ \text{a } \textit{parabola} & \text{when } e = 1, \\ \text{a } \textit{hyperbola} & \text{when } e > 1. \end{cases}$$



Central Conics

Standard equations:

a) Circle : $x^2 + y^2 = r^2$
b) Ellipse : $b^2x^2 + a^2y^2 = a^2b^2$
c) Hyperbola : $\pm b^2x^2 \pm a^2y^2 = a^2b^2$

Center at the origin, axes are on the coordi[nate] axes

d) Parabola : $y^2 = 2px$ (or $x^2 = 2py$)

Vertex at the origin, axis on a coordinate axis

If the vertex of the parabola, and centers of the othe[rs] are at the point (h, k) with the same orientations of the axes, the above equations become:

a) $(x - h)^2 + (y - k)^2 = r^2$
b) $b^2(x - h)^2 + (a^2(y - k)^2 = a^2b^2$
c) $\pm b^2(x - h)^2 \pm a^2(y - k)^2 = a^2b^2$
d) $(y - k)^2 = 4p(x - h)$ (or $(x - h)^2 = 4p(y - k)$)

where r is the radius of the circle, max {a, b}, min {a, b} are the *semi major* and *semi minor axes* of the ellipse, and in $b^2x^2 - a^2y^2 = ab^2$ the elements a, b (in $a^2y^2 - b^2x^2 = a^2b^2$) the elements b, a) are *semi transverse* and *semi conjugate axes* of the hyperbola.

The two hyperbolas given in (c) are called *conjugate hyperbolas* having the same pair of asymptotes given by

$$b^2(x - h)^2 - a^2(y - k)^2 = 0$$

An ellipse becomes a circle, and a hyperbola becomes a *rectangular (equilateral) hyperbola* when $a = b$.

Linear family of conics:

Let $F(x, y) = 0$ and $G(x, y) = 0$ be standard equations of two conics. Then for $\lambda, \mu \in R$, the equation

$\lambda F(x, y) + \mu G(x, y) = 0$ (at least one of $\lambda, \mu \neq 0$)

represents the linear family of conics (passing through the points of intersecting of the original conics when they are intersecting).

$\lambda = 0$ $G(x, y) = 0$, and $\mu = 0 \Rightarrow F(x, y) = 0$.

If $\lambda \neq 0$, then the family can be represented by the use of single parameter:

$$F(x, y) + k(G(x, y) = 0 \quad (k = \mu/\lambda)$$

What happens when $F = 0$, $G = 0$ represent any two curve?

Example. Given two conics C, C' defined by their directrices, foci and eccentrisities

$$D: x = 6, \quad F(1, 2), \quad e = 2/3$$
$$D: y = -3, \quad F(-1, 2) \quad e = 3/2,$$

find

a) the standard equations and determine

p, a, c, b for C and C'

b) the equation of the conic which is a member of the linear family of C, C' and passing through the point $(-3, 0)$,

c) does this family contain a parabola? Discuss.

Solution.

a) C: $d(P, F)/d(P, D) = 2/3 \quad \Rightarrow$

$$\frac{(x-1)^2+(y-2)^2}{(x-6)^2}=\frac{4}{9} \Rightarrow \frac{(x+3)^2}{36}+\frac{(y-2)^2}{20}=1$$

$\Rightarrow$ $p=5$, $a=6$, $c=ae=4$, $b=2\sqrt{5}$.

$$C': \frac{(x+1)^2+(y-2)^2}{(y+3)^2}=(\tfrac{3}{2})^2 \Rightarrow \frac{(y+7)^2}{36}-\frac{(x+1)^2}{45}=1$$

$\Rightarrow$ $p=5$, $a=6$ (semi transverse axis),
$c=ae=9$, $b=3\sqrt{5}$

b) $\lambda\left[5(x+3)^2+9(y-2)^2-180\right]+\mu\left[5(y+7)^2-4(x+1)^2-180\right]=0$

Setting the point $(-3, 0)$ we have

$$\mu = \frac{144}{49}\lambda$$

and

$$49\left[5(x+3)^2+9(y-2)^2-180\right]+144\left[5(y+7)^2-4(x+1)^2-180\right]=0$$

c) Coefficient of x^2 (or of y^2) is to be zero:

$$5\lambda-4\mu=0 \quad \text{or} \quad 9\lambda+5\mu=0.$$

Yes, the family contains two parabolas, one having axis // to x-axis.

Example 2. Discuss the following equations:

1. a) $b^2x^2+a^2y^2=-a^2b^2$ $(ab\neq 0)$ b) $b^2x^2+a^2y^2=0$ $(ab=0)$
2. $b^2x^2-a^2y^2=0$ $(ab\neq 0)$
3. $x^2=a$ (or $y^2=b$)

Solution.

1. a) Since the left hand side is non negative and the right hand side is negative, no point in R^2 satisfies this equation (Imaginary ellipse, no graph)

   b) The only solution is $(0, 0)$ which is the origin (*degenerate ellipse*), the graph is a point).

2. $b^2x^2 - a^2y^2 = 0$ $(bx-ay)(bx+ay) = 0$ $bx-ay = 0$ or $bx + ay = 0$ (*degenerate hyperbola;*) the graph is two intersecting lines which are the asymptotes of the hyperbolas $\pm b^2x^2 \mp a^2y^2 = a^2b^2$) .

3. No real point / Two coincident lines / Two parallel lines — when a (or b) <0 imagenary parabola; $=0$; >0 degenerate parabola

No real point	when a (or b)	<0 imagenary parabola
Two coincident lines		=0
Two parallel lines		>0 degenerate parabola

The discussion is also valid when x and y are replaced by $x - h$ and $y - k$.

EXERCISES (4, I)

1. Find the projections on the coordinate axes and the length of the line segment joining the following points:
   a) (-4, -4) and (1, 3) b) $(-\sqrt{2}, \sqrt{3})$ and $(\sqrt{3}, \sqrt{2})$

2. Find the midpoint of the line segment with the following and points:
   a) (a-b, d-c) and (a+b, c+d) b) (-4, 6) and (2, 4)

3. Find the coordinates of the point which divides internally the line segment joining (-1, 4) and (-5, -8) in the ratio 1/3.

4. Find the locus of the points equidistant from A(3, -4) and B(-1, 6).

5. One end of a line segment whose length is 13 is the point (-4, 8), the ordinate of the other end is -3. What is its abscissa?

6. Show that the diagonals of a rectangle are equal.

7. Show that the points

a) (3, 0), (6, 4), (-1, 3) are the vertices of a right triangle (1) by means of slope, (2) by the use of Pythagorean theorem,

b) (2, 2), (-2, -2), $(2\sqrt{3}, -2\sqrt{3})$ are the vertices of an equilateral triangle.

8. Prove by means of slope that the points (10, 0), (5, 5), (5, -5) and (-5, 5) are the vertices of a trapezoid.

9. Prove that the points

a) (2, 3), (1, -3), (3, 9) are collinear: (1) by means of slope, (2) by means of distance, (3) by the use of equations, (4) by determinants.

b) (1, -2), (2, 3) and (-2, -17) are collinear.

10. If the points (a, 3), (3, -6) and (4, 7) are collinear, find a.

11. Show that the line

$$t(2x - y - 9) + k(x - 3y - 17) = 0$$

passes through a fixed point for all values of t and k. What is this fixed point?

12. Show that

$$\begin{vmatrix} x & y & 1 \\ -1 & 3 & 1 \\ 3 & 5 & 1 \end{vmatrix} = 0$$

is the equation of the line through (-1, 3) and (3, 5).

13. If the vertices of a triangle are A(2, 3), B(5, 7), C(3, 9), show that the area of the triangle is

$$A = \frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ 5 & 7 & 1 \\ 3 & 9 & 1 \end{vmatrix}$$

14. Prove that the points (6, 6), (7, -1), (0, -2), (-2, 2) lie on a circle whose center is (3, 2).

15. Find the centers and radii of the following circles:

 a) $x^2 + 2y - 3x + y^2 = 0$ b) $x^2 + y^2 + 4y = 5$

16. Write the equation of the circle whose center and radius are:

 a) (2, 5), $r = 7$ b) (3, -4), $r = 3$

17. Classify the following curves:

 a) $x^2 - 2x + y^2 + 4y + 1 = 0$ b) $x^2 + y^2 - 4y + 5 = 0$

 c) $x^2 + 2x + y^2 - 2y + 2 = 0$ d) $(x - 2y)(2x + 3y - 5) = 0$

18. Write the standard equation of the parabola whose vertex A and directrix D are:

 a) A(2, 5), D: $y = 3$ b) A(3, -4), D: $y = -2$

19. Write the standard equation of the ellipse whose center, eccentricity and a are:

 a) (1, -2), $e = 2/3$, $a=6$ b) (-3, 0). $e = 4/5$, $a=5$

20. Same question if the given curve is a hyperbola:

 a) (2, 3), $e = 5/3$, $a=3$ b) (-2, 1), $e = 5/4$, $a=8$

ANSWERS TO EVEN NUMBERED EXERCISES

2. a) (a, d), b) (-1, 5)

4. $2x - 5y + 3 = 0$

10. 4

16. a) $(x-2)^2 + (y-5)^2 = 49$, b) $(x-3)^2 + (y+4)^2 = 9$

18. a) $8(y-5)=(x-2)^2$, b) $8(y+4)=(x-3)^2$.

20. a) $16(x-2)^2-9(y-3)^2=144$, b) $36(x+2)^2-64(y-1)^2=2324$.

4. 2. Second Degree Curves (SDC):

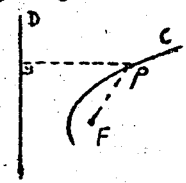
A. Definitions and Classification:

The equation of a conic C in the general case is obtained by taking the directrix D and focus F arbitrarily in the analytic plane as

$$D:\ f(x,\ y)=ax+by+c=0,\quad (a^2+b^2=1),\quad F(x_0,\ y_0)$$

From

$$C=\{P(x,\ y):\ d(P,\ F)/d(P,\ D)=e\}$$



we have

$$\left[(x-x_0)^2+(y-y_0)^2\right]\ /\ |ax+by+c|^2=e^2$$

which when expanded and arranged gives the general second degree equation (SDE):

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0,\quad (A^2+B^2+C^2\neq 0)\qquad (1)$$

as the equation of C where the coefficient $A, B, \ldots, F$ are functions of constants $a, b; c, x_0, y_0$ and e. It follows that the equation of any conic is of second degree, since $A^2+B^2+C^2\neq 0$ But the equation (1) may represent curves other than the conics (ellipse, hyperbola, parabola) as seen from

$$(2x-y+1)(x+y-3)=0$$

which is of second degree and represents two intersecting lines (a degenerate conic).

A curve represented by second degree equation (1) in the variables x, y is called a *second degree curve*.

The following are second degree curves:

$$2x^2 - 4xy + x + 3y - 1 = 0$$
$$2x^2 + xy - 4y^2 - 6x + y = 0$$

The number

$$\Delta = B^2 - 4AC,$$

called the *discriminant* of (1), permits to classify the second degree curves as elliptic, hyperbolic and parabolic by asymptotic directions. The equation $y = mx + n$ of an asymptote of a second degree curve can be determined as follows. (See Part I, sketching the graph of an algebraic function)

$$Ax^2 + Bx(mx + n) + C(mx + n)^2 + Dx + E(mx + n) + F = 0$$
$$(A + Bm + Cm^2)x^2 + (Bn + 2Cmm + D + Em)x + Cn^2 + En + F = 0$$

$Cm^2 + Bm + A = 0$ gives the slope m (the asymptotic direction). We have the cases:

Elliptic case	when Δ	< 0	(no real m)
Parabolic case		$= 0$	(Two equal real m)
Hyperbolic case		> 0	(Two distinct real m)

The discriminant Δ does not give any information about degeneracy of a second degree curve. The following theorem state under what conditions a second degree curve is degenerate (non degenerate):

<u>Theorem</u>. A second degree curve, real or imaginary, given by

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \quad (A^2 + B^2 + C^2 \neq 0) \qquad (1)$$

degenerate or non degenerate according as $T = 0$ or $T \neq 0$

$$T = \begin{vmatrix} 2A & B & D \\ B & 2C & E \\ D & E & 2F \end{vmatrix} = 2(4ACF + BDE - AE^2 - CD^2 - FB^2)$$

Proof. Let $A \neq 0$. Multiplying (1) by 4A and completing to (perfect) square:

$$(2Ax + By + D)^2 - (By + D)^2 + 4A(y^2 + 4AEy) + 4AF = 0$$

$$(2Ax + By + D)^2 - \left[(B^2 - 4AC)y^2 + 2(BD - AE)y + (D^2 - 4AF)\right] = 0$$

To be factorable iff the bracket is a perfect square. Then

$$\delta = (BD - 2AE)^2 - (B^2 - 4AC)(D^2 - 4AF) = 0$$

$$\delta = -4A(4ACF + BDE - AE^2 - CD^2 - FB^2) = -2AT \Rightarrow T = 0$$

When $A = 0$, T becomes $2(BDE - CD^2 - FB^2)$; and (1) reduces to

$$Bxy + Cy^2 + Dx + Ey + F = 0$$

Multiplying it by 4C and completing to square we have:

$$(2Cy + Bx + E)^2 - \left[(Bx + E)^2 - 4CDx - 4CF\right] = 0$$

where bracket is to be a perfect square implying $T = 0$.

The proof can be done considering the coefficient C instead of A, in a similar manner.

The following theorem states the cases where the second degree curve is real or imaginary.

Theorem. A second degree curve given by

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \quad (A^2 + B^2 + C^2 \neq 0) \quad (1)$$

is imaginary (has no graph) iff

(1) (Elliptic case) $\Delta < 0$ and $HT > 0$ $(H = A + C)$

(2) (Parabolic case) $\Delta = 0$ and $\psi < 0$ $(\psi = \Delta' + \Delta'' = D^2 - 4AF + E^2 - 4CF)$

Proof. If the curve is real it contains at least one point in $\mathbb{R}^2$. Therefore the family $y = k$ $(k \in \mathbb{R})$ intersects it, at one or more point. Setting $y = k$ in (1) one obtains the equation:

$$Ax^2 + (Bk + D)x + (Ck^2 + Ek + F) = 0 \qquad (2)$$

with discriminant

$$\delta = (B^2 - 4AC)k^2 - 2(BD - 2AE)k + (D^2 - 4AF). \qquad (3)$$

Hence the curve has no real point only when $\delta < 0$ (for all $k \in R$) implying

(1) $\Delta = B^2 - 4AC < 0$ and $\delta' = -2AT < 0$ (δ' is the discriminant of (3))

If one considers the family $x = h$ instead of $y = k$, one obtains $\delta'' = -2CT$. Combining δ' and δ'' we have $HT > 0$.

(2) For $\Delta = 0$, δ becomes

$$-2(BD - 2AE)k + (D^2 - 4AF)$$

and $\delta < 0$ (for all k) $\Rightarrow$

$$BD - 2AE = 0, \quad \Delta' = D^2 - 4AF < 0.$$

Similarly

$$BE - 2CD = 0, \quad \Delta'' = E^2 - 4CF < 0$$

Combining these we get $\psi = \Delta' + \Delta'' < 0$.

The converse of the theorem is true since it depends on the discussion of the nature of the roots of the quadratic equation.

In all other cases the curve is a real one.

A CLASSIFICATION TABLE

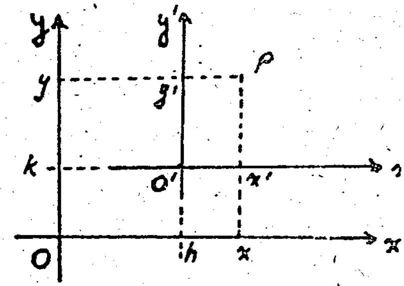
$\Delta < 0$: Elliptic case	$T \neq 0$. Non degenerate	Real ellipse if $HT < 0$ Imaginary ellipse if $HT > 0$
	$T = 0$. Degenerate	A point ($HT = 0$)
$\Delta = 0$: Parabolic case	$T \neq 0$. Non degenerate	Real parabola
	$T = 0$. Degenerate	Two parallel lines if $\psi > 0$ Two coincient lines if $\psi = 0$ Imaginary parabola if $\psi < 0$

B. TRANSLATION OF COORDINATE AXES AND ITS APPLICATION

A transformation which moves all points of a figure through tha same dintance in the same direction and the same sense is called a *translation*.

A translation which moves the origin $O(0, 0)$ to the point $O'(h, k)$ translates the coordinate system Oxy to the coordinate system $O'x'y'$.

From the Figure one can obviously obtains the following transforming formulas:



From old to new	From new to old
$x' = x - h$	$x = x' + h$
$y' = y - k$	$y = y' + h$

Application to SDE:

Let $f(x, y) = 0$ be an equation. Then $f(x' + h, y' + k) = 0$ is the equation in the new system $Ox'y'$ under the translation $O(0, 0) \to O'(h, k)$.

If

$$f(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \qquad (1)$$

after the same translation, (1) becomes

$$A(x' + h)^2 + B(x' + h)(y' + k) + C(y' + k)^2 + D(x' + h) + E(y' + k) + F = 0$$

which is in the form

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0 \qquad (1')$$

where

$$A' = A, \quad B' = B, \quad C' = C$$

$$D' = 2Ah + Bk + D, \quad E' = Bh - 2Ch + E, \quad F' = f(h, k).$$

To eliminate the linear terms one gets

$$h = \frac{2CD - BE}{\Delta} \qquad k = \frac{2AE - BD}{\Delta}$$

Examples.

1. Given $y = \frac{x}{x-2}$ find the new equation after the translation $0(0, 0) \to 0'(2, 1)$.
2. Given $2x + 3y = 4$ and $3x - y = -5$, translate $0(0, 0)$ to their point of intersection $0'$ and find the new equations of the lines.
3. Given $x^2 - 3xy + y^2 - y = 0$
   a) Classify the conic,
   b) eleminate the linear term by a convenient translation.

Solution.

1. Setting $x = x' + 2$, $y = y' + 1$, we have

$$y' + 1 = \frac{x' + 2}{x'} \Rightarrow y' = \frac{2}{x'} \quad (x'y' = 2)$$

2. Simultaneous solution gives $0'(-1, 2)$ and setting $x = x' - 1$, $y = y' + 2$, one gets $2x' + 3y' = 0$, $3x' - y' = 0$.

3. a) $\Delta = 5 > 0$ (hyperbolic case)

$$T = \begin{vmatrix} 2 & -3 & 0 \\ -3 & 2 & -1 \\ 0 & -1 & 0 \end{vmatrix} = -2 \neq 0 \quad \text{(non degenerate: . nyperbola)}$$

b) Setting $x = x' + h$, $y = y' + k$, we have

$$(x' + h)^2 - 3(x' + h)(y' + k) + (y' + k)^2 - (y' + k) = 0$$

$$D' = 2h - 3k = 0, \qquad E' = -3h + 2k - 1 = 0$$

$$\Rightarrow \qquad h = -3/5, \qquad k = -2/5.$$

Since the coefficients of second degree terms are unchanged under a translation and $F' = f(h, k)$, we have

$$x'^2 - 3x'y' + y'^2 + (-\tfrac{3}{5})^2 - 3(-\tfrac{3}{5})(-\tfrac{2}{5}) + (-\tfrac{2}{5})^2 - (-\tfrac{2}{5}) = 0$$

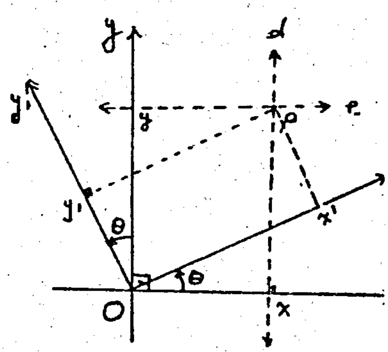
$$\Rightarrow \quad x'^2 - 3x'y' + y'^2 + \frac{1}{5} = 0.$$

C. ROTATION OF COORDINATE AXES AND APPLICATION:

A transformation which rotates (turns) all points of a figure through the same angle Θ about a given point O is called a *rotation*.

The point O is the *center of rotatian* and Θ the *angle of rotation*. Θ is considered positive (negative) when measured counterclockwise (clockwise).

A rotation with center at the origin and with angle Θ rotates the coordinate system Oxy into a new coordinate system Ox'y'



To obtain the transforming formulas for coordinates x, y and x', y' of the same point P in the systems Oxy and Ox'y', consider the lines d and e through P and perpendicular to Ox and Oy respectively.

The ormal equations of d and e in Oxy system are

$$\begin{cases} d: x' \cos(-\Theta) + y' \sin(-\Theta) - x = 0 \\ e: x' \cos(\frac{\pi}{2} - \Theta) + y' \sin(\frac{\pi}{2} - \Theta) - y = 0 \end{cases}$$

or

$$\begin{cases} x'\cos\Theta - y'\sin\Theta - x = 0 \\ x'\sin\Theta + y'\cos\Theta - y = 0 \end{cases}$$

from which we have the transforming formulas:

<u>From new to old</u>	<u>From old to new</u>
$x = x'\cos\Theta - y'\sin\Theta$	$x' = x\cos\Theta + y\sin\Theta$
$y = x'\sin\Theta + y'\cos\Theta$	$y' = -x\sin\Theta + y\cos\Theta$

<u>Application to SDE</u>:

Let $f(x, y) = 0$ be an equation. Then $f(x'\cos\Theta - y'\sin\Theta,$

$x'\sin\Theta + y'\cos\Theta) = 0$ is the equation in the new system under the rotation of axes with center $O(0, 0)$ and angle Θ.

If

$$f(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \qquad (1)$$

after the rotation (1) becomes

$$A(x'\cos\Theta - y'\sin\Theta)^2 + B(x'\cos\Theta - y'\sin\Theta)(x'\sin\Theta + y'\cos\Theta)$$
$$+ C(x'\sin\Theta + y'\cos\Theta) + D(x'\cos\Theta - y'\sin\Theta) + E(x'\sin\Theta + y'\cos\Theta) + F = 0$$

which is in the form

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0 \qquad (2)$$

where

$$A' = A\cos^2\Theta + B\cos\Theta\sin\Theta + C\sin^2\Theta$$
$$B' = -2A\cos\Theta + B(\cos^2\Theta - \sin^2\Theta) +$$
$$+2C\sin\Theta\cos\Theta = B\cos 2\Theta - (A-C)\cos 2\Theta$$
$$C' = A\sin^2\Theta - B\cos\Theta\sin\Theta + C\cos^2\Theta$$
$$D' = D\cos\Theta + E\sin\Theta$$
$$E' = -D\sin\Theta + E\cos\Theta$$
$$F' = F$$

A rotation $R(\Theta)$ can be determined such that cross term $B'x'y'$ vanishes $(B' = 0)$ when $B \neq 0$:

$$B\cos 2\Theta - (A-C)\sin 2\Theta = 0 \Rightarrow \tan 2\Theta = \frac{B}{A-C} \quad \text{if} \quad A \neq C$$

If $A = C$, then $B' = B\cos 2\Theta = 0 \Rightarrow \cos 2\Theta = 0 \Rightarrow \Theta = \frac{\pi}{4}$ which is the same result obtained from $\tan 2\Theta = B/(A-C)$ when $A = C$.

Hence the angle of rotation, to eleminate the cross term, can be determined by

$$\tan 2\Theta = \frac{B}{A-C} \ (= k)$$

which gives

$$\frac{2t}{1-t^2} = k \Rightarrow t = \frac{-1 \pm \sqrt{1+k^2}}{k} \quad (t = \tan\Theta)$$

After having one of the values of $t = \tan\theta$ (preferably the positive one) the values of $\cos\theta$ and $\sin\theta$ can be obtained by the use of a right triangle with legs t and 1.

To obtain $\cos\theta$ and $\sin\theta$ one may proceed in a different way as follows:

$$\cos 2\theta = \frac{1}{\sqrt{1 + \tan^2 2\theta}} \Rightarrow \begin{cases} \cos\theta = \sqrt{\dfrac{1 + \cos 2}{2}} \\ \sin\theta = \sqrt{\dfrac{1 - \cos 2}{2}} \end{cases}$$

where positive signs are taken for convenience.

<u>Examples.</u>

1. Given the line $y = \sqrt{3}x + 2$, rotate the coordinate axes by the angle $\pi/3$ and obtain the new equation of the line.
2. Given the parabola $y = x^2$ and the line $y = x - 3$, find the new equation of the parabola after the rotation of the coordinate axes such that x'-axis be parallel to the given line.
3. Given $x^2 - 3xy + y^2 - y = 0$, apply rotation to eliminate the cross term and obtain the new equation of the conic.

<u>Solution.</u>

1. The transforming formulas being

$$x = x'\cos\frac{\pi}{3} - y'\sin\frac{\pi}{3} = \frac{1}{2}x' - \frac{\sqrt{3}}{2}y'$$

$$y = x'\sin\frac{\pi}{3} + y'\cos\frac{\pi}{3} = \frac{\sqrt{3}}{2}x' + \frac{1}{2}y',$$

we have

$$y = \sqrt{3}x + 2 \Rightarrow \frac{\sqrt{3}}{2}x' + \frac{1}{2}y' = \sqrt{3}\left(\frac{1}{2}x' - \frac{\sqrt{3}}{2}y'\right) + 2 \Rightarrow y' = 1.$$

2. The angle of rotation is $\pi/4$. Then the transforming formulas are

$$x = \frac{1}{\sqrt{2}}(x' - y')$$

$$y = \frac{1}{\sqrt{2}}(x' + y')$$

and

$$y = x^2 \Rightarrow \frac{1}{\sqrt{2}}(x'+y') = \frac{1}{2}(x'-y')^2$$

$$\Rightarrow \quad x'^2 - 2x'y' + y'^2 - 2x' - 2y' = 0.$$

3. $\tan 2\theta = \frac{-3}{1-1} = \infty \Rightarrow \theta = \pi/4$, and $x^2 - 3xy + y^2 - y = 0.$

$$\Rightarrow \quad \frac{1}{2}(x'-y')^2 - \frac{3}{2}(x'-y')(x'+y') + \frac{1}{2}(x'+y')^2 - \frac{1}{2}(x'+y') = 0$$

$$\Rightarrow x'^2 - 2x'y' + y'^2 - 3(x'^2 - y'^2) + (x'^2 + 2x'y' + y'^2) - 2(x'+y') = 0$$

$$\Rightarrow -x'^2 + 5y'^2 - \sqrt{2}x' - \sqrt{2}y' = 0$$

Example. Translate the coordinate axes to eleminate the linear terms in (1) and compute Δ, H, T before and after the translation.

$$f(x, y) = x^2 + 4xy + 2y^2 + x - y - 5 = 0 \qquad (1)$$

Solution. Setting $x = x' + h$, $y = y' + k$ in (1) we have

$$x'^2 + 4x'y' + 2y'^2 + (2h + 4k + 1)x' + (4h + 4k - 1)y' + f(h, k) = 0$$

$$2h + 4k = -1, \quad 4h + 4k = 1 \qquad h = 1, \quad k = -3/4$$

Then (1) becomes

$$x'^2 + 4x'y' + 2y'^2 - \frac{33}{8} = 0.$$

Now

$$H = A + C = 3, \qquad H' = A' + C' = 3$$

$$\Delta = B^2 - 4AC = 8, \qquad \Delta' = B'^2 - 4A'C' = 8$$

$$T = \begin{vmatrix} 2 & 4 & 1 \\ 4 & 4 & -1 \\ 1 & -1 & -10 \end{vmatrix} = 66, \qquad T' = \begin{vmatrix} 2 & 4 & 0 \\ 4 & 4 & 0 \\ 0 & 0 & -\frac{33}{4} \end{vmatrix} = 66.$$

The equalities $H = H'$, $\Delta = \Delta'$ and $T = T'$ observed above are general as proved in the second of the following two theorem

<u>Example</u>. Obtain the standard form of the equation

$$2x^2 + 3xy - 2y^2 + 8 = 0 \qquad (1)$$

and compute H, Δ and [illegible] before and after the rotation.

<u>Solution</u>. The angle of rotation is obtained from

$$\tan 2\Theta = \frac{B}{A-C} = \frac{3}{2+2} = \frac{3}{4}$$

which gives

$$\cos 2\Theta = \frac{1}{\sqrt{1 + \tan^2 2\Theta}} = \frac{1}{\sqrt{1 + 9/16}} = 4/5$$

$$\Rightarrow \quad \cos\Theta = \sqrt{\frac{1 + \cos 2\Theta}{2}} = \sqrt{\frac{1 + 4/5}{2}} = 3/\sqrt{10}$$

$$\sin\Theta = \sqrt{\frac{1 - \cos 2\Theta}{2}} = \sqrt{\frac{1 - 4/5}{2}} = 1/\sqrt{10}$$

Then substituting

$$x = \frac{1}{\sqrt{10}}(3x' - y')$$

$$y = \frac{1}{\sqrt{10}}(x' + 3y')$$

into (1) we have

$$\frac{2}{10}(3x' - y')^2 + \frac{3}{10}(3x' - y')(x' + 3y') - \frac{2}{10}(x' + 3y')^2 + 8 = 0$$

or

$$2(3x' - y')^2 + 3(3x' - y')(x' + 3y') - 2(x' + 3y')^2 + 80 = 0$$

$$\Rightarrow (18 + 9 - 2)x'^2 + (2 - 9 - 18)y'^2 + 80 = 0$$

$$\Rightarrow 25x'^2 - 25y'^2 + 80 = 0 \Rightarrow \frac{y'^2}{80/25} - \frac{x'^2}{80/25} = 1 \Rightarrow a = b = \frac{4}{5}\sqrt{5}.$$

Note that

$$H = A + C = 0, \qquad H' = A' + C' = 0$$

$$\Delta = B^2 - 4AC = 25, \qquad \Delta' = B'^2 - 4A'C' = -4 \cdot \frac{25}{10}\left(-\frac{25}{10}\right) = 25$$

(Since $F = F' = 8$ for a rotation)

$$T = \begin{vmatrix} 4 & 3 & 0 \\ 3 & -4 & 0 \\ 0 & 0 & 16 \end{vmatrix} = -400, \quad T' = \begin{vmatrix} 2\cdot\frac{25}{10} & 0 & 0 \\ 0 & -2\cdot\frac{25}{10} & 0 \\ 0 & 0 & 16 \end{vmatrix} = -400$$

The equalities $H = H'$, $\Delta = \Delta'$ and $T = T'$ observed above are general ones as proved in the second of the following two theorems.

D. Invariants Under Translation and Rotation:

Translation and rotation are among Euclidean transformations. Under these Euclidean transformations distance and angle are invariant (unaltered). These type of transformations are called *distance preserving* and *angle preserving transformations*.

Theorem. Translation and rotation are distance and angle preserving transformations.

Proof. Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two distinct points with

$$d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

If P_1, P_2 are mapped to $P'_1(x'_1, y'_1)$ and $P'_2(x'_2, y'_2)$ under the translation $x' = x - h$, $y' = y - k$ of axes we have

$$d(P'_1, P'_2) = \sqrt{(x'_1 - x'_2)^2 + (y'_1 - y'_2)^2}$$
$$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = d(P_1, P_2)$$

showing that translation of axes preserve distances.

If P_1, P_2 are mapped to P'_1, P'_2 under the rotation

$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta$$

of axes, we have

$$d(P'_1, P'_2) = \sqrt{(x_1 - x_2)^2 \cos^2\theta + (y_1 - y_2) \sin^2\theta + (x_1 - x_2)\sin^2\theta + (y_1 - y_2)\cos^2\theta}$$

$$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = d(P_1, P_2)$$

showing that rotation of axes preserves distance.

Invariance of angles is a consequence of that of the distances, since a triangle is mapped into a congruent triangle under a translation or rotation.

Other invariants to be mentioned are ones in connection with second degree equations used in the classification.

Theorem. *Given the second degree equation*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

the quantities

$$H = A + C, \quad \Delta = B^2 - 4AC, \quad T = \begin{vmatrix} 2A & B & D \\ B & 2C & E \\ D & E & 2F \end{vmatrix}$$

are invariant under translation and rotation.

According to this theorem, one may classify the conics given by a second degree equation before or after the transformation.

Proof. Under a translation, the coefficients A, B, C being invariant, the quantities H and Δ are obviously invariant.

In the transformed equation, having

$$D' = 2Ah + Bk + D, \quad E' = Bh + 2Ck + E, \quad F' = f(h, k),$$

the determinant T' is

$$T' = \begin{vmatrix} 2A & B & 2Ah + Bk + D \\ B & 2C & Bh + 2Ck + E \\ 2Ah + Bk + D & Bh + 2Ck + E & 2f(h, k) \end{vmatrix}$$

To prove $T = T'$, multiplying the first row by $-h$ and the second by $-k$ and adding to the last row, we have

$$T' = \begin{vmatrix} 2A & B & 2Ah + Bk + D \\ B & 2C & Bh + 2Ch + E \\ D & E & 2f(h,\ k) - D'h - E'k \end{vmatrix}$$

Now multiplying the first column by $-h$ and the second by $-k$ and adding to the last column, we get

$$T' = \begin{vmatrix} 2A & B & D \\ B & 2C & E \\ D & E & 2f(h,\ k) - D'h - E'k - Dh - Ek \end{vmatrix}$$

where the element a_{33} is simplified to 2F. Hence $T' = T$.

Recalling that the coefficients of the transformed equation under a rotation are

$$\begin{aligned} A' &= Ac^2 + Bcs + Cs^2, & D' &= Dc + Es, \\ B' &= B(c^2 - s^2) - 2(A-C)cs, & E' &= -Ds + Ec, \\ C' &= As^2 - Bcs + Cc^2, & F' &= F, \end{aligned}$$

where c, s stand for $\cos\Theta$, $\sin\Theta$, we have

$$H' = A' + C' = A(c^2 + s^2) + C(s^2 + c^2) = A + C = H.$$

To prove the invariance of Δ, we write A', B', C' in terms of $\cos 2\Theta$, $\sin 2\Theta$ by the use of $2\cos^2\Theta = 1 + \cos 2\Theta$, $2\sin^2\Theta = 1 - \cos 2\Theta$:

$$\begin{aligned} 2A' &= B\sin 2\Theta + (A-C)\cos 2\Theta + A + C \\ B' &= B\cos 2\Theta - (A-C)\sin 2\Theta \\ 2C' &= B\sin 2\Theta - (A-C)\cos 2\Theta + A + C, \end{aligned}$$

we have

$$\begin{aligned} \Delta' &= B'^2 - 4A'C' \\ &= \left[B\cos 2\Theta - (A-C)\sin 2\Theta\right]^2 - \left[(A+C)^2 - (B\sin 2\Theta + (A-C)\cos 2\Theta)^2\right] \\ &= B^2 + (A-C)^2 - (A+C)^2 = B^2 - 4AC^2 = \Delta. \end{aligned}$$

The invariance of T is proved as follows:

$$T' = \begin{vmatrix} 2A' & B' & D' \\ 6' & 2C' & E' \\ D' & E' & 2F' \end{vmatrix} = D' \begin{vmatrix} B' & 2C' \\ D' & E' \end{vmatrix} - E' \begin{vmatrix} 2A' & b' \\ D' & E' \end{vmatrix} + 2F' \begin{vmatrix} 2A' & B' \\ B' & 2C' \end{vmatrix}$$

where the last term, being equal to F, is invariant.

The first two terms when expanded give

$$2\left(B'D'E' - D'^2C' - E'^2A'\right)$$

which is obvious to be equal to $2\left(BDE - D^2C - E^2A\right)$. ■

Example. Apply translation of axes to eleminate the linear terms in

$$f(x, y) = 2x^2 + \sqrt{3}\, xy + y^2 - x + y - 2 = 0 \qquad (1)$$

and then apply rotation of axes to the transformed equation to eliminate the cross term $(B'x'y')$. Also verify invariance of H, Δ and T under the transformation.

Solution. Setting $x = x' + h$, $y = y' + k$ into (1), we have

$$2x'^2 + \sqrt{3}x'y' + y'^2 + (4h + 3k - 1)x' + (\sqrt{3}h + 2k + 1)y' + f(h, k) = 0$$

and

$$\begin{aligned} 4h + \sqrt{3}k &= 1 \\ \sqrt{3}h + 2k &= -1 \end{aligned} \quad \Rightarrow \quad h = \frac{2 + \sqrt{3}}{5}, \quad k = -\frac{4 + \sqrt{3}}{5}$$

$$\Rightarrow \quad F' = f(h, k) = -\frac{13 + \sqrt{3}}{5}$$

So the transformed equation is

$$2x'^2 + \sqrt{3}x'y' + y'^2 - \frac{1}{5}(13 + \sqrt{3}) = 0 \qquad (1')$$

The angle of rotation being given by $\tan 2\theta = B'/(A'-C')$, we have $2\theta = \pi/3$, $\theta = \pi/6$ and transformating relations are

$$x' = \frac{1}{2}(x''\sqrt{3} - y''), \quad y' = \frac{1}{2}(x'' + y''\sqrt{3})$$

which when substituted in (1') give

$$\frac{5}{2}x''^2 + \frac{1}{2}y''^2 - \frac{1}{5}(13 + \sqrt{3}) = 0 \qquad (1'')$$

Evaluating H, Δ and T for (1), (1') and (1") we have

$$H = 2+1 = 3, \quad \Delta = 3 - 8 = -5, \quad T = \begin{vmatrix} 4 & \sqrt{3} & -1 \\ \sqrt{3} & 2 & 1 \\ -1 & 1 & -4 \end{vmatrix} = -2(13+\sqrt{3})$$

$$H' = 2+1 = 3, \quad \Delta' = 3 - 8 = -5, \quad T' = \begin{vmatrix} 4 & \sqrt{3} & 0 \\ \sqrt{3} & 1 & 0 \\ 0 & 0 & 2(13+\sqrt{3}) \end{vmatrix} = -2(13+\sqrt{3})$$

$$H'' = \frac{5}{2} + \frac{1}{2} = 3, \quad \Delta'' = 0 - 4 \cdot \frac{5}{2} \cdot \frac{1}{2} = -5, \quad T'' = \begin{vmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{2}{5}(13+\sqrt{3}) \end{vmatrix} = -2(13+\sqrt{3})$$

Hence $H = H' = H''$, $\Delta = \Delta' = \Delta''$ and $T = T' = T''$

E. Determinantal Equations of Second Degree Curves:

A second degree curve has the equation

$$f(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \quad (A^2 + B^2 + C^2 \neq 0) \quad (1)$$

where at least one of A, B, C is not zero. Dividing every term by that non zero coefficient, the equation will involve in general 5 parameters. So a second degree curve is completely determined, in general, by five conditions.

Let the equation of a second degree curve passing through the five distinct points $A_i(x_i, y_i)$, $i = 1, 2, 3, 4, 5$ be determined. Since (1) is satisfied by a general point $P(x, y)$ and $A_i(x_i, y_i)$, $i = 1, \ldots, 5$, we have six homogeneous linear equations (corresponding to given five conditions):

$$f(x, y) = 0, \quad f(x_1, y_1) = 0, \quad \ldots, \quad f(x_5, y_5) = 0$$

in the six unknown coefficients A, B, C, D, E, F. The homogeneous system to have a non trivial solution it is necessary that the determinant of the coefficients of unknowns vanish:

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{vmatrix} = 0$$

and this is the *determinantal equation* of the second degree curv through five distinct points.

Example. Find the equation of the second degree curve passing through the five points (0, 0), (1, 1), (1, 0), (0, 1), (-1, 2) and expand it.

Solution. For the given points the determinantal equation is

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & -2 & 4 & -1 & 2 & 1 \end{vmatrix} = 0$$

which when expanded gives

$$x^2 - y^2 - x + y = 0 \quad \text{or} \quad (x-y)(x+y-1) = 0.$$

Example. Obtain the determinantal equation of the curve passing through the points (0, 1), (2, -1), (4, 2) and having the equation of the form: $Ax^2 + Bxy + Cx + D = 0$

Solution. The required equation is

$$\begin{vmatrix} x^2 & xy & x & 1 \\ 0 & 0 & 0 & 1 \\ 4 & -2 & 2 & 1 \\ 16 & 8 & 4 & 1 \end{vmatrix} = 0.$$

EXERCISES (.., 2)

21. Show that, in the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

of the conic $C = \{P(x, y): d(P, F)/d(P, D) = e\}$ where $D: ax + by + c = 0$ $(a^2 + b^2 = 1)$ is the directrix and $F(x_0, y_0)$ is the focus and e the eccentricty, the relation

$$A^2 + B^2 + C^2 = 1 + (1 - e^2)^2 + 2e^4 a^2 b^2$$

holds, hence A, B, C cannot vanish simultaneously.

22. Sketch the degenerate conics:

a) $(x - y + 1)(x + y) = 0$ b) $(2x + y - 2)(x - 2y + 2) = 0$

23. Show that each of the following sets is a point:

a) $\{(x, y): (x - y + 3)^2 + (x + 3y - 1)^2 = 0\}$

b) $\{(x, y): 2x^2 - 4xy + 4y^2 + 8x - 16y + 16 = 0\}$

24. Are the curves of the following equations degenerate or non degenerate?

a) $x^2 - 2xy + y^2 - 2y - 1 = 0$ b) $x^2 + 2xy + y^2 + x + y - 2 = 0$

c) $xy + x - y + 2 = 0$ d) $xy + 2x - y - 2 = 0$

25. Find the real values of k, for which the curves of the following equations are degenerate:

a) $kx^2 + (1 - k)y^2 - (a + k) = 0$ b) $xy + k(x^2 - y^2) = 0$

26. For what values of λ the linear family

$$(x^2 - 2xy + 4y^2 - 4x + 1) + \lambda(-x^2 + 3xy - y^2 + 2y - 2) = 0$$

corresponds to

a) elliptic case, b) parabolic case, c) hyperbolic case?

27. Show that the following sets are empty:

a) $\{(x, y): 5x^2 - 10xy + 10y^2 + 5 = 0\}$

b) $\{(x, y): 2x^2 + 2xy + y^2 - 2x + 6 = 0\}$

28. After the translation of coordinate axes by indicated h and k, obtain the new equation of the following curves:

a) $xy + 2x - 2y - 2 = 0$, $h=2$, $k = -2$ b) $x^2 + 4y = 0$, $h=0$, $k = -4$

29. After the translation of coordinate axes by indicated h and k, obtain the new equations of the following curves:

a) $y^2 - 4x - 4y - 8 = 0$, $h = -3$, $k = 2$ b) $x^2 + 9y^2 - 30y = 0$, $h=0$, $k=5$,

30. Translate the coordinate axes by $h = -4$, $k = 0$ followed by the rotation with an angle $-\pi/2$; find the new equation of $f(x, y) = y^2 - 4x + 16 = 0$.

31. Transform the following equations when the axes are rotated through the indicated angle:

a) $x^2 + 2xy + y^2 = 8$, $\pi/4$,   b) $x^2 + 4xy + y^2 = 16$, $\pi/4$

32. Same question for

a) $x^2 + y^2 = r^2$, θ b) $x^2 + 2xy + y^2 + 4x - 4y = 0$, $-\pi/4$.

33. Determine the condition on A, B, C such that the second degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

represents

a) a circle,   b) an equilateral hyperbola.

34. Show that $\pm C/(A^2 + B^2)$ is an invariant of the line $Ax + By + C = 0$ under a rotation of the coordinate axes.

35. Find the equation of the conic passing through the five points (0, 0), (2, 0), (0, 2), (4, 2), (2, 4).

36. Same question for (1, 0), (-5, 0), (2, 2), (-6, 0) and

(-2, -2) and expand.

37. Same question for (0, 0), (2, 1), (-2, 4), (-4, -2), (2, -4).

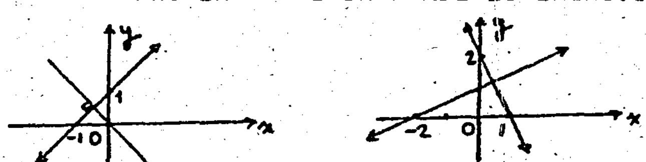
38. Find the equation of the conic passing through (0, 4), (5, 0), and symmetrical with respect to both axes.

39. Same question for (0, 5), (10, 0)

40. Find the equation of the conic passing through (-4, 0), (0, 4), (0, -4), (5, 6) if $\Delta = B^2 - 4AC = 0$.

ANSWERS TO EVEN NUMBERED EXERCISES

22.



(a) (b)

24. a) Non-degenerate, b) Degenerate
c) Non-degenerate, d) Degenerate.

26. a) $\lambda \epsilon (-1, 12/5)$, b) $\lambda = -1; 12/5$, c) $\lambda \epsilon \mathbb{R} - [-1, 12/5]$.

28. a) $x'y' = 6$, b) $x'^2 + 4y' - 16 = 0$

30. $x'^2 - 4y' = 0$

32. a) $x'^2 + y'^2 = r^2$, b) $2y'^2 + 4x' = 0$

36. $12x^2 + 11y^2 + 48x - 108y - 60 = 0$

38. $16x^2 + 25y^2 = 400$

40. $y^2 - 4x - 16 = 0$

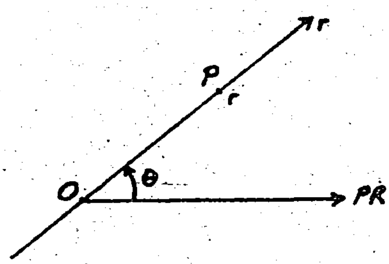
1. 3. POLAR COORDINATES

A. DEFINITIONS AND DISTANCE

One takes on a plane a fixed point O, called the *pole*, and a fixed ray with initial point O, called the *polar ray* (PR).

A point P(≠ O) in the plane will be determined by two coordinates θ and r as follows:

Pass through O and P an axis with origin at the pole. There are two such axes which are oppositely directed. Let Or be one of these axes.

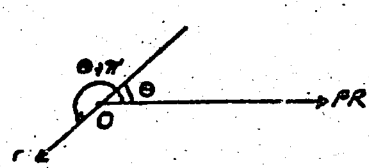


The oriented angle θ between the polar ray and the positive side of Or is called the *polar angle* of P. It is positive when measured counterclockwise and negative otherwise.

The abscissa r of P on the axis Or is called (improperly) the *radius vector* of P (r is not a vector).

The numbers θ and r thus defined determine uniquely a point P and called the *polar coordinates* of P and one writes[1] P(θ, r) or P(θ + 2kπ, r).

If the oppositely directed axis through O and P were taken, the polar angle will be θ+π and the radius vector -r, so that a second representation of P will be P(θ + π , -r) or P(θ + π + 2kπ, -r).



Thus we have essentially two representations of the same point P(≠ O), namely

(1) Many authors write P(r, θ) instead of P(θ, r). The reason for adopting P(θ, r) will become clear in the text.

$$P(\theta, r) = P(\theta + 2k\pi, r) \text{ and}$$
$$P(\theta + \pi, -r) = P(\theta + \pi + 2k\pi, -r)$$

and one is obtained from the other by increasing the polar angle by π and changing the sign of the radius vector.

If P is the pole, the polar angle is indeterminate and $r = 0$, that is, the pole has no definite polar angle and is given only by $r = 0$, or as $O(\theta, 0)$.

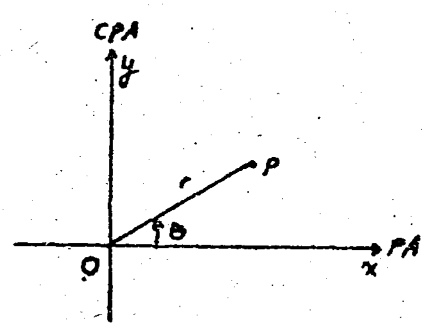
Transforming Relations:

In a polar coordinate system, one sometimes uses a polar axis (PA) instead of the polar ray, and also for convenience the axis with origin O and perpendicular to polar axis is taken into consideration and called the *copolar axis* (CPA).

Taking the polar axis as x-axis and the copolar axis as y-axis we may esteblish relations between the cartesian coordinates x, y and polar coordinates θ, r of the same point.

If P is taken in the first quadrant with $x \geqslant 0$, $y \geqslant 0$ and $r \geqslant 0$, $0 \leqslant \theta \leqslant \pi/2$, we have the relations

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \text{or} \quad \begin{cases} r^2 = x^2 + y^2 \\ \theta = \arctan\frac{y}{x} \end{cases}$$



One may check the validty of these relations for any position of P in the plane and for any determination in polar coordinates.

These relations are called the *transforming relations* which may be used to transform an equation in one system to the other.

Example 1. Transform the cartesian equation to polar one:

a) $ax + by + c = 0$ (line) b) $x^2 + y^2 - 2ax = 0$ (circle)

Solution. Setting $x = r \cos \Theta$, $y = r \sin \Theta$, we have

a) $ar \cos \Theta + br \sin \Theta + c = 0 \Rightarrow r = -\dfrac{c}{a \cos \Theta + b \sin \Theta}$

b) $r^2 - 2ar \cos \Theta = 0 \Rightarrow r(r - 2a \cos \Theta) = 0 \Rightarrow$

$r = 0$ (pole) or $r = 2a \cos \Theta$. But $r = 0$ is contained in the second for $\Theta = \pi/2$. Hence the transformed equation is $r = 2a \cos \Theta$.

Example 2. Transform the polar equation into cartesian:

a) $r = a(1 + \cos \Theta)$ (cardioid) b) $r^2 = a^2 \cos 2\Theta$ (lemniscate)

Solution.

a) We express first $\cos \Theta$ in terms of x and r, and then replace r^2 by $x^2 + y^2$:

$$r = a(1 + \cos \Theta) \Rightarrow r = a(1 + \frac{x}{r}) \Rightarrow r^2 = a(r + x)$$

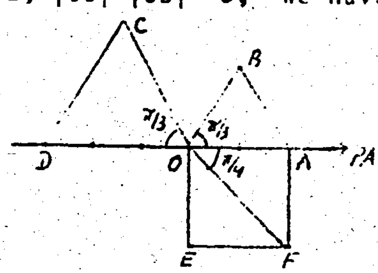
$$x^2 + y^2 = ar + ax \Rightarrow (x^2 + y^2 - ax)^2 = a^2(x^2 + y^2).$$

b) $x^2 = a^2 \cos 2\Theta \Rightarrow x^2 + y^2 = a^2(\cos^2\Theta - \sin^2\Theta)$

$$x^2 + y^2 = a^2(\frac{x^2}{r^2} - \frac{y^2}{r^2}) \Rightarrow (x^2 + y^2)^2 = a^2(x^2 - y^2).$$

Example 3. Write two representations of points A, B, C, D, E, F in the given figure, where OAB, OCD are equilateral triangles, and OEFA is a square ($|OA| = 2$, $|OD| = 3$)

Solution. Since $|OA| = |OB| = 2$, $|OC| = |OD| = 3$, we have from the Figure,



$A(0, 2) = A(\pi, -2)$

$B(\frac{\pi}{3}, 2) = B(\frac{\pi}{3} + \pi, -2)$

$C(\frac{2\pi}{3}, 3) = C(-\frac{\pi}{3}, -3)$

$D(\pi, 3) = D(0, -3)$

$E(-\pi/2,\ 2) = E(\frac{\pi}{2}\ ,\ -2)$

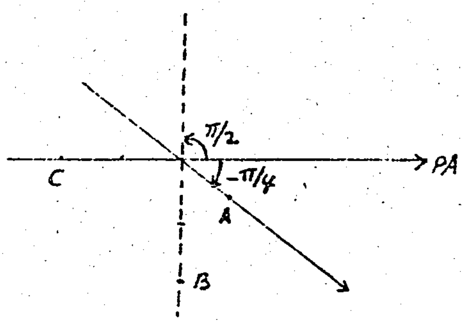
$F(-\ \frac{\pi}{4}\ ,\ 2\sqrt{2}) = F(-\ \frac{\pi}{4} + \pi,\ -2\sqrt{2})$

Example 4. Plot the points $A(-\ \frac{\pi}{4}\ ,\ 1)$, $B(\frac{\pi}{2}\ ,\ -2)$. $C(\pi,2)$.

Solution. Since the polar angle of A is $-\pi/2$, we draw the axis Or making the oriented angle $-\pi/2$ with the PA, and mark on this axis Or the point A with abscissa 1.

Similarly, to plot B, one draws the axis making the angle $\pi/2$ with polar axis, and on it one takes B with abscissa -2.

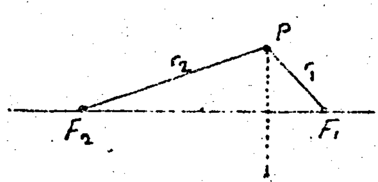
The point C is plotted in similar manner.



Bipolar coordinates:

Besides the polar system, we mention shortly another coordinate system, called *bipolar system* in which a point P in the plane is determined by the pair (r_1, r_2) where r_1, r_2 are distance (> 0) of P from two given distinct fixed point F_1, F_2 called the *poles* or *foci*.

It is to be noted that a pair (r_1, r_2) in bipolar system determines two points that are symmetrically placed with respect to the focal line F_1F_2.



The representation is unique if the point is restricted to one of the half planes, say, upper half plane.

Some curves have very simple equations in bipolar system. Thus the sets (1), (2) and (4) below

1. $\{(r_1, r_2): r_1 + r_2 = 2a\}$ 2. $\{(r_1, r_2): |r_1 - r_2| = 2a\}$
3. $\{(r_1, r_2): r_1 \cdot r_2 = \ell^2\}$ 4. $\{(r_1, r_2): r_1 : r_2 = k \neq 1\}$

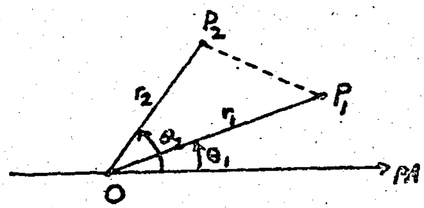
represent respectively an ellipse, a hyperbola and a circle (circle of APOLLONIUS); as to (3), it represents a curve called *oval to CASSINI* including the curve named *lemniseate of BERNOULLI*.

Distance between two points.

The distance $d = d(P_1, P_2)$ between two given points $P_1(\theta_1, r_1)$, $P_2(\theta_2, r_2)$ is obtained by the use of cosine law applied to the triangle OP_1P_2:

$$d^2 = r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_2 - \theta_1)$$

where $\cos(\theta_2 - \theta_1) = \cos(\theta_1 - \theta_2)$



It is valid for any determinations of P_1, P_2, since the right hand side remains unaltered when θ_i is replaced by $\theta_i + \pi$ and r_i by $-r_i$

Example. Find the distance between the given pair of points:

a) $A(\frac{\pi}{3}, -2)$, $B(\frac{\pi}{6}, 4)$ b) $C(-\frac{\pi}{4}, 3)$, $D(3\pi/4, -3)$

Solution.

a) $d^2 = |AB|^2 = 4 + 16 - 2(-2)(4) \cos(\frac{\pi}{3} - \frac{\pi}{6})$

$= 20 + 16 \cos\frac{\pi}{6} = 20 + 8\sqrt{3}$

$d = |AB| = \sqrt{20 + 8\sqrt{3}}$

b) $d^2 = |CD|^2 = 9 + 9 - 2(3)(-3) \cos(\frac{3\pi}{4} + \frac{\pi}{4})$

$= 18 + 18 \cos\pi = 0$

$d = 0$ (explain this result!)

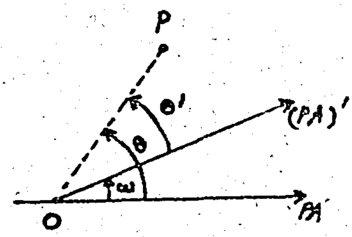
B. ROTATION AND TRANSLATION IN POLAR SYSTEM.

In polar system rotation is more easier and more usefull than the translation.

Rotation of polar axis:

A rotation with center at the pole and angle ω, rotates the polar axis PA to a new polar axis (PA)

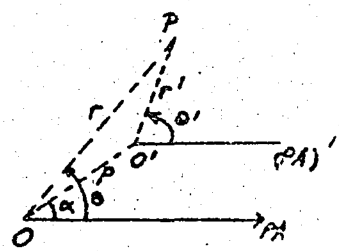
From the Figure one obtains the following transforming formulas:



From old to new	From new to old
$\theta' = \theta - \omega$	$\theta = \theta' + \omega$
$r' = r$	$r = r'$

Translation of polar axis:

When the pole is translated to the point $O'(\alpha,\ p)$ the transforming relation are obtained by the use of sine and cosine laws applied to the triangle $OO'P$:



$$\frac{r'}{\sin(\theta-\alpha)} = \frac{r}{\sin(\theta'-\alpha)}\ ,\quad r'^2 = r^2 + p^2 - 2pr\cos(\theta-\alpha)$$

To get relation between θ, θ' (between r, r') one has to eleminate r, r' (θ, θ'). Results being very involved one does not expect any usefullness of translation in general.

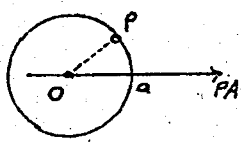
Transforming a polar relation under a rotation:

Let $F(\theta,\ r) = 0$ be a relation in a polar system O, PA. If one rotates the polar axis PA by an angle ω about the pole O, one obtains the relation $F(\theta' + \omega,\ r') = 0$ upon the application of the transforming formulas given above.

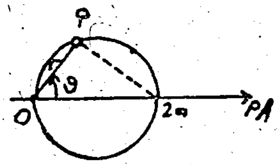
C. CIRCLES AND LINES:

1. Circles.

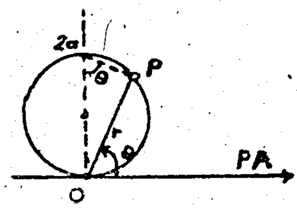
The equations of the following circles with special positions are immediately obtainable:



Center at the pole
radius $a > 0$
$r = a$ or $r = -a$



Center on PA,
radius $a>0$
$r = 2a . \cos\theta$



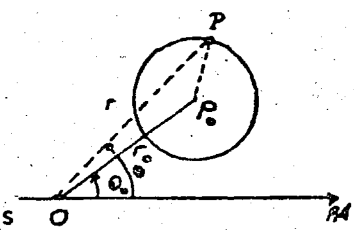
Center at CPA,
radius $a>0$
$r = 2a . \sin\theta$

The equation of the circle centered at $P_0(\theta_0, r_0)$ and radius "a" is obtained from $a = d(P_0, P)$ where $P(\theta, r)$ is any point on the circle:

$$a^2 = r^2 + r_0^2 - 2r_0 r \cos(\theta - \theta_0)$$

or

$$r^2 - 2r_0 r \cos(\theta - \theta_0) + r_0^2 - a^2 = 0 \quad (1)$$



In particular, if the circle passes through the pole ($r_0 = a$) the equation becomes

$$r\left[r - 2a \cos(\theta - \theta_0)\right] = 0$$

$$r = 0 \quad \text{or} \quad r - 2a \cos(\theta - \theta_0) = 0$$

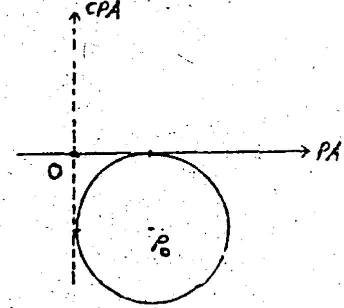
where the first equation (the pole) is contained in the second, hence

$$r = 2a \cos(\theta - \theta_0)$$

is the required equation which could be obtained from $r=2a \cos\theta$ by a rotation of PA with angle $-\theta_0$

The three equations given below the Figure, can be obtained from (1) for centers $(\theta, 0)$, $(0, a)$ and $(\pi/2, a)$ respectively.

Example 1. Obtain the polar equation of the circle, radius 3, tangent to PA and CPA lying in the fourth quadrant.



Solution. Since $P_0(-\frac{\pi}{4}, 3\sqrt{2})$ and $a = 3$, (1) gives

$$r^2 - 2.3\sqrt{2}\ r\cos(\theta + \frac{\pi}{4}) + 18 - 9 = 0$$

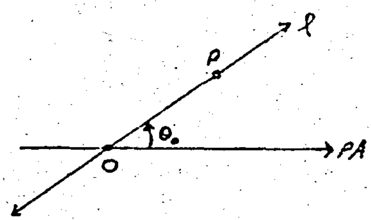
$$r^2 - 6\sqrt{2}\ r\cos(\theta + \frac{\pi}{4}) + 9 = 0.$$

2. Lines.

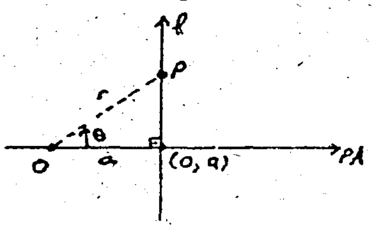
The equation of the line ℓ passing through the pole and making an angle θ_0 with PA is

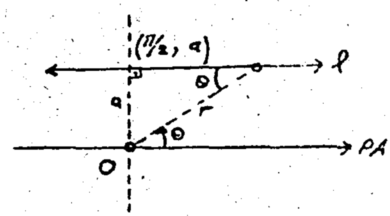
$$\theta = \theta_0 \text{ (or } \theta = \theta_0 + \pi),$$

since it holds true for any r on the axis corresponding to θ_0



The following are two special cases for a line:





$r\cos\theta = a$ ($r = a\sec\theta$) $\qquad$ $r\sin\theta = a$ ($r = a\csc\theta$)

Normal form of the equation of a line:

The normal form of the equation of a line ℓ is one in involving the coordinates of the (vertical) projection of the pole on ℓ.

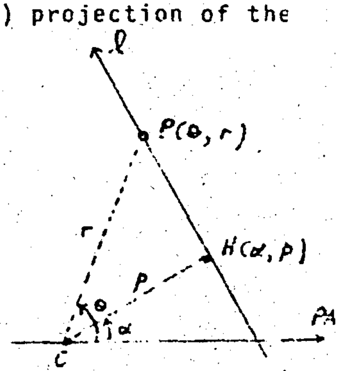
If $H(\alpha, p)$ is the projection of O on ℓ, we have the normal form

$$r\cos(\theta - \alpha) = p$$

or $\qquad\qquad\qquad\qquad\qquad\qquad$ (1)

$$F(\theta, r) = r\cos(\theta - \alpha) - p = 0$$

which can be shown to be valid for the



second determination $(\alpha + \pi, -p)$.

The above three equations for lines in special positions, can be obtained from (1) by taking (α, p) as $(\theta_0 + \frac{\pi}{2}, 0)$, $(0, a)$ and $(\frac{\pi}{2}, a)$ respectively.

The equation (1) can be obtained from $r \cos\theta = a$ by rotation of PA with angle $-\alpha$, and also from the normal cartesian form $x \cos\alpha + y \sin\alpha - p = 0$ by transforming formula as follows:

$$x \cos\alpha + y \sin\alpha - p = 0 \Rightarrow r(\cos\theta \cos\alpha + \sin\theta \sin\alpha) - p = 0$$

$$\Rightarrow r \cos(\theta - \alpha) - p = 0.$$

<u>General form</u>: Transforming the general form $Ax + By + C = 0$ to polar, we have

$$r(A \cos\theta + B \sin\theta) + C = 0 \qquad (2)$$

where $A^2 + B^2 \neq 1$ in general. If $A^2 + B^2 = 1$, then (2) becomes a normal form, and a general form can be normalized by dividing every term by $\sqrt{A^2 + B^2}$.

<u>Distance of a point from a line</u>:

Normal equation of a line permits us to express its distance $d(P_0, \ell)$ from a point P_0 in a simple from:

$$d(P_0, \ell) = \left| r_0 \cos(\theta_0 - \alpha) - p \right|$$

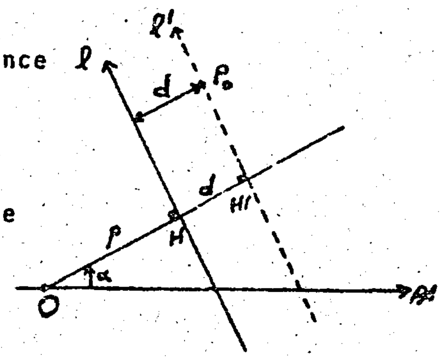
Indeed, consider the point $H'(\alpha, p \pm d)$ and the line $\ell' // \ell$ and through H', where d is the distance $d(\ell, \ell')$. Then:

$$\ell' : r \cos(\theta - \alpha) = p \pm d.$$

Since $P_0(\theta_0, r_0)$ is on ℓ', we have

$$r_0 \cos(\theta_0 - \alpha) = p \pm d$$

for which we get



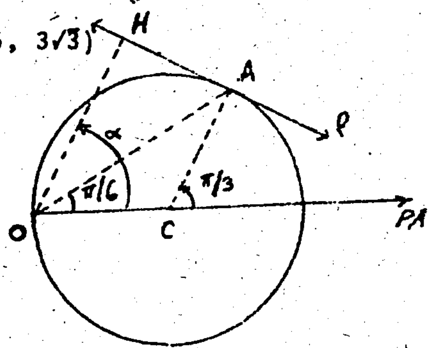
$$d = \pm(r_0 \cos(\theta_0 - \alpha) - p)$$

which is (3). The sign of the expression in the parantheses given information on the position of P_0:

$$F(\theta_0, r_0) = r_0 \cos(\theta_1 - \alpha) - p$$

$$\begin{cases} > 0 \text{ when } P_0 \text{ and } O \text{ are on opposite sides of } \ell, \\ = 0 \text{ when } P_0 \text{ is an } \ell, \\ < 0 \text{ when } P_0 \text{ and } O \text{ are on the same side of } \ell. \end{cases}$$

Example 1. Find the equation of the tangent line ℓ to the circle $r = 6 \cos\theta$ at its point $A(\pi/6, 3\sqrt{3})$



Solution. From the Figure

$PA, OH = PA, CA \quad \alpha = \pi/3$

and

$$p = \overline{OH} = \overline{OA} \cos(\alpha - \tfrac{\pi}{6}) = 9/2.$$

Then

$$\ell: r \cos(\theta - \tfrac{\pi}{3}) - \tfrac{9}{2} = 0$$

Example 2. Given the equation $r(\cos\theta - \sqrt{3} \sin\theta) = 3$ of a line ℓ,

a) normalize the equation, get $H(\alpha, p)$,

b) find the distance of $A(\pi/2, 2)$ from ℓ.

Solution.

a) Since $A = 1$, $B = -\sqrt{3}$, $A^2 + B^2 = 2$, the normal equation is $r(\frac{1}{2} \cos\theta - \frac{\sqrt{3}}{2} \sin\theta) = 3/2$ or $r \cos(\theta + \frac{\pi}{3}) - \frac{3}{2} = 0$,

and $H(-\pi/3, 3/2)$.

b) $d(P_0, \ell) = \left|F(\theta_0, r_0)\right| = \left|r_0 \cos(\theta_0 + \frac{\pi}{3}) - \frac{3}{2}\right|$

$$= \left|2 \cos(\tfrac{\pi}{2} + \tfrac{\pi}{3}) - \tfrac{3}{2}\right| = \frac{3 + 2\sqrt{3}}{2}$$

Since $F(\frac{\pi}{2}, 2) < 0$, the points A and O are on the same side of ℓ.

D. Curves.

Circle and line are some curves having polar equations

$$r = a,\quad r = a\cos\theta,\quad r^2 - 2r_0 r\cos(\theta - \theta_0) + r_0^2 - a^2 = 0$$

$$\theta = \theta_0,\quad r = a\sec\theta,\quad r(A\cos\theta + B\sin\theta) + C = 0$$

which are in the form $r = f(\theta)$, or $F(\theta, r) = 0$.

An equation $r = f(\theta)$, $\theta = g(r)$ or $F(\theta, r) = 0$ represents a curve in general, where f may be a periodic function.

<u>Sketching</u>:

A general procedure may be outlined as follows:

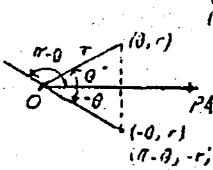
1) *Determination of the domain* D of $r = f(\theta)$ or of $F(\theta, r) = 0$.

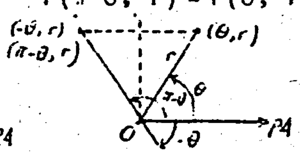
If f is not periodic, vary θ in D. If f has period T, vary θ in $[0, T]$ and complete the curve by rotations through multiples of T about O.

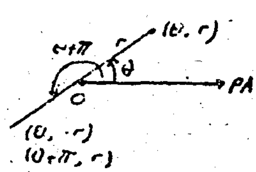
2) *Determination of symmetries* with respect to PA, CPA and the pole.

If a symmetry exists the domain is reduced. Conditions for symmetries with respect to polar, copolar axes and the pole are given in following table:

PA	CPA	Pole
$F(-\theta, r) = F(\theta, r)$	$F(-\theta, -r) = F(\theta, r)$	$F(\theta, -r) = F(\theta, r)$
or	or	or
$F(\pi-\theta, -r) = F(\theta, r)$	$F(\pi-\theta, r) = F(\theta, r)$	$F(\theta+\pi, r) = F(\theta, r)$



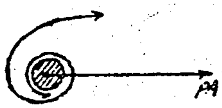


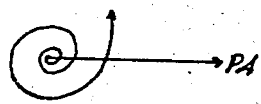


3) If necessary, determine the interval of increase and decrease for r.

4) *Determination of asymptotes*: In polar coordinates one distinguishes two kinds of asymptotes as *circle-asymptote* (in particular a point-asymptote as a degenerate case) and a *line-asymptote*.

Circle-asymptote: It exists when $r \to a$ as $\theta \to \infty$.





Line-asymptote: (When necessary):

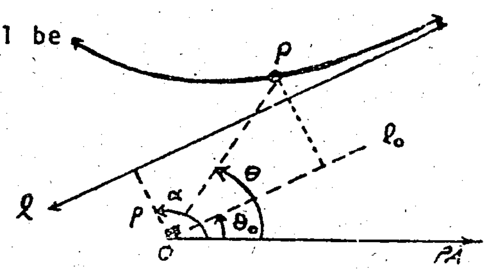
The direction of a line asymptote exists when $r \to \infty$ as $\theta \to \theta_0$. If ℓ is such an asymptote (See Fig.), its normal equation will be

$$r\cos(\theta - \alpha) - p = 0$$

where

$$\alpha = \theta_0 + \pi/2$$

$$p = \lim_{\substack{\theta \to \theta_0 \\ (r \to \infty)}} \left[r\sin(\theta - \theta_0) \right].$$



Note that $\theta_0 = k\pi$ and $\theta_0 = (2k+1)\pi/2$ correspond to horizontal and vertical asymptotes respectively.

5) Intercepts other than the pole. For PA-intercepts one sets $\theta = k\pi$ and for CPA-intercepts $\theta = (2k+1)\pi/2$ in the equation. The curve passes through the pole when $r = 0$.

Example. Sketch the curve of $r = 2\tan\theta$.

Solution.

1) $D = (-\infty, \infty) - \{\theta = (2k+1)\frac{\pi}{2} : k \in Z\}$

$T = \pi \Rightarrow D_1 = [0, \pi) - \{\pi/2\}$

2) $\Theta \to -\Theta \Rightarrow r \to -r$: Symmetry with respect to PA

$\Theta \to \pi - \Theta \Rightarrow r \to -r$: Symmetry with respect to PA

$\Theta \to \pi + \Theta \Rightarrow r \to r$: Symmetry with respect to the pole.

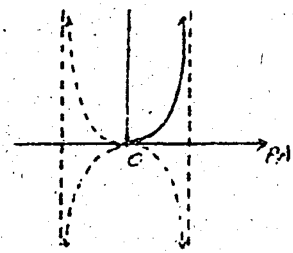
It is therefore sufficient to sketch the curve for $\Theta \varepsilon [0, \pi/2)$ instead of for $\Theta \varepsilon D$ since there are symmetries.

3) No-circle asymptote, since no limit exists for r as $\Theta \to \infty$. Line-asymptote: $r \to \infty$ when $\Theta \to \pi/2$ $(= \Theta_0)$. $\alpha = \Theta_0 + \pi/2 = \pi$

$$p = \lim_{\Theta \to \pi/2} r \sin(\Theta - \pi/2) = \lim_{\Theta \to \pi/2} = (2 \tan\Theta(-\cos\Theta)) = -2$$

5) $r = 2 \tan k\pi = 0$ (pole), $r = 2 \tan(2k + 1)\frac{\pi}{2}$

(No CPA-intercept)



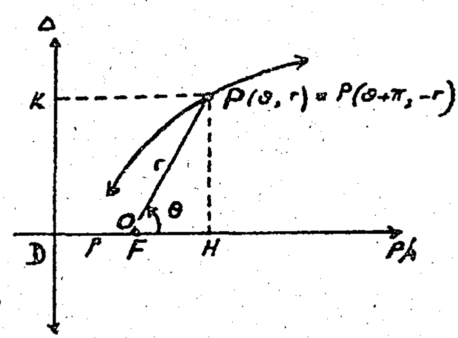
The whole curve is obtained by the use of symmetries from the dark curve.

CONICS.

Let the conic be defined by a focus F, a directrix Δ, and the eccentricity e, or by the set

$$C = \{P: d(P, F)/d(P, \Delta) = e\}$$

Taking F as pole and focal axis a polar axis and setting $p = d(F, \Delta)$ we have from



$$d(P, F) = e\, d(P, \Delta)$$

$$|r| = e|DH|$$

$$|r| = e|p + r \cos\Theta|$$

$$r = \pm e(p + r \cos\Theta)$$

$$r(1 \pm e \cos\Theta) = \pm ep$$

$$r = \frac{\pm e\ p}{1 \pm e \cos} = \begin{cases} \dfrac{ep}{1 - e \cos\theta} & (1) \\ \\ \dfrac{ep}{1 + e \cos\theta} & (1') \end{cases}$$

The equations (1) and (1') represent the same conic, since they correspond to two determinations of P.

Hence the desired equation of C is given by

$$r = \frac{ep}{1 - e\cos\theta} \quad \begin{cases} \text{an ellipse (E) if } e < 1 \\ \text{a parabola (P) if } e = 1 \\ \text{a hyperbola (H) if } e > 1 \end{cases}$$

The equation (1) in which the denominator contains the constant term "1" is called the *standard equation* of a conic, with directrix Δ perpendicular to polar axis, and focus at the pole. In the standard equation (1) the coefficient, in absolute value, of $\cos\Theta$ is the eccentricity of the conic.

Rotating the figure about F(pole) with angle $\pi/2$, π, and $3\pi/2$, we have the following equations of conics

$$r = \frac{ep}{1 - e\sin\theta}, \quad r = \frac{ep}{1 + e\cos\theta}, \quad r = \frac{ep}{1 + e\sin\theta}$$

respectively.

In a standard equation, the presence of $\cos\Theta$ ($\sin\Theta$) indicates that the directrix is perpendicular to PR(CPR) and the presence of + sign (-sign) shows that the directrix intersects (does not intersect) the corresponding ray.

When the numerator and denominator of the right hand side in a standard equation, say (1), is multiplied by a constant $A (\neq 0)$ we have

$$r = \frac{Ape}{A - Ae\cos\theta} \quad \left(r = \frac{D}{A + B_1\cos\theta}\right), \quad (A \neq 0)$$

representing the same curve. In this case, identification is done, dividing all the terms of the fraction by A.

If the polar axis is rotated by an angle α, the last equation becomes (dropping the primes):

$$r = \frac{D}{A + B_1 \cos(\Theta - \alpha)}$$

which is of the form

$$r = \frac{D}{A + B \cos\Theta + C \sin\Theta} \qquad (A \neq 0)$$

where

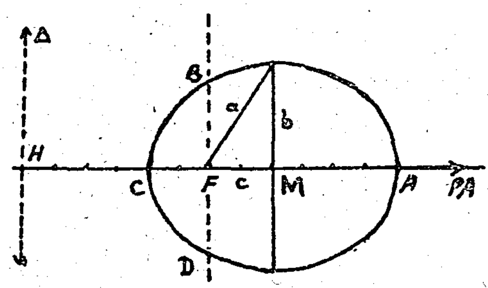
$$B = B_1 \cos\alpha, \quad C = B_1 \sin\alpha \qquad B^2 + C^2 = B_1^2, \; \tan\alpha = C/B$$

$$e = \frac{\sqrt{B^2 + C^2}}{A}$$

Example. Identity and sketch the following conics:

a) $r = \dfrac{6}{2 - \cos\Theta}$ b) $r = \dfrac{10}{2 + 3\sin\Theta}$

Solution.



a) $e = \frac{1}{2} < 1$: Ellipse.

$ep = 6/2 \Rightarrow p = 6$.

Intercepts:

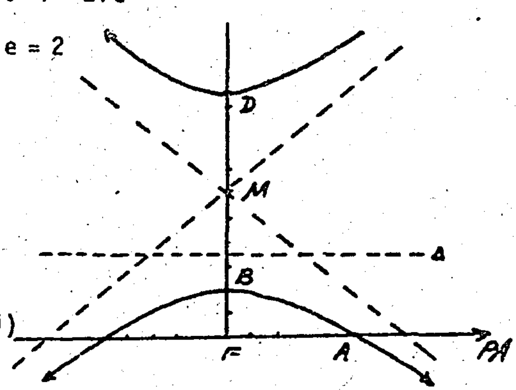
$A(0, 6)$, $B(\pi/2, 3)$

$C(\pi, 2)$, $D(-\frac{\pi}{2}, 3)$

$2a = |CA| = 8 \Rightarrow a = 4$

$c = ae = 2$, $b = 16 - 4 = 2\sqrt{3}$

$|MH| = a/e = 8$, $|MF| = ae = 2$



b) $e = \frac{3}{2} > 1$: Hyperbola.

$ep = 5 \Rightarrow p = 10/3$

Intercepts:

$A(0, 5)$, $B(\pi/2, 2)$

$C(\pi, 5)$, $D(-\pi/2, -10)$

Asymptotes: $r \to \infty$

$\Rightarrow \quad 2 + 3 \sin \Theta = 0$

$\Rightarrow \quad \tan \Theta = \pm \frac{\sqrt{5}}{2}$

$2a = |BB'| = 8 \quad \Rightarrow a = 4,$

$c = ae = 6 \qquad b = \sqrt{36-16} = 2\sqrt{5}$

$|MH| = a/e = 8/3, \quad |MF| = ae = 6$

The following two subjects (limaçons of PASCAL and curves of CASSINI) are given in some detail, but for the reader not interested in the details, their equations and shape of the curves may be important.

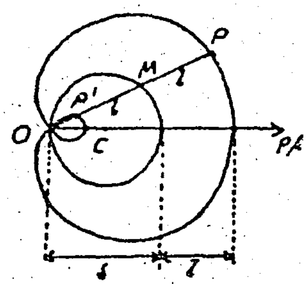
Limaçons of PASCAL

These curves are defined as follows:

Consider a circle (center at C) of diameter δ and a fixed point O on it, and consider a line through the fixed point O meeting the circle again at a point M.

On this line lay-off segments (MP), (MP') of constant length ℓ.

When the point M describes the circle, the points P describe a closed curve called a *limaçon of PASCAL* or simply a *limaçon*.



To obtain the polar equation of this limaçon, we choose the fixed point O as pole and the line OC as polar axis.

The equation of the locus of $P(\Theta, r)$ is

$$r = \ell + \delta \cos \Theta \qquad (1)$$

while that of $P'(\Theta, r)$ is

$$r = -\ell + \delta \cos \Theta \qquad (1')$$

Since (1') can be obtained from (1) by changing Θ to $\Theta + \pi$ and r to $-r$, the limaçon has then the equation

$$r = a + \delta \cos \Theta \qquad (1'')$$

where $|a| = \ell$.

Rotating the limaçon about the pole by the angles $\pi/2$, π $3\pi/2$ we have the following equations:

$$r = a + \delta \sin \Theta, \quad r = a - \delta \cos \Theta, \quad r = a - \delta \sin \Theta$$

Hence an equation of the form

$$r = a + b_1 \cos \Theta \quad \text{or} \quad r = a + b_1 \sin \Theta$$

represents a limaçon of PASCAL with $\ell = |a|$ and $\delta = |b_1|$.

Now rotating the PA, by an angle α, the equation

$$r = a + b_1 \cos(\Theta - \alpha) \quad \text{or} \quad r = a + b_1 \sin(\Theta - \alpha)$$

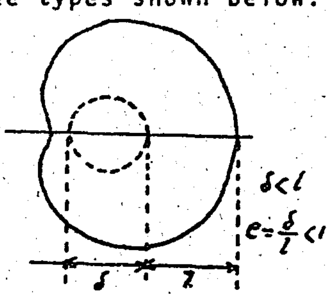
is obtained which is in the form (primes being dropped):

$$r = a + b \cos \Theta + c \sin \Theta \qquad (2)$$

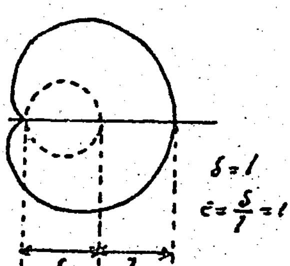
where $\ell = |a|$ and $\delta = \sqrt{b^2 + c^2}$.

A limaçon can be traced continuously as follows: Drawing a circle and marking on a ruler three points P', M, P such that $|P'M| = |MP|$, and moving the ruler in such a way that it always passes through a fixed point O of the circle and M lying always on the circle (do it!)

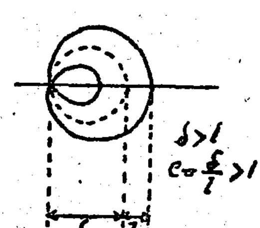
For various values of δ and ℓ, limaçons appear to be of three types shown below:



Elliptic case



Parabolic case
(a cardioid)



Hyperbolic case

They differ from each other by the values of the ratio $e = \delta/\ell$ as in the case of conics. The limaçon is elliptic, parabolic or hyperbolic according as e is less than, equal to or greater than 1

These types are also distinquished by the numbers of tangent lines at the pole 0. This number is 0 in elliptic case since the pole is not on the curve, and 1 and 2 for parabolic and hyperbolic case respectively.

The type corresponding to $e = 1$ is called a *cardioid*, because of its ressemblance to heart and its equations appear in the forms:

$$r = a(1 + \cos\Theta), \quad r = a(1 - \cos\Theta)$$
$$r = a(1 + \sin\Theta), \quad r = a(1 - \sin\Theta)$$

with $|a| = \delta = \ell$.

Comparing the general equation of conics and limaçons one may observe that if $r = f(\Theta)$ is the equation of a conic, then $r = a/f(\Theta)$ is the equation of a limaçon.

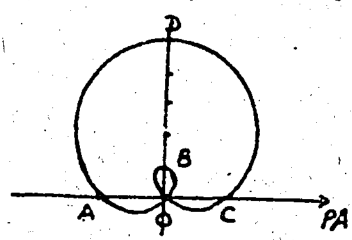
<u>Example 1</u>. Plot the limaçon $r = -2 + 3\sin\Theta$.

<u>Solution</u>. $\ell = |-2| = 2$, $\delta = 3$, $e = \delta/\ell = 3/2 > 1$ (hyperbolic case). It passes through the pole having two tangent lines there.

Intercepts:

$A(0, -2)$, $B(\frac{\pi}{2}, 1)$

$C(\pi, -2)$, $D(-\frac{\pi}{2}, -5)$



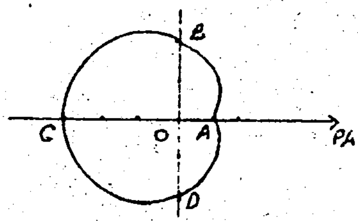
<u>Example 2</u>. Plot the limaçon $r = 2 - \cos\Theta$

<u>Solution</u>. $\ell = 2$, $\delta = |-1| = 1$, $e = \delta/\ell = 1/2$ (Elliptic case) It does not pass through the pole.

Intercepts:

A(0, 1), $B(\frac{\pi}{2}, 2)$

$C(\pi, 3)$, $D(-\frac{\pi}{2}, 2)$

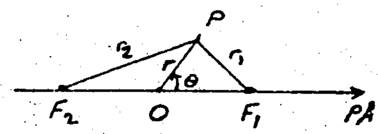


Curves of CASSINI

A curve of CASSINI is the locus of points P whose product of distances from two distinct points F_1, F_2 is constant. F_1, F_2 are said to be the *foci* of the curve.

The equation of a curve of CASSINI in bipolar coordinates is $r_1 r_2 = \ell^2$, where $r_1 = |PF_1|$, $r_2 = |PF_2|$

We obtain the polar equation by taking the midpoint O of $[F_1F_2]$ as pole and OF_1 as polar axis, with



$c = |OF_1| = |OF_2|$:

$$r_1^2 r_2^2 = \ell^4 \Rightarrow (r_1^2 + c^2 - 2cr\cos\theta)(r^2 + c^2 + 2cr\cos\theta) = \ell^4$$

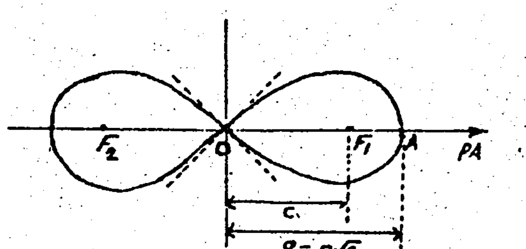
$$\Rightarrow (r^2 + c^2)^2 - 4c^2r^2\cos^2\theta - \ell^4 = 0$$

$$\Rightarrow \quad r^4 - 2c^2(\cos 2\theta)r^2 + c^4 - \ell^4 = 0. \qquad (1)$$

Case 1. $\ell = c$: The curve passes through the pole ($r = 0$) only when $\ell = c$ yielding the equation

$$r^2 = 2c^2\cos 2\theta \quad \text{or} \quad r^2 = a^2\cos 2\theta \qquad (2)$$

and this particular curve is called the *lemniscate of Bermoull* or simply *lemniscate*.



Since in (2), $r = 0$ when $\theta = \pm \frac{\pi}{4}$ it follows that the tangents to lenniscate at the pole are inclined to PA by angles $\pi/4$.

Case 2. $\ell \neq c$: PA-intercepts: Setting $\theta = 0$ in (2), we have

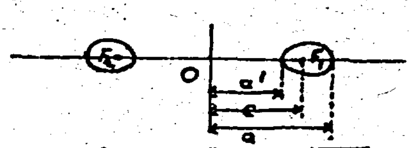
$$r^4 - 2c^2r^2 + c^4 - \ell^4 = 0$$

$$\Rightarrow \quad r^2 = c^2 \pm \ell^2 \quad \begin{cases} r = \sqrt{c^2 + \ell^2} \\ r = -\sqrt{c^2 - \ell^2} \quad \text{when} \quad \ell < c \end{cases}$$

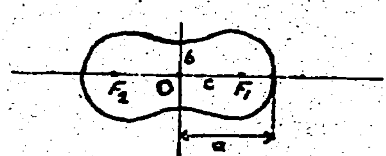
CPA-intercepts: Setting $\theta = \frac{\pi}{2}$ in (1), we have

$$r^4 + 2c^2 + c^4 - \ell^4 = 0$$

$$\Rightarrow \quad r^2 = -c^2 \pm \ell^2 \Rightarrow |r| = \sqrt{\ell^2 - c^2} \quad \text{when } \ell > c.$$



$a' = \sqrt{c^2 - \ell^2}$, $a = \sqrt{c^2 + \ell^2}$
(Two ovals)

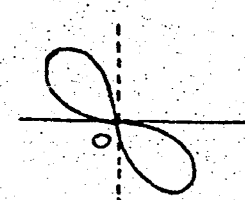


$a = \sqrt{\ell^2 + c^2}$, $b = \sqrt{\ell^2 - c^2}$
(A closed curve)

As to the closed curve corresponding to $\ell > c$, it can be shown using calculus that when $c < \ell \leq c\sqrt{2}$, the curve is an oval (any intersecting line meets the curve at 2 point utmost).

Example. Sketch the lemniscate $r^2 = -4 \sin 2\theta$.

Solution. $|r| = 2$ is maximum when $\sin 2\theta = -1$ implying $2\theta = 3\pi/2$, $\theta = 3\pi/4$. The line $\theta = 3\pi/4$ is the transverse axis of symmetry. The directions along which $r \to 0$, are $\theta = 0$, $\theta = \pi/2$, $\theta = \pi$, and $\theta = 3\pi/2$.



Spirals.

A spiral in a plane is a curve winding (circling) round a circle (or a center) and gradually receding from or approaching it.

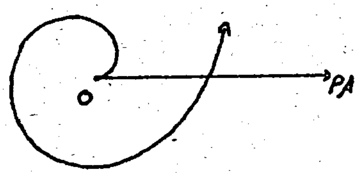
Two examples of spirals are

1) $r = a\Theta$ (ARCHIMEDES' Spiral)

2) $a\Theta = \ell n\ r$ or $r = e^{a\Theta}$ (Logarithmic Spiral)(See Chapter 6).

1. An ARCHIMEDES' Spiral is the trajectory of a point P moving uniformly on a line which in turn rotating uniformly about about a point.

Taking the center 0 of rotation as pole and that line when P is at 0 as polar axis, we have



$r = bt$ (linear motion on 0r)

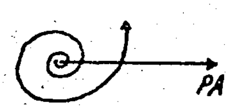
$\Theta = \omega t$ (uniform rotation)

$r = bt = b \cdot \frac{\Theta}{\omega} = \frac{b}{\omega}\Theta = a\Theta$

If $a > 0$, having $r \to \infty$ as $\Theta \to \infty$, the curve is a spiral admitting CPA as axis of symmetry (since $r \to -r$ when $\Theta \to -\Theta$), with PA as tangent at 0. (When $a < 0$, the motion takes place in clockwise sense).

In the history of mathematics this curve is the first curve other than the circle to which tangent line has been constructed. Also Archimedes used this curve to trisect an angle and squaring a circle.

2. The curve $r = e^{a\Theta}$ $(a > 0)$ is another spiral since $r \to \infty$ as $\Theta \to \infty$. Furthermore since $r \to 0$ as $\Theta \to -\infty$, he pole is a point-asymptote.



Among other spirals we mention the following.

3. $r\Theta = a$ (hyperbolic or reciprocal spiral)

4. $r^2\theta = a^2$ (lituus)

5. $r^n = a\theta$ (FERMAT's spiral)

<u>Roses.</u>

A rose is a figure composed of some number of loops (leaves) arranged regularly around a point.

The curves given by the equation

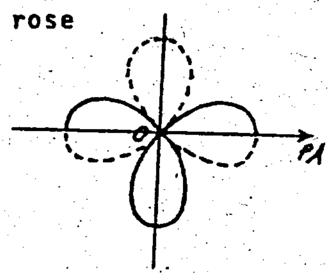
$$r = a \cos n\theta \quad (\text{or } r = a \sin n\theta) \quad (n \in N^+)$$

are examples of roses:

$n = 1$: $r = a \cos \theta$ (circle) 1-leaved rose

$n = 2$: $r = a \cos 2\theta$ $(a > 0)$

$T = \frac{2\pi}{2} = \pi$

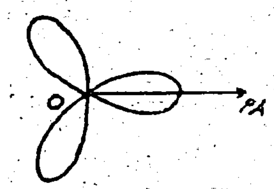


θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
r	a	0	$-a$	0	a

The complete curve is then obtained by rotation through π about 0. The curve is a *4-leaved rose.*

$n = 3$: $r = a \cos 3\theta$ $(a > 0)$

$T = \frac{2\pi}{3}$



θ	0	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$
r	a	0	$-a$	0	a

The curve is a *3-leaved rose.*

The curves $r = a \cos n\theta$ (or $r = a \sin n\theta$) are *n-leaved roses* when n is odd, and *2n-leaved* ones when n is even.

The curves given by

$$r^n = a^n \cos n\theta \quad (\text{or } r^n = a^n \sin n\theta)$$

are *n-leaved roses* ($n = 1$ gives a circle and $n = 2$ a lemniscate).

E. INTERSECTION OF CURVES:

Let

$$r = f(\Theta), \quad r = g(\Theta)$$

be the polar equations of two curves.

To find their points of intersection one first examines the pole as possible point of intersection. If $f(\Theta)$, $g(\Theta)$ both vanish for any Θ_1, Θ_2 respectively, equal or distinct, the pole is a point of intersection, otnerwise it is not.

The other intersection points are obtained frequently by solving $f(\Theta) = g(\Theta)$ for Θ. If Θ_1, Θ_2, ... are the roots, then (Θ_1, r_1), (Θ_2, r_2), ... are the required points of intersection which may include the pole also. It may not give the all points,

To cover all, one has to solve also

$$f(\Theta) = -g(\Theta + (2k+1)\pi),$$

since (Θ, r) is the same as $(\Theta \quad (2k+1)\pi, -r)$

Example. Find the points of intersection of

$$r = \cos\Theta \text{ (circle) and } r = 1 - \cos\Theta \text{ (cardioid)}$$

Solution. r's vanish for $\Theta = \pi/2$ and $\Theta = 0$ respectively. Hence the pole is a point of intersection. The other points are obtained from

$$\cos\Theta = 1 - \cos\Theta$$

$$\Rightarrow \quad 2\cos\Theta = 1 \Rightarrow \cos\Theta = \frac{1}{2} \Rightarrow \Theta = \pm\frac{\pi}{3} + 2k\pi$$

$$\Rightarrow \quad \Theta_1 = \pi/3, \qquad \Theta_2 = -\pi/3$$

$$\Rightarrow \quad r_1 = \cos\frac{\pi}{3} = \frac{1}{2}, \qquad r_2 = \cos(-\frac{\pi}{3}) = \frac{1}{2} \quad \Rightarrow$$

$$A_1(\frac{\pi}{3}, \frac{1}{2}), \qquad A_2(-\frac{\pi}{3}, \frac{1}{2}).$$

Example 2. Find the points of intersection of

$$r^2 = 16\cos 2\Theta, \qquad r = 2 + \sin\Theta$$

Solution. Both r do not vanish. Pole is not a point of intersection.

$$16 \cos 2\theta = (2 + \sin\theta)^2 \Rightarrow 16 \cos 2\theta = 4 + 4 \sin\theta + \sin^2\theta$$

$$\Rightarrow \quad 16(1 - 2 \sin^2\theta) = \sin^2\theta + 4 \sin\theta + 4$$

$$\Rightarrow \quad 33 \sin^2\theta + 4 \sin\theta - 12 = 0$$

$$\Rightarrow \quad \sin\theta = \frac{-2 \pm \sqrt{4 + 396}}{33} = \frac{-2 \pm 20}{33} = \begin{cases} -\frac{2}{3} \\ \frac{6}{11} \end{cases}$$

$$\Rightarrow \begin{cases} \theta_1 = \arcsin(-\frac{2}{3}), & \theta'_1 = \pi - \theta_1 & r_1 = 4/3 \\ \theta_2 = \arcsin \frac{6}{11}, & \theta'_2 = \pi - \theta_2 & r_2 = 28/11 \end{cases}$$

Hence the points of intersection are:

$A(\theta_1, 4/3)$, $B(\theta'_1, 4/3)$, $C(\theta_2, 28/11)$, $D(\theta'_2, 28/11)$.

To see whether or not these are all points of intersection it will be convenient to sketch the given curves. (Sketch the curves and observe that these are all the points of intersection).

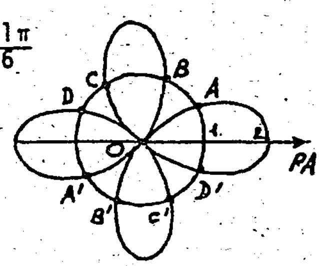
Example. Find the points of intersection of the curves

$$r = 1, \qquad r = 2 \cos 2\theta$$

Solution. The curves are a circle and a 4-leaved rose given in the Figure, showing 8 points of intersection.

$$2 \cos 2\theta = 1 \Rightarrow \cos 2\theta = \frac{1}{2}$$

$$\Rightarrow \quad \theta_1 = \frac{\pi}{6}, \quad \theta_2 = \frac{5\pi}{6}, \quad \theta_3 = \frac{7\pi}{6}, \quad \theta_4 = \frac{11\pi}{6}$$



giving A, D, A', D' only.

Then we must solve

$-2 \cos 2(\theta + \pi) = 1$, implying

$$\cos 2\theta = -\frac{1}{2} \quad \text{and}$$

$$\theta_5 = \frac{\pi}{3}, \quad \theta_6 = \frac{4\pi}{4}, \quad \theta_7 = \frac{2\pi}{3}, \quad \theta_8 = \frac{5\pi}{3} \quad (r = 1)$$

giving B, B', C, C'.

EXERCISES (4. 3)

41. Show that the two determinations of a point P in polar coordinates can be expressed in the form

$$P(\theta + m\pi;\ (-1)^m r),\quad m\in\mathbb{Z}$$

42. Plot the following points in the polar system of coordinates:

a) $A(\pi/3, 0)$, $B(\pi/6, 0)$, $C(0, 1)$, $D(\pi/2, 2)$, $E(-\pi/2, 2)$, $F(0, 3)$, $G(-\pi, -1)$, $H(-\pi, 3)$, $J(\pi, -2)$, $K(\pi/2, -2)$.

b) Write their second determinations with $0 < \theta < 2\pi$.

43. Verify the relations $x = r\cos\theta$, $y = r\sin\theta$ for any P in the polar system and for any determination (θ, r).

44. Transform the given cartesian equation to polar one:

a) $x^2 y = 4a^2(2a - y)$ (*witch of AGNESI*)

b) $(x^2 + y^2)(x - a)^2 = b^2 x^2$

45. Transform the given polar equations into cartesian ones:

a) $r^2 = a^2 \sin 2\theta$ b) $r = a(1 + \sin\theta)$

c) $r = \sin 2\theta$ d) $r = \cos 2\theta$

46. Find the distance between the given points:

a) $A(\pi, 4)$, $B(\pi/6, 2/3)$, b) $C(0, 4)$, $D(\pi/4, -4)$,

c) $E(-\pi/4, 3)$, $F(3\pi/4, -3)$ d) $O(\theta, 0)$, $P(\theta, r)$.

47. Obtain the new equation of the curve $r = ep/(1 - e\cos\theta)$ when the polar equation is rotated by an angle φ.

48. Show that a rotation about the pole preserves

a) distance between two points b) angle

49. Write the equation of the circle:

a) Center: O, radius: -2 b) Center: $(\pi/4, 2)$, radius: 2.

50. Sketch the set of points:

a) $\{(\theta, r): r=2 \text{ or } r=-3\}$ b) $\{(\theta, r): r^2+r-2=0\}$

c) $\{(\theta, r): \theta=\pi/4 \text{ or } \theta=\pi+\pi/4\}$ d) $\{(\theta, r): (\theta-\pi/3)(r+\pi/3)=0\}$

51. Given the equations

a) $r^2-10\,r\cos(\theta-\pi/3)+9=0$, b) $r^2+8\,r\cos(\theta+\pi/4)+2=0$

of circles, find the centers and radii, and then sketch them.

52. Find the distance between the given point and line:

a) $A(0, 4)$, ℓ: $r\cos(\theta-\pi/3)-2=0$

b) $A(\pi/6, 2\sqrt{3})$, ℓ: $r=2/(\cos\theta+\sin\theta)$

53. Show that the equation of line passing through the given distinct points $P_1(\theta_1, r_1)$, $P_2(\theta_2, r_2)$ is

$$\frac{\sin(\theta_1-\theta_2)}{r} = \frac{\sin(\theta-\theta_2)}{r_1} - \frac{\sin(\theta-\theta_1)}{r_2}$$

Hint: Use determinantal equation of the line in cartesian coordinates and transform it into polar coordinates, and then expand the determinant.

54. Derive the polar equation of the conic with focus at the pole, eccentricity and directrix Δ as stipulated:

a) $e=2$, $\Delta \perp PA$ and through $(\pi, 4)$

b) $e=6$, $\Delta /\!/ PA$ and through $(3\pi/2, 1)$

55. Same question, if

a) $e=1/3$, $\Delta /\!/ PA$ and through $(\pi/2, 2)$,

b) $e=1$, $\Delta \perp PA$ and through $(0, 4)$.

56. Show that the equation

$$\frac{e}{r} = \frac{\cos\theta}{a} \pm \frac{\sin\theta}{b}$$

represents the asymptotes of the hyperbola $r = ep/(1-e\cos\theta)$ where $2a$, $2b$ are the lengths of the transverse and conjugate diameters $(e > 1)$. <u>Hint</u>: Recall that the asymptotes of

$$b^2(x-h)^2 - a^2(y-k)^2 = a^2b^2 \quad \text{are}$$
$$b^2(x-h)^2 - a^2(y-k)^2 = 0$$

57. Plot and discuss the following curves. Find e, p and draw the conics:

a) $r = \dfrac{5}{2 - 2\cos\theta}$ b) $r = \dfrac{3}{3 - \cos\theta}$

58. Same question for:

a) $r = \dfrac{6}{2 - 3\cos\theta}$ b) $r = \dfrac{12}{3 - 4\cos\theta}$.

59. Same question for:

a) $r = \dfrac{6}{1 - \sin\theta}$, b) $r = \dfrac{5}{3 - \sin\theta}$

60. Plot the conics:

a) $r = \dfrac{2}{1 - \cos\theta}$ b) $r(2 + 4\sin\theta) = 0$

c) $r(3 - 2\sin\theta) = 2$ d) $r = \dfrac{5}{2 - 3\sin\theta}$

61. A chord (OP_1) of the circle $r = 2a\cos\theta$ is extended a distance $|P_1P| = 2a$. Find the locus of P.

62. Plot the cardioids:

a) $r = 2\cos^2\frac{\theta}{2}$ b) $r = 2\sin^2\frac{\theta}{2}$

63. Find the intersection of the curves: $r = \sin\theta + 1$, $r = \cos\theta - 1$

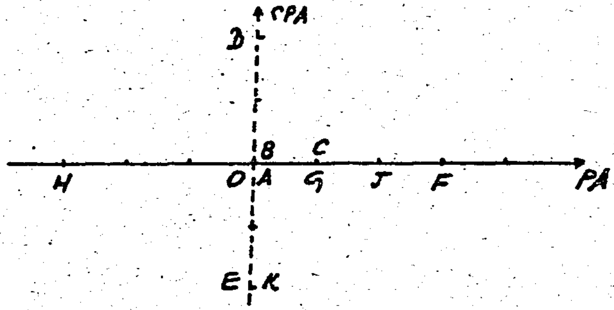
64. Sketch and find the points of intersection:

$r = 3\sin 3\theta$, $r\cos(\theta - \pi/6) - 3 = 0$

65. Same question for: $r = 2\cos 2\theta$, $r = 2\cos\theta$.

ANSWERS TO EVEN NUMBERED EXERCISES

42. a)



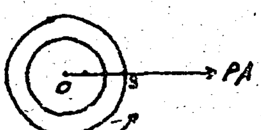
b) $A(\frac{\pi}{3} + \pi, 0)$, $B(\frac{\pi}{6} + \pi, 0)$, $C(\pi, -1)$, $D(-\frac{\pi}{2}, -2)$
$E(\frac{\pi}{2}, -2)$, $F(\pi, -3)$, $G(0, 1)$, $H(0, -3)$, $J(0, 2)$, $K(-\frac{\pi}{2}, 2)$

44. a) $r^3 \sin^2\theta \cos\theta = 4a^2(2a - r\cos\theta)$

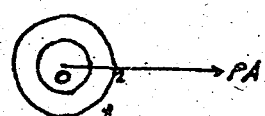
b) $r = 0$ or $r\cos\theta = a + b\sin\theta$ or $r\cos\theta = a - b\sin\theta$

46. a) $2\sqrt{13}$, b) $4\sqrt{2 + \sqrt{2}}$ c) 0 d) $|r|$

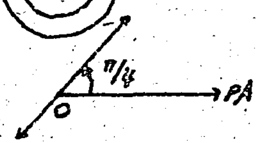
50. a)



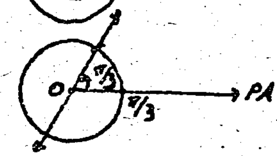
b)



c)



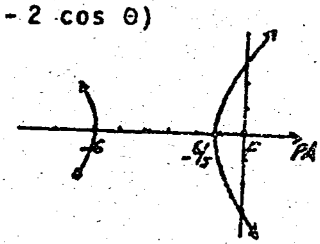
d)



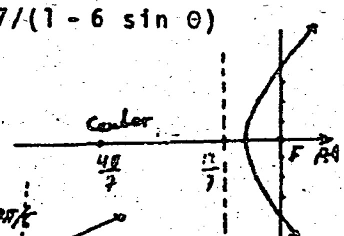
52. a) 0, b) $\left|2\sqrt{3}\cos\frac{\pi}{12} - \sqrt{2}\right|$

54. $r = 12/(1 - 2\cos\theta)$ b) $r = 7/(1 - 6\sin\theta)$

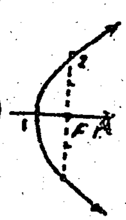
58. a)



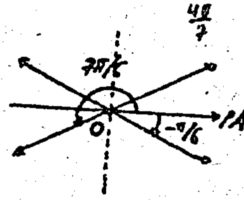
b)



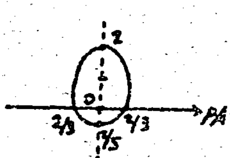
60. a)



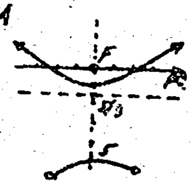
b)

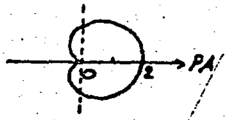


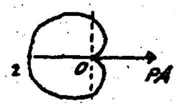
c)

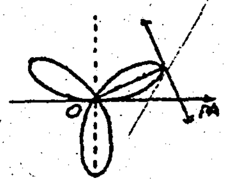


d)



62. a) 

b) 

64.  $(\pi/6,\ 3)$

4. 4. COMPLEX NUMBERS (Polar form)

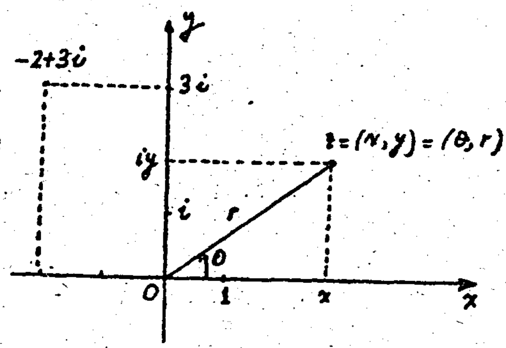
A. DEFINITIONS:

A complex number $z = x + iy$, when denoted by the ordered pair (x, y), represents a point in the analytic plane. Then the equality

$$z = x + iy = (x, y)$$

esteblishes a one to one correspondance between complex numbers and points of R^2.

The analytic plane in which complex numbers are represented is called the *complex plane (ARGAND plane* or *z-plane)*. The x-axis contains the real numbers $x + o\,i = (x, 0)$, while y-axis contains only zero and only pure imaginary numbers $0 + iy = (0, y)$ and accordingly are called the *real axis* and *imaginary axis* respectively.



The distance $\sqrt{x^2 + y^2} > 0$ of the point $z = x + iy$ from the origin is defined to be the *modulus* (or the *absolute value*) of z, written

$$\text{mod } z = |z| = r$$

which becomes the absolute value of a real number when $y = 0$.

Introducing the angle θ as in polar coordinates (see Fig.)

we obtain

$$x = r\cos\theta, \quad y = r\sin\theta \quad (r \geq 0)$$

which when substituted in $x + iy$ gives

$$r(\cos\theta + i\sin\theta),$$

called the *polar form* of z.

The angle θ such that $0 \leq \theta \leq 2\pi$ is called the *principal argument* of z, written Arg z. Any other argument of z is given by $\text{Arg}\, z + 2k\pi$, and

$$\theta = \arctan \frac{y}{x}$$

which is satisfied by two principal values of argument, one of which corresponds to z.

Example. Transform

a) $z = \sqrt{3} - i$, b) $z = -1 + i$

into polar form.

Solution.

a) $r = |z| = 2$, $\theta = \arctan \frac{-1}{\sqrt{3}}$. The solution for θ as principal values are $\theta_1 = 5\pi/6$ and $\theta_2 = 11\pi/6$. Then $\text{Arg}\, z = 11\pi/6$ since z lies in the fourth quadrant. Hence

$$z = 2(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}).$$

b) $r = \sqrt{2}$, $\theta = \arctan(-1) \Rightarrow \theta_1 = 3\pi/4$, $\theta_2 = 7\pi/4$. $\text{Arg}\, z = 3\pi/4$ since z lies in the second quadrant, and we have

$$z = \sqrt{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$$

Example. Write the cartesian form of the complex number with modulus 3 and principal argument $4\pi/3$.

Solution. From $r = 3$ and $\theta = 4\pi/3$, we have

$$z = 3(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}) = -\frac{3}{2} - i\,\frac{3\sqrt{3}}{2}$$

B. Multiplication and Division.

Polar form of complex numbers is very suitable for the operations of multiplication and division.

__Multiplication__: Let

$$a = r(\cos\alpha + i\ \sin\alpha),\ b = s(\cos\beta + i\ \sin\beta)$$

be two complex numbers. Then

$$\begin{aligned} ab &= rs(\cos\alpha + i\ \sin\alpha)(\cos\beta + i\ \sin\beta) \\ &= rs\big((\cos\alpha\ \cos\beta - \sin\alpha\ \sin\beta) + i(\sin\alpha\ \cos\beta + \cos\alpha\ \sin\beta)\big) \\ &= rs\big(\cos(\alpha + \beta) + i\ \sin(\alpha + \beta)\big) \end{aligned}$$

where

$$rs = |a||b| = |ab|,\ \alpha + \beta = \arg a + \arg b = \arg(ab) \qquad (1)$$

Stating (1) in words, *the modulus of the product of two complex numbers is the product of their moduli, and the argument is the sum of their arguments.*

The generalization to n numbers is immediate: If $z_1, \ldots, z_n$ are n complex numbers with $r_1, \ldots, r_n$ as their moduli and $\theta_1, \ldots, \theta_n$ as arguments, one has

$$z_1 \ldots z_n = r_1 \ldots r_n\big[\cos(\theta_1 + \ldots + \theta_n) + i\ \sin(\theta_1 + \ldots + \theta_n)\big]$$

which can be proved by induction.

In particular, for $z_1 = \ldots = z_n = z = r(\cos\theta + i\ \sin\theta)$ the above equality reduces to

$$\big[r(\cos\theta + i\ \sin\theta)\big]^n = r^n(\cos n\theta + i\ \sin n\theta)$$

and for $r \neq 0$, to

$$(\cos\theta + i\ \sin\theta)^n = \cos n\theta + i\ \sin n\theta, \quad n \in N$$

which is known as *De MOIVRE's Formula*.

__Division__.

If for complex numbers a and b one sets

$$\frac{a}{b} = c \quad \text{or} \quad a = bc,$$

in view of (1), one gets

$$|a| = |b||c|, \quad \arg a = \arg b + \arg c$$

or

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|} \quad \text{and} \quad \arg \frac{a}{b} = \arg a - \arg b \qquad (2)$$

In words, *the modulus of the ratio of two complex numbers is the ratio of their moduli, and argument is the difference of their arguments.*

Example. Given the complex numbers

$$u = 6 + 2i, \qquad v = 4 + 2i$$

find the polar form of their product

a) by the property (1)

b) first finding the cartesian product, and then transforming to polar form.

Solution.

a) $|uv| = |u||v| \qquad |uv| = \sqrt{40}\sqrt{20} = 20\sqrt{2},$

$\arg(uv) = \arg u + \arg v = \arctan \frac{1}{3} + \arctan \frac{1}{2}.$

$\tan(\arctan \frac{1}{3} + \arctan \frac{1}{2}) = \frac{1/3 + 1/2}{1 - 1/6} = 1$

$\Rightarrow \arg(uv) = \frac{\pi}{4} + k\pi$

$\Rightarrow \operatorname{Arg}(uv) = \pi/4$, since $\operatorname{Im}(uv) > 0$.

$\Rightarrow uv = 20\sqrt{2}\,(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}).$

b) $uv = 20 + 20i \Rightarrow |uv| = 20\sqrt{2},$

$\arg(uv) = \arctan 1$

$\Rightarrow \arg(uv) = \frac{\pi}{4} + k\pi = \frac{\pi}{4}$, since $\operatorname{Re}(uv) > 0$, $\operatorname{Im}(uv) > 0$.

$$\Rightarrow \quad uv = 20\sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$$

Example. Given the complex numbers

$$u = 2 + 2\sqrt{3}\, i, \quad v = \sqrt{3} - i$$

find the polar form of their ratio u/v

a) by the property (2)

b) first finding the cartesian ratio, and then transforming to polar form.

Solution.

a) $\left|\frac{u}{v}\right| = \frac{|u|}{|v|} = \frac{4}{2} = 2$, $\arg \frac{u}{v} = \arg u - \arg v$

$$= \frac{\pi}{3} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{2}$$

$$\Rightarrow \quad \frac{u}{v} = 2\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right).$$

b) $\frac{u}{v} = \frac{2 + 2\sqrt{3}i}{\sqrt{3} - i} = \frac{(2 + 2\sqrt{3})(\sqrt{3} + i)}{(\sqrt{3} - i)(\sqrt{3} + i)} = \frac{8i}{4} = 2i$

$$\Rightarrow \left|\frac{u}{v}\right| = 2, \quad \operatorname{Arg} \frac{u}{v} = \frac{\pi}{2}$$

$$\Rightarrow \frac{u}{v} = 2\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right).$$

Example. By the use of De Moirve's formula, compute $\cos 5\theta$, $\sin 5\theta$ in terms of $\cos\theta$ and $\sin\theta$.

Solution.

$$\cos 5\theta + i \sin 5\theta = (\cos\theta + i \sin\theta)^5$$

$$= \cos^5\theta + 5 \cos^4\theta(i \sin\theta) + 10 \cos^3\theta(i \sin\theta)^2$$

$$+ 10 \cos^2\theta(i \sin\theta)^3 + 5 \cos\theta(i \sin\theta)^4 + (i \sin\theta)^5$$

$$= (\cos^5\theta - 10 \cos^3\theta \sin^2\theta + 5 \cos\theta \sin^4\theta)$$

$$+ (5 \cos^4\theta \sin\theta - 10 \cos^2\theta \sin^3\theta + \sin^5\theta)i$$

$$\begin{cases} \cos 5\theta = \cos^5\theta - 10 \cos^3\theta \sin^2\theta + 5 \cos\theta \sin^4\theta \\ \sin 5\theta = \sin^5\theta - 10 \sin^3\theta \cos^2\theta + 5 \sin\theta \cos^4\theta \end{cases}$$

C. Roots of Numbers

By an *n th root* of a complex number is meant a complex number whose nth power is equal to the given number. If

$$\xi = \rho(\cos\varphi + i \sin\varphi)$$

is an nth root of the complex number

$$z = r(\cos\Theta + i \sin\Theta),$$

it satisfies the equation $\xi^n = z$ which, by the use of De Moivre's formula, gives

$$\rho^n(\cos n\varphi + i \sin n\varphi) = r \cos(\Theta + 2k\pi) + i \sin(\Theta + 2k\pi)$$

and ther

$$\rho^n = r, \qquad n\varphi \quad \Theta + 2k\pi$$

$$\rho = \sqrt[n]{r}, \qquad \varphi = \frac{\Theta}{n} + k\frac{2\pi}{n}, \quad k\in\mathbb{Z}.$$

Hence z has n distinct nth roots given by

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\Theta}{n} + k\frac{2\pi}{n}\right) + i \sin\left(\frac{\Theta}{n} + k\frac{2\pi}{n}\right)\right], \ k=1, 2, \ldots, n.$$

Since $|z_k| = \sqrt[n]{r}$, all these roots $z_1, z_2, \ldots, z_n$ lie on the circle with center at the origin and radius $\sqrt[n]{r}$ as vertices of a regular n-gon.

In particular, the nth roots of 1 (unity) are

$$\varepsilon_k = \cos k\frac{2\pi}{n} + i \sin k\frac{2\pi}{n}, \quad k=1, \ldots, n$$

one of which, namely ε_n, is the number 1 (Note that $\varepsilon_n = \varepsilon_o$).

If one of the nth roots of z is z_1, then all the nth roots of z are obtained multiplying z_1 by $\varepsilon_1, \ldots, \varepsilon_n$.

The roots of the polynomial equation $z^n - 1 = 0$ being $\varepsilon_1, \ldots, \varepsilon_n$, the following properties are the consequence of the relations between the roots and coefficients:

$$\sigma_1 = \sum \varepsilon_k = \varepsilon_1 + \ldots + \varepsilon_n = 0,$$

$$\sigma_2 = \sum_{k<\ell} \varepsilon_k\varepsilon_\ell = \varepsilon_1 \varepsilon_2 + \ldots + \varepsilon_1\varepsilon_n + \ldots + \varepsilon_{n-1} \varepsilon_n = 0,$$

$$\sigma_3 = \sum_{k<\ell<m} \varepsilon_k \varepsilon_l \varepsilon_m = 0,$$

$$\vdots$$

$$\sigma_{n-1} = 0,$$

$$\sigma_n = \varepsilon_1 \cdots \varepsilon_n = 1.$$

Example. Prove: the nth roots of unity can be represented as powers of $\varepsilon (= \varepsilon_1)$ as

$$\varepsilon, \varepsilon^2, \ldots, \varepsilon^n$$

Proof. Since $\varepsilon_k = \cos k \frac{2\pi}{n} + i \sin k \frac{2\pi}{n}$ and $\varepsilon = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$, from De Moivre's formula, we have

$$\varepsilon_k = (\cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n})^k = \varepsilon^k, \quad k=1,\ldots,n.$$

EXERCISES (4. 4)

66. Find $|z|$ and Arg z for the following complex numbers:

    a) $-3 + i$ b) $1 - \sqrt{3}i$ c) $-3i$ d) 7

67. Show that $i^n = i^r$ if $n = r \pmod 4$

68. Write the polar form of:

    a) $3 - 3i$ b) $2 + 2\sqrt{3}i$ c) $3\sqrt{3} - 3i$ d) -5

69. Find the polar form of the product of the complex numbers $z_1 = 2 + i$ and $z_2 = 3 + i$ without finding their product.

70. Find the polar form of the ratio $(3-i)/(1-3i)$ of two complex numbers without performing the division.

71. Sketch the following loci of points:

    a) $\{z: |z| = 2, \ z \in \mathbb{C}\}$, b) $\{z: \frac{|z-1|}{|z-2i|} = 1, \ z \in \mathbb{C}\}$

72. Same question for:

    a) $\{z: |z| > 3, \ z \in \mathbb{C}\}$, b) $\{z: \frac{|z-1|}{|z-2|} = \frac{1}{2}, \ z \in \mathbb{C}\}$

73. Sketch the following sets in Arg and plane:

    a) $\{z: 1 < |z| < 4, \ z \in \mathbb{C}\}$, b) $\{z: |z-1| + |z+1| = 3, \ z \in \mathbb{C}\}$

74. Sketch the following loci of points in z-plane:

a) $\frac{\operatorname{Re} z}{|z-1|} = 1$, b) $\frac{|z-3|}{\operatorname{Re} z} = \frac{1}{2}$

75. Same question for:

a) $\frac{\operatorname{Re} z}{|z-3|} = \frac{1}{2}$, b) $|z-2| - |z+2| = 2$

76. Same question for:

a) $\{z: \frac{\pi}{6} < \operatorname{Arg} z < \frac{\pi}{3},\ z \in \mathbb{C}\}$ b) $\{z: 1 < |z| < 2,\ \frac{\pi}{6} < \operatorname{Arg} z\ \frac{\pi}{2},\ z \in \mathbb{C}\}$

77. Find the roots of $z^3 + i = 0$

78. Find $\sqrt[5]{1+i}$ and plot these five roots in the complex plane.

79. Find $\sqrt[7]{-128}$.

80. If the roots of the polynomial equation $z^3 + 7 = 0$ are z_1, z_2, z_3 show that

a) $z_1 + z_2 + z_3 = 0$ b) $z_2 z_3 + z_3 z_1 + z_1 z_2 = 0$

c) $z_1 z_2 z_3 = -7$.

ANSWERS TO EVEN NUMBERED EXERCISES

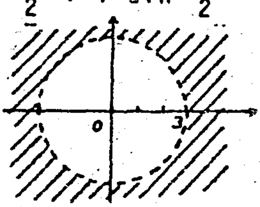
66. a) 2, $5\pi/6$, b) 2, $5\pi/3$, c) 3, $3\pi/2$, d) 7, 0.

68. a) $3\sqrt{2}(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4})$, b) $4(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$

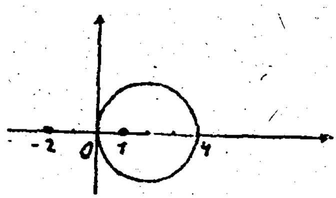
c) $6(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6})$, d) $5(\cos\pi + i\sin\pi)$

70. $\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$

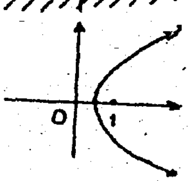
72. a)



b)



74. a)



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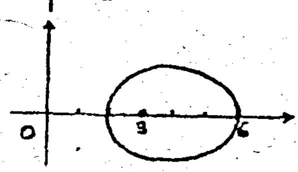
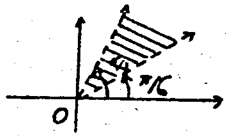
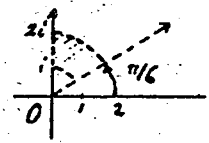
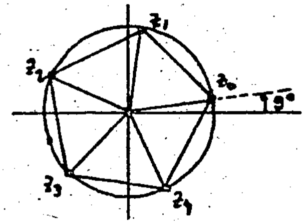


76. a) 

b) 

78. $z_k = \sqrt[10]{2}\Big[\cos(\frac{\pi}{20} + k\,\frac{2\pi}{5})$

$+ i\,\sin(\frac{\pi}{20} + k\,\frac{2\pi}{5})\Big]$



A SUMMARY

(CHAPTER 4)

4. 2 For a second degree equation

$$Ax^2 + bxy + Cy^2 + Dx + Ey + F = 0$$

we have the table of classification:

Δ 0 Elliptic case	$T \neq 0$ Non deg.	Real ellipse if $HT < 0$ Imag.ellipse if $HT > 0$
	$T = 0$ Degen.	A point(real)if $HT = 0$
$\Delta = 0$ Parabolic case	$T \neq 0$ Non deg.	Real parabola
	$T = 0$ Degen.	Two parallel line if $\psi > 0$ A double line if $\psi = 0$ Imag. parabola if $\psi < 0$
$\Delta > 0$ Hyperbolic case	$T \neq 0$ Non deg.	Real hyperbola
	$T = 0$ Degen.	Two intersecting lines

where

$$\Delta = B^2 = 4AC, \quad T = \begin{vmatrix} 2A & B & D \\ B & 2C & E \\ D & E & 2F \end{vmatrix}, \qquad \begin{aligned} H &= A + C \\ \psi &= (D^2-4AF)+(E^2-4CF) \end{aligned}$$

which are invariant under translation and rotation.

4. 3. Some curves and their equations in polar coordinates:

<u>Circles</u>: $r^2 - 2r_o r\cos(\theta - \theta_o) + r_o^2 - a^2 = 0$

Center at (θ_o, r_o), radius a.

<u>Lines</u>: $r\cos(\theta - \alpha) - p = 0$ (Normal form)

where (α, p) is the foot of the perpendicular from the pole to the line.

Conics: $r = ep/(1 - e\cos(\theta - \alpha))$

e is the eccentricity, p is the distance of the pole from the directrix

Limaçons: $r = a + b\cos(\theta - \alpha)$, (*cardioid* if $a = b$)

Lemniscate: $r^2 = a^2 \cos 2(\theta - \alpha)$

Spirals: $r = a\theta$ (Archimedes spiral)

$r = a^{\theta}$ (Logarithmic spiral)

4. 4. Complex numbers in polar form: $z = r(\cos\theta + i\sin\theta)$

where $r = |z| = \text{mod } z$, $\theta = \arg z$.

De Moivre's formula: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

MISCELLANEOUS EXERCISES

81. Find the point which divides the line segment joining the following pairs of the points in the given ratio:

a) $(2, 6)$ and $(-4, 8)$; $-\frac{4}{3}$ b) $(-3, 4)$, $(5, 2)$; $-\frac{2}{3}$

82. Prove that the points $(a, b+c)$, $(b, c+a)$ and $(c, a+b)$ are collinear.

a) by means of slope, b) by means of distance

c) by use of equation, d) by determinant.

83. Find the point (x'_0, y'_0) which is symmetric of the point (x_0, y_0) with respect to the line $F(x, y) = Ax + By + C = 0$.

84. Obtain the equation of the conic with the following data:

a) Δ: $2x - y + 3 = 0$, $F(1, 3)$, $e = 3/4$

b) Δ: $x + 2y - 3 = 0$, $F(1, 3)$, $e = 5/3$

c) Δ: $2x + 3y - 4 = 0$, $F(2, -1)$, $e = 1$

85. If $a = 5$, $b = 3$, find c, e, p.

a) for an ellipse, b) for a hyperbola

86. If $p = 4$, find a, b, c

a) when $e = 3/5$ b) when $e = 5/3$

87: Show that the circle which is orthogonal to the circles:
$x^2 + y^2 + 4x + 6y - 5 = 0$, $x^2 + y^2 + 8x + y - 20$,
$x^2 + y^2 + 6x + 2y - 14 = 0$ is orthogonal to the circle:
$x^2 + y^2 - 6x + 16y + 30 = 0$.

88. Find the equation of the circle which is orthogonal to the circles: $x^2 + y^2 + 3x - 6y - 5 = 0$, $x^2 + y^2 - 7x - y = 0$ and passing through the point $(-3, 0)$.

89. If $ax^2 + 3xy - 2y^2 - 5x + 5y + c = 0$ represents two perpendicular lines; find a and c.

90. Transform $2x^2 - 3xy - 2y^2 + 2x + 11y - 12 = 0$, first translating O to $O'(1, 2)$ and then rotating the axes through the acute angle $\Theta = \arctan 3$.

91. Show that each of the equations

$$3x^2 + 2xy - y^2 + 10x + 6y + 7 = 0$$

and

$$2x^2 + 7xy - 15y^2 + x - 44y - 21 = 0$$

represents a pair of lines; prove that these four lines are concurrent.

92. Transform $12x^2 - 7xy - 12y^2 - 32x - 24y = 0$, rotating the axes through the acute angle $\Theta = \arctan 3/4$.

93. Find the points of intersection of the following curves:

$$r = 4 \cos 3\Theta, \quad r = 2$$

94. Find the polar equation of the following curves whose parametric equations are given:

a) $x = 2t \quad 3$, $y = 3t - 1$ b) $x = a \cos t$, $y = b \sin t$

95. Transform the cartesian equation into polar form:

a) $x^2 + y^2 = 1$ b) $x^2 - y^2 = 1$

c) $x^2 + y^2 - 4y = 0$ d) $(x^2 + y^2)^2 - (x^2 - y^2) = 0$

e) $(x^2 + y^2 - ax)^2 = a^2(x^2 + y^2)$

96. Find the cartesian equation of the locus of the mid-points of the segments between the coordinate axes through (2, 3) and then transform it into polar coordinates.

97. Sketch the graphs of the following function:

a) $r = \begin{cases} 2 & \text{when} \\ \cos\theta & \text{when} \end{cases}$ b) $r = \begin{cases} \sec\theta & \text{when } 0 \leq \theta < \pi/2 \\ \csc\theta & \text{when } \frac{\pi}{2} \leq \theta < \pi \end{cases}$

98. A circle passing through the foci of the conic

$$\frac{ep}{r} \quad 1 - e \cos \theta$$

cuts the conic in points whose radius vectors are r_1, r_2, r_3 and r_4. Show that

$$1/r_1 + 1/r_2 + 1/r_3 + 1/r_4 = \frac{2}{ep}$$

99. Find the eccentricity e of the conic $r = \frac{A}{2 + B \sin \theta}$ if it passes through the points $(\pi/2, -2)$ and $(-\pi/6, 1)$.

100. Identify and sketch the conics, determining a, b, c.

a) $r(2 - \sin\theta) = 3$ b) $r(3 + 4 \cos\theta) = 8$

101. Show that a focal line intersects a conic at two points P_1, P_2 such that,

$$\frac{1}{|FP_1|} + \frac{1}{|FP_2|} = \frac{2}{ep}$$

102. Sketch the graph and determine the asymptotes if any of the following curve:

$$r = \frac{10}{2 + 3\cos\theta}$$

103. Find the polar form of $\frac{1+2i}{2+i}$ without performing the division.

104. Compute $(\sqrt{3} + i)^6$.

105. Express the following in a simpler form:

$1 + \cos\theta + \ldots + \cos n\theta, \quad \sin\theta + \ldots + \sin n\theta.$

ANSWERS TO EVEN NUMBERED EXERCISES

84. a) $44x^2 + 36xy + 71y^2 - 268x - 426y + 719 = 0$

b) $4x^2 - 20xy - 11y^2 + 12x + 6y + 45 = 0$

c) $9x^2 - 12xy + 4y^2 - 36x + 50y + 49 = 0$

86. a) $a = 15/4, \quad b = 3, \; c = 9/4$ b) $a = 15/4, \; b = 5, \; c = 75/12$

88. $3x^2 + 3y^2 + 4x + 2y - 15 = 0$

90. $x^2 - y^2 = 0$

92. $5xy + 8x = 0$

94. a) $r(3\cos\theta - 2\sin\theta) - 9 = 0$ b) $r^2(b^2\cos^2\theta + a^2\sin^2\theta) = a^2b^2$

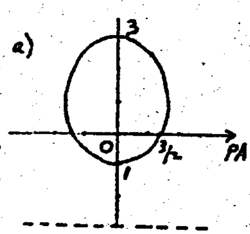
96. $2xy - 3x - 2y = 0; \quad r = 0, \; r\sin 2\theta = 3\cos\theta + 2\sin\theta.$

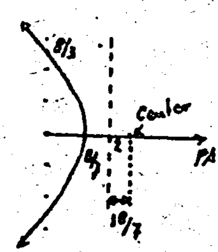
100. a) an ellipse $e = 1/2$

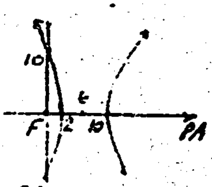
$p = 3, \quad a = 2, \quad b = \sqrt{3}$

b) a hyperbola $e = 4/3$

$p = 2, \quad a = \frac{24}{7}, \quad b = \frac{8\sqrt{7}}{7}$

a) 

b) 

102. 

$r\cos[\theta - \arccos(-2/3)] - 2\sqrt{5} = 0$

104. -64

CHAPTER 5
INTEGRATION

5. 1. INDEFINITE INTEGRAL (PRIMITIVE FUNCTION):

A. DEFINITIONS:

$F(x)$ is called a *primitive function* or simply a *primitive* of the function $f(x)$ if $f(x)$ is the derivative of $F(x)$

According to the definition, one can write

$$D\,F(x) = f(x) \Rightarrow F(x) = \frac{1}{D} f(x)$$

where $\frac{1}{D}$ $(= D^{-1})$ is the inverse of the derivative otorator D and accordingly $F(x)$ is said to be an *antiderivative* of $f(x)$.

In terms of differentials we have

$$D\,F(x) = f(x) \Rightarrow dF(x) = f(x)dx \Rightarrow F(x) = \frac{1}{d}\,(f(x)dx)$$

where $\frac{1}{d}$ $(= d^{-1})$ is the inverse of the differential operator d, denoted by the symbol $\int$ which is called *integral sign*. Then

$F(x) = \int f(x)dx$ (Read: *indefinite integral* of $f(x)dx$)

The following theorem justifies the term "indefinite":

Theorem. Any two primitives of a given function differ by a constant.

Proof. Let $F(x)$, $G(x)$ be any two primitives of the given function $f(x)$.

$$D\,F(x) = D\,G(x) = f(x) \Rightarrow D\big(F(x) - G(x)\big) = 0$$

$$\Rightarrow F(x) - G(x) = c \quad (c \text{ is an arbitrary constant}).$$

Corollary. If $F(x)$ is a primitive of $f(x)$, then any primitive is in the form $F(x) + c$ where $c \in R$, namely

$$\int f(x)dx + c$$

where c is called the *constant of integration*, and $f(x)$ the integrand, and dx indicates that the integral is to be taken with respect to the variable x.

Example 1. Complete the table where F(x) is a primitive of f(x):

f(x)	4x + 3			cos x	$\frac{1}{1+x^2}$	sin 2x
F(x)		x^3	tan x			

Solution: $D\,x^3 = 3x^2$, $D\tan x = \sec^2 x$, and since $D(2x^2 + 3x) = 4x + 3$, $D\sin x = \cos x$, $D\arctan x = \frac{1}{1+x^2}$, $\frac{1}{2}D\cos 2x = \sin 2x$, we have

f(x)	4x + 3	$3x^2$	$\sec^2 x$	cosx	$\frac{1}{1+x^2}$	sin 2x
F(x)	$2x^2+3x+c$	x^3	tan x	sinx + c	arctanx + c	$-\frac{1}{2}\cos 2x+c$

Example 2. Find the primitive of $f(x) = \frac{1}{\sqrt{1-x^2}}$, if $F(\frac{1}{2}) = 0$.

Solution. $F(x) = \int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + c$

$$\Rightarrow \quad F(\tfrac{1}{2}) = \arcsin\tfrac{1}{2} + c = 0 \quad \Rightarrow c = -\frac{\pi}{6}$$

$$\Rightarrow \quad F(x) = \arcsin x - \frac{\pi}{6}$$

Properties.

1. $\int du(x) = u(x) + c$ 2. $D\int f(x)dx = f(x)$

3. $\int(f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$

4. $\int \lambda f(x)dx = \lambda\int f(x)dx$ if and only if λ is a scalar.

Proof.

.. $\int du(x) = \int u'(x)dx = u(x) + c$,

2. Let $F(x) = \int f(x)dx$. Then $D\int f(x)dx = D\,F(x) = f(x)$,

3. Let $\phi(x) = \int(f(x) + g(x))dx$. Then $D\phi(x) = f(x) + g(x)$.

If F(x), G(x) are any primitives of f(x), g(x) respectively:

$$D\phi(x) = DF(x) + DG(x) \qquad \phi(x) = F(x) + G(x).$$

4. Let $\lambda = \lambda(x)$. Then differentiatinq both sides, we have

$$\lambda(x).f(x) = \lambda'(x) \int f(x)dx + \lambda(x)\ f(x) \qquad \text{(a)}$$

$$\lambda'(x) \int f(x)dx = 0 \Rightarrow \lambda'(x) = 0 \Rightarrow \lambda(x) = \text{const.}$$

If λ is a constant (4) is true from (a).■

Example 1. Evaluate the following indefinite integrals:

a) $\int \sec x \tan x\ dx$. b) $\int (4x^3 + 1 + 5\cos 2x)dx$

Solution.

a) $\int \sec x\ \text{tna}\ x\ dx = \int d \sec x = \sec x + c$,

b) $\int (4x^3 + 1 + 5\cos 2x)dx = \int 4x^3dx + \int dx + \int 5 \cos 2x\ dx$

$= x^4 + x + \frac{5}{2} \sin 2x + c.$

Example 2. Derive a formula for $\int x^\alpha\ dx$ $(\alpha \in R)$ and discuss the result.

Solution. We have $D\ x^{\alpha+1} = (\alpha + 1)x^\alpha$ implying

$$x^{\alpha+1} = (\alpha + 1) \int x^\alpha dx \Rightarrow \int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} \quad \text{if} \quad \alpha \neq -1.$$

If $\alpha = -1$, $\int x^\alpha dx = \frac{dx}{x} = \ln x + c$ where $\ln x$ is the natural logarithm of x that we discuss in the next Chapter:

$$\int x^\alpha dx = \begin{cases} \frac{x^{\alpha+1}}{\alpha+1} & \text{if} \quad \alpha \neq -1, \\ \ln x & \text{if} \quad \alpha = -1. \end{cases}$$

Example 3. Using the result in Example 2, evaluate

a) $\int \sqrt[3]{x^2}\ dx$ b) $\int \frac{dx}{\sqrt[3]{x^2}}$

Solution.

a) $\int \sqrt[3]{x^2}\ dx = \int x^{2/3}dx = \frac{x^{5/2}}{5/2} + c = \frac{2}{5} x^{5/2} + c,$

b) $\int \frac{dx}{\sqrt[3]{x^2}} = \int x^{-2/3}dx = \frac{x^{1/3}}{1/3} + c = 3\ x^{1/3} + c.$

B. METHODS OF INTEGRATION.

In calculus there are essentially two methods of integration: "change of variable" and "by parts"

Integration by change of variable (substitution)

Let the indefinite integral

$$I = \int f(x)dx$$

to be evaluated.

One makes (tries) the substitution

$$x = u(t) \quad \text{or} \quad t = u^{-1}(x) = v(x):$$

$$I = \int f(x)dx = \int f(u(t)).u'(t)dt = \int g(t)dt.$$

If the substitution is properly selected the new integral is more easily integrable than the original one, getting $G(t) + c$ and replacing t by $v(x)$ one has.

$$I = G(t) + c = G(v(x)) + c = F(x) + c$$

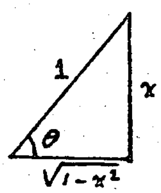
Example 1. Evaluate

$$I = \int \frac{dx}{(1 - x^2)^{3/2}}$$

Solution. Since square root is involved, $1 - x^2 > 0$ follows and the substitution $x = \sin\theta$ may work:

$$I = \int \frac{\cos\theta \, d\theta}{(1-\sin^2\theta)^{3/2}} = \int \frac{\cos\theta \, d\theta}{\cos^3\theta} = \int \sec^2\theta \, d\theta = \tan\theta + c.$$

The result is to be written in terms of x. Using the relation $x = \sin\theta$ we have



$$I = \tan\theta + c = \frac{x}{\sqrt{1 - x^2}} + c$$

Example 2. Evaluate $I = \int (x^3 - 2x + 3)^{15}(3x^2 - 2)dx$

Solution. Observing that $D(x^3 - 2x + 3) = 3x^2 - 2$ the proper substitution is

$$u = x^3 - 2x + 3, \quad du = (3x^2 - 2)dx,$$

$$I = \int u^{15} du = \frac{u^{16}}{16} + c = \frac{1}{16}(x^3 - 2x + 3)^{16} + c.$$

<u>Example 3</u>. Evaluate $I = \int \sec^2 x \tan x \, dx$

<u>Solution</u>. Since $\sec^2 x = D \tan x$, the substitution $u = \tan x$ works:

$$I = \int \tan x \sec^2 x \, dx = \int u \, du = \frac{1}{2} u^2 + c = \frac{1}{2} \tan^2 x + c$$

<u>Integration by parts.</u>

This method is based on the equality

$$d(uv) = udv + vdu$$

where u and v are functions of x. Writing this as

$$udv = d(uv) - vdu$$

and integrating each side, we have the method, namely

$$\int udv = uv - \int vdu \qquad (*)$$

where $\int udv$ is the given integral in the form $\int f(x)dx$. At the right hand side, for the properly selected u and dv, the evaluation of the integral $\int vdu$ is more easier than the given integral.

This method is generally applied when the integrand contains one or more transcendental functions like $\sin x$, $(x^2 - x)$. $\arcsin x$.

If the selection of u and dv gives a complicated integral one tries another selection.

<u>Example 1</u>. Integrate $I = \int \arcsin x \, dx$.

<u>Solution</u>. Setting

$$u = \arcsin x, \; dv = dx$$

one has

$$du = \frac{dx}{\sqrt{1 - x^2}}, \quad v = x.$$

Then

$$I = uv - \int v\,du = x \arcsin x - \int \frac{x\,dx}{\sqrt{1-x^2}}$$

The new integral is easily evaluated by the substitution $t = 1 - x^2$, $dt = -2x\,dx$:

$$\int \frac{x\,dx}{\sqrt{1-x^2}} = -\frac{1}{2}\int t^{-1/2}\,dt = -\frac{1}{2}\,\frac{t^{1/2}}{1/2} + c = -\sqrt{t} + c,$$

$$I = x \arcsin x - \sqrt{1-x^2} + c.$$

Example 2. Evaluate $I(x) = \int x^2 \sin x\,dx$ with $I(\frac{\pi}{2}) = 3$.

Solution. The setting $u = \sin x$, $dv = x^2 dx$ obviously does not work, since $v = x^3/3$ leads to a more complicated integral.

Trying

$$u = x^2, \qquad dv = \sin x\,dx$$

$$du = 2x\,dx, \qquad v = -\cos x,$$

we have

$$I(x) = -x^2 \cos x + 2\int x \cos x\,dx$$

where for new integral "by parts" is needed again:

$$u = x, \qquad dv = \cos x\,dx$$
$$du = dx, \qquad v = \sin x$$

$$I(x) = -x^2 \cos x + 2\left(x \sin x - \int \sin x\,dx\right)$$

$$= -x^2 \cos x + 2\,x \sin x + 2\cos x + c,$$

$$I(\pi/2) = 3 \Rightarrow 0 + \pi + 0 + c = 3 \Rightarrow c = 3 - \pi$$

and

$$I(x) = -x^2 \cos x + 2x \sin x + 2\cos x + 3 - \pi.$$

The next example differs from previous ones by arrival at the original integral during process of integration:

Example 3. $I = \int \cos^4 x\,dx$

Solution.

$$u = \cos^3 x, \qquad dv = \cos x\,dx$$

$$du = -3\cos^2 x \sin x\,dx. \quad v = \sin x$$

$$I = \cos^3 x \sin x + 3\int \cos^2 x \sin^2 x\, dx$$
$$= \cos^3 x \sin x + 3\int \cos^2 x(1-\cos^2 x)dx$$
$$= \cos^3 x \sin x + 3\int \cos^2 x\, dx - 3\underbrace{\int\cos^4 x\, dx}_{I}.$$
$$4I = \cos^3 x \sin x + \frac{3}{2}\int (1+\cos 2x)dx$$
$$= \cos^3 x \sin x + \frac{3}{2}\left(x + \frac{\sin 2x}{2}\right) + c_1$$
$$I = \frac{1}{4}\cos^3 x \sin x + \frac{3}{16}\sin 2x + \frac{3}{8}x + c.$$

__Properties__. (Indefinite integrals of even, odd functions)

Let $e_i(x)$, $\omega_i(x)$ be even, odd functions respectively. Then recalling the properties $D\,e_1(x) = \omega_1(x)$, $D\,\omega_2(x) = e_2(x)$ (§2. 1, Exercise 20) we may have the converse. Indeed the following properties hold:

1. $\int e_1(x)dx = \omega_1(x) + c$, 2. $\int \omega_2(x)dx = e_2(x) + c$

__Proof__.

1. Let $F(x) = \int e_1(x)dx$ without constant of integration. Then

$$F(-x) = \int e_1(-x)d(-x) = -\int e_1(-x)dx = -\int e_1(x)dx = -F(x),$$

Showing that $F(x)$ is an odd function, namely $\omega_i(x)$

2. Proved similarly. ■

Discuss periodicity of the integral of a periodic function.

EXERCISES (5. 1)

1. Simplify the following

a) $\int df(x)$ b) $d\int f(x)dx$ c) $\frac{d}{dx}\int \arccos x\, dx$

d) $\int \frac{d}{df} \arccos x\, dx$ e) $\int d(x^7+x+7)^7$ f) $\frac{d}{dx}\int \frac{d}{dx} \operatorname{arcsec} x$

2. If $F_1(x)$, $F_2(x)$ are two primitives of $f(x)$, then show that $c_1F_1(x) + c_2F_2(x)$ is a primitive of $f(x)$ when c_1, c_2

are arbitrary constants.

3. Solve for y:

a) $Dy = \sqrt{x}$, $y(0) = 1$,   b) $Dy = \frac{1}{(x-1)^2}$, $y(2) = 2$

c) $D^2y = \sqrt{x} + \sqrt[3]{x} + 2$, $y(0) = 1$, $y'(0) = 2$

d) $D^3y = 1/\sqrt[3]{x}$, $y(1) = 1$.

4. Evaluate the following by the use of definition:

a) $\int \tan^{21}x \sec^2x\, dx$ b) $\int \sin x.(1 + \cos x)dx$

c) $\int f'(u) f''(u)u'dx$ d) $\int \arctan x . \frac{dx}{1+x^2}$

5. Evaluate the following by the use of definition:

a) $\int \sin x\, dx$ b) $\int \sin^5x \cos x\, dx$

c) $\int (x^3+2x)^7(3x^2+2)dx$,   d) $\int f(u) f'(u) u'\, dx$

6. Find the function whose primitive is:

a) $x^3 + 5x^2 - 7x + 3$ b) $x^2 - \sec^2x + x^{1/2}$

c) $\frac{1 + x}{1 - x} + 2$ d) $\sin x \cos x + 1$

7. Find the primitive of the given function which satisfies the given condition:

a) $f(x) = \sin x \cos x$, $F(\pi/4)=1$ b) $f(x) = \frac{x}{4} + \sqrt[3]{x}$, $F(8) = 10$

c) $f(x)= \frac{\arctan x}{1 + x^2}$, $F(\sqrt{3})=0$ d) $f(x) = \sin^3x \cos x$, $F(\pi/2) = -1$.

8. Evaluate the following integrals:

a) $\int \sin x \cos 2x\, dx + \int \sin 2x \cos x\, dx$

b) $\int \frac{dx}{1+x} + \int \frac{x\, dx}{1 + x}$,   c) $\int \frac{dx}{1 - x} - \int \frac{x^2dx}{1 - x}$ d) $\int \cos^2xdx - \int dx$

9. Evaluate the following integrals

a) $\int \cos x\, (\sin x + 1)dx$ b) $\int (\cos^2x - \sin^2x)dx$

c) $\int (\tan x + x\sec^2x)dx$ d) $\int \sec x\, (\sec x \tan x + \cos x)dx$

10. Evaluate the following indefinite integrals:

a) $\int \frac{1}{x} \cdot (-\frac{1}{x^2})dx$ b) $\int \tan x \sec^2 x\, dx$

c) $\int \arcsin x \cdot (\frac{1}{\sqrt{1-x^2}})dx$ d) $\int \frac{1+x}{1-x} \cdot \frac{-2x}{(1-x)^2}\, dx$

11. Integrate by substitution:

a) $\int 6(x^2 + 3x)^3 (2x + 3)dx$ b) $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx$

c) $\int \frac{1}{x^2} \sin \frac{1}{x}\, dx$ d) $\int (3x + 1) \cos(3x^2 + 2x)dx$

12. Integrate by parts

a) $\int x \sin \frac{x}{2}\, dx$ b) $\int x \sin^2 x\, dx$

c) $\int x \cos^2 x\, dx$ d) $\int x^2 \cos x\, dx$

13. Verify

$$\int \sin x \cos x\, dx = \begin{cases} \frac{\sin^2 x}{2} + c. \\ -\frac{\cos 2x}{2} + c \end{cases}$$

without integrating, and explain distinct appearance of results.

14. Given $F(x) = \int G(x)dx$, $G(x) = \int H(x)dx$, and $H(x) = \int F(x)dx$ find a relation between one of these functions and its derivatives.

15. Find a condition between $f(x)$, $g(x)$, $f'(x)$, $g'(x)$, $f''(x)$, $g''(x)$ such that

$$\int f(x)\, g(x)dx = \int f(x)dx \cdot \int g(x)dx$$

ANSWERS TO EVEN NUMBERED EXERCISES

4. a) $(\tan^{22} x)/22 + c$,   b) $-\cos x + (\sin^2 x)/2 + c$

c) $\frac{1}{2}\, (f'(u))^2 + c$,   d) $\frac{1}{2} \arctan^2 x + c$

6. a) $3x^2 + 10x - 7$,   b) $2x - \tan x + 1/(2\sqrt{x})$ c) $2/(1-x)^2$

d) $\cos 2x$

8. a) $-\frac{1}{3}\cos 3x + c$, b) $x + c$, c) $\frac{1}{2}x^2 + x + c$, d) $\frac{1}{2}\sin 2x + c$

10. a) $1/(2x^2) + c$, b) $\frac{1}{2}\tan^2 x + c$

c) $\frac{1}{2}\arcsin x + c$, d) $(1 + x)/(1 - x)^2 + c$

12. a) $-2x \sin\frac{x}{2} + 4\sin\frac{x}{2} + c$ b) $\frac{x^2}{4} - \frac{x}{4}\sin 2x - \frac{\cos 2x}{8} + c$

c) $\frac{x^2}{4} + \frac{x^2}{4}\sin 2x + \frac{\cos 2x}{8} + c$, d) $x^2\sin x + 2x\cos x - 2\sin x + c$

14. $F(x) = -F'''(x)$.

5. 2. The Definite Integral

A. Definitions:

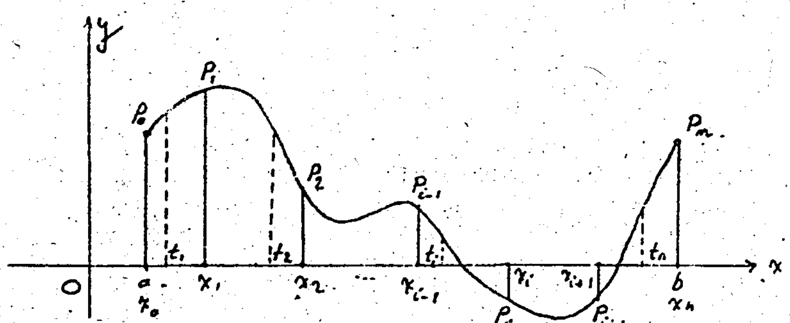
Historically the definite integral arose in an effort to formulate the area under a curve of a positive function over a closed interval.

The definite integral is defined by RIEMANN as follows:

Let $f(x) \in C[a, b]$ which may be positive, zero or negative on $[a, b]$, and let $[a, b]$ be partitioned (subdivided) into n subintervals by points of the set

$$P = \{x_0(= a), x_1, \ldots, x_{i-1}, x_i, \ldots, x_{n-1}, x_n(= b)\}$$

such that $a < x_1 < \ldots < x_{n-1} < b$ where P is called a *partition*[1]



(1) A partition is called *regular* if it partitions the given interval into subintervals in equal length. In case of a regular partition the equality (1) becomes

$$I_n = \Delta x_i \sum_{i=1}^{n} f(t_i)$$

Let

$$\Delta x_i = x_i - x_{i-1}, \quad t_i \varepsilon \left[x_{i-1}, x_i\right], \quad i = 1, \ldots, n$$

and consider the sum

$$I_n = \sum_{i=1}^{n} f(t_i)\Delta x_i \qquad (1)$$

called the <u>RIEMANN Sum</u> (<u>DARBOUX sum</u>)

Let $m_i = \min f(x)$, $M_i = \max f(x)$ on $\left[x_{i-1}, x_i\right]$ so that we have

$$m_i \leqslant f(x) \leqslant M_i \quad \text{for} \quad x\varepsilon\left[x_{i-1}, x_i\right]$$

and

$$\sum_{i=1}^{n} m_i\Delta x_i \leqslant I_n \leqslant \sum_{i=1}^{n} M_i\Delta x_i$$

where we call the left hand and right hand summuations the *lover sum* and *upper sum* respectively for the given partition P, that we denote by L_n and U_n:

$$L_n \leqslant I_n \leqslant U_n$$

If L_n and U_n have the same limit for all partitions as $n\to\infty$ and $\max \Delta x_i \to 0$, then I_n tends to this common limit, and this common limit is denoted by

$$\int_a^b f(x)dx$$ (Read:- integral from a to b of f(x)dx)

As to the existence of limit we have the following theorem whose proof is given in Advanced Calculus:

<u>Theorem</u>. $f(x)\varepsilon C(a, b) \Rightarrow \int_a^b f(x)dx$ exists.

This definite integral is the *RIEMANN integral* of f(x) over the closed interval (a, b), and f(x) is said to be *RIEMANN integrable* function, where a and b are the *lover limit* and *upper limits* of the integral, respectively.

<u>Example 1</u>. Evaluate $\int_a^b dx$

Solution. The integrand is $f(x) = 1$. For any partition, having

$$I_n = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} (x_i - x_{i-1})$$

$$= (x_1 - x_o) + (x_2 - x_1) + \ldots + (x_n - x_{n-1})$$

$$= x_n - x_o = b - a,$$

and

$$\int_a^b dx = \lim_{\substack{n \to \infty \\ \max\Delta x_i \to 0}} (b-a) = b-a \quad \text{(for any partition)}$$

which is the area of the rectangle with boundaries

$y = f(x) = 1$, $y = 0$, $x = a$, $x = b$, since $f(x) = 1 > 0$.

This and some other simple properties of definite integral are listed below whose proofs can be done by the use of the definition of definite integral and some properties of limits:

Properties. $f(x)$, $g(x) \varepsilon C[a, b] \Rightarrow$

1. $\int_a^b dx = b - a$ 2. $\int_a^a f(x)dx = 0$

3. $\int_a^b f(x)dx = \int_a^b f(t)dt$

4. $m(b-a) \leqslant \int_a^b f(x)dx \leqslant M(b-a)$, $\begin{cases} m = \min f(x) \\ M = \max f(x) \end{cases}$ in $[a, b]$

5. $\int_a^b \left[f(x) + g(x)\right]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

6. $\int_a^b \lambda f(x)dx = \lambda \int_a^b f(x)dx$ (λ is a constant)

7. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

8. $\int_\alpha^\beta f(x)dx + \int_\beta^\gamma f(x)dx = \int_\alpha^\gamma f(x)dx$

for any α, β, γ in $[a, b]$.

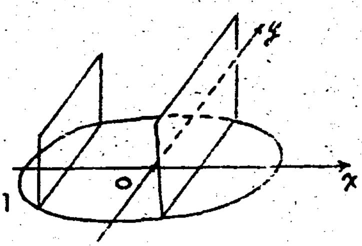
9. $\int_a^b f(x)dx \leqslant \left|\int_a^b f(x)dx\right| \leqslant \int_a^b |f(x)|dx$

10. $\int_a^b f(x)dx \leq \int_a^b g(x)dx$ if $f(x) \leq g(x)$

11. $\int_a^b f(x)dx$ is the area bounded by the curves of $y = f(x)$, $y = 0$ and $x = a$, $x = b$ if $f(x) > 0$ on (a,b)

Example 2. Find the volume of the solid with circular base of radius 5 m, and each cross section perpendicular to a definite diameter is a square.

Solution. Let us take the definite diameter as x-axis and the one perpendicular to it as y-axis. Then the equation of the circle is $x^2 + y^2 = 25$. From the symmetry of the solid with respect to the cross section through y-axis the volume V will be twice as that for $0 \leq x \leq 5$.



For a regular partition of $(0; 5)$ we have congruent subintervals of lengths 5/n. The area of the cross section through the point (x_k, y_k) being $(2y_k)^2$ the volume V_k of the slice with thickness $5/n$ is $4y_k^2\, 5/n$:

$$V_k = \frac{20}{n} y_k^2 = \frac{20}{n}(25 - x_k^2) = \frac{20}{n}(25 - (k\frac{5}{n})^2) = \frac{500}{n}\cdot\left(1 - \frac{k^2}{n^2}\right)$$

$$\sum_{k=1}^{n} V_k = \frac{500}{n}\sum_{k=1}^{n}(1 - \frac{k^2}{n^2}) = \frac{500}{n}(n - \frac{\Sigma k^2}{n^2})$$

$$= 500\left(1 - \frac{n(n+1)(2n+1)}{6n^3}\right)$$

$$V = 2\lim_{n\to\infty}\sum_{k=1}^{n} V_k = 1000\left(1 - \lim\frac{n(n+1)(2n+1)}{6n^3}\right)$$

$$= 1000\,(1 - \frac{1}{3}) = (2000/3)m^3.$$

B. The Fundamental Theorems

We state here two fundamental theorems (F.T.) the proofs of which are based on the following mean value theorem for integrals:

Theorem. (MVT for integral). If $f(x) \varepsilon C(a, b)$, then there exists an interior point $c \varepsilon (a, b)$ such that

$$\int_a^b f(x)dx = (b-a)\, f(c)$$

Proof. If the function is constant, say $f(x) = y_0$, then

$$\int_a^b f(x)dx = \int_a^b y_0\, dx = y_0 \int_a^b dx = (b-a)y_0 = (b-a)\, f(c)$$

for any $c \varepsilon (a, b)$.

Let then $f(x)$ be a non constant function. By its continuity it attains $m = \min f(x)$, $M = \max f(x)$ on $[a, b]$ so that

$$\int_a^b m\, dx \leqslant \int_a^b f(x)dx \leqslant \int_a^b M\, dx$$

$$m(b-a) \leqslant \int_a^b f(x)dx \leqslant M(b-a)$$

$$m \leqslant \frac{\int_a^b f(x)dx}{b-a} \leqslant M.$$

Again from continuity of $f(x)$ the intermediate value

$$\bar{y} = \frac{\int_a^b f(x)dx}{b-a}$$

is attained at a point c which is certainly between a and b, so that

$$\bar{y} = \frac{\int_a^b f(x)dx}{b-a} = f(c). \qquad (a)$$

The value $\bar{y}$ defined by (a) or by

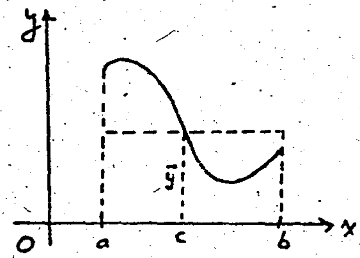
$$\bar{y} = \frac{\int_a^b f(x)dx}{\int_a^b dx}$$

is called the *average value* of $f(x)$ on $[a, b]$ with respect to x.

<u>Geometric Interpretation of the MVT for Integral:</u>

The definite integral

$$\int_a^b f(x)dx$$

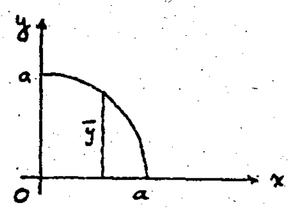


is equal to the length of interval times the average value of the function.

In particular when $f(x) > 0$, the area given by the integral is equal to the area of the rectangle with base $b-a$ and altitude $\bar{y} = f(c)$.

<u>Example</u>. Find the average value of $y = \sqrt{a^2 - x^2}$ in $[0, a]$ with respect to x.

<u>Solution</u>. $\int_0^b \sqrt{a^2 - x^2} = \frac{\pi}{4} a^2$,



since it is the area of the quarter of the circle with radius a.

The length of the interval being a, we have

$$\bar{y} = \frac{\pi}{4} a^2/a = \frac{\pi}{4} a \ (< a).$$

When $f(x)$ is interpreted as a physical quantity such as density, energy, force, etc, $\bar{y}$ will mean averages of these quantities.

<u>Theorem</u>. (F.T. of Calculus). If $f(x) \in C[a, b]$ then $\int_a^x f(t)dt$ is differentiable function, and

$$\frac{d}{dx} \int_a^x f(t)dt = f(x)$$

Proof. Let $F(x) = \int_a^x f(t)dt$. Then

$$F(x+h) - F(x) = \int_a^{x+h} f(t)dt - \int_a^x f(t)dt$$

$$= \int_a^{x+h} f(t)dt + \int_x^a f(t)dt$$

$$= \int_x^{x+h} f(t)dt$$

$$= \left[(x+h) - x\right] f(c), \quad x < c < x+h$$

by the MVT for integral. Hence

$$\frac{F(x+h) - F(x)}{h} = f(c) = f(x + \Theta h), \quad 0 < \Theta < 1,$$

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} f(x + \Theta h) = f(x). \blacksquare$$

Theorem. (F.T. of integral calculus). If $f(x) \varepsilon C[a, b]$ and $F(x)$ is a primitive of $f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Proof. Since $D \int_a^x f(t)dt = f(x)$ by previous theorem, and $DF(x) = f(x)$, then $\int_a^x f(t)dt$ differs from $F(x)$ by a constant:

$$\int_a^x f(t)dt = F(x) + c$$

Now

$$x = a \Rightarrow 0 = F(a) + c \Rightarrow c = -F(a),$$

$$x = b \Rightarrow \int_a^b f(t)dt = F(b) + c = F(b) - F(a).$$

Notations. $\int_a^b f(t)dt = F(b) - F(a) = F(x)\Big|_{x=a}^{x=b} = F(x)\Big|_a^b$

In view of this theorem, evaluation of a definite integral reduces to that of an indefinite integral. It is to be noted that if the evaluation is done by substitution, the new limits of integration are to be written in the integrals in which the new variable is used or indefinite integral is computed in original variable, and then F(b) - F(a) is computed.

If the evaluation is done by parts, one should note

$$\int_a^b u(x)dv(x) = \int_a^b d\, u(x)v(x) - \int_a^b v(x)du(x)$$

and hence use the equality

$$\int_a^b udv = uv\Big|_a^b - \int_a^b vdu$$

<u>Example 1</u>. Evaluate $A = \int_0^a \sqrt{a^2 - x^2}\, dx \quad (a > 0)$

<u>Solution</u>. Substituting $x = a \sin\theta$ with $x = 0 \Rightarrow \theta = 0$, and $x = a \Rightarrow \theta = \pi/2$, we have

$$A = \int_0^{\pi/2} \sqrt{a^2 - a^2 . \sin^2\theta} \,.\, a \cos\theta\, d\theta$$

$$= a^2 \int_0^{\pi/2} |\cos\theta| \cos\theta\, d\theta$$

$$= a^2 \int_0^{\pi/2} \cos^2\theta\, d\theta, \ (|\cos\theta| = \cos\theta \text{ in } (0, \pi/2))$$

$$= \frac{1}{2} a^2 \int_0^{\pi/2} (1 + \cos 2\theta) d\theta$$

$$= \frac{1}{2} a^2 \left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_0^{\pi/2} = \frac{1}{4}\pi a^2.$$

<u>Example 2</u>. Evaluate $B = \int_{\pi/4}^{\pi/2} x \sin x\, dx$

<u>Solution</u>. $u = x, \quad dv = \sin x\, dx$

$du = dx, \quad v = -\cos x$

$$\Rightarrow \quad B = -\left(x \cos x\right)\Big|_{\pi/4}^{\pi/2} + \int_{\pi/4}^{\pi/2} \cos x\, dx$$

$$= -\left(0 - \frac{\pi}{4} \cdot \frac{1}{\sqrt{2}}\right) + \left(\sin x\right)\Big|_{\pi/4}^{\pi/2} = \frac{\pi}{4\sqrt{2}} + 1 - \frac{1}{2}.$$

An extension of the F.T. of calculus is the following where limits of integrations are differentiable functions.

Corollary. If $f(x) \in C(a, b)$, and $a(x)$, $b(x)$ are differentiable functions, then

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(t)dt = f(b(x))\, b'(x) - f(a(x))\, a'(x).$$

Proof. Let $F(t) = \int f(t)dt$. Then

$$\int_{a(x)}^{b(x)} f(t)dt = F(b(x)) - F(a(x)),$$

and by chain rule

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(t)dt = F'(b(x)).b'(x) - F'(a(x)).a'(x)$$
$$= f(b(x))\, b'(x) - f(a(x))\, a'(x).$$

When a function $\phi(x)$ is defined by an integral as

$$\phi(x) = \int_{a(x)}^{b(x)} f(t)dt.$$

one can obtain all properties related to derivative of any order without evaluting the integral.

Example. Find and identify the critical points, if any, of

$$u(x) = \int_{\arcsin x}^{\arctan x} \tan t\, dt.$$

Solution. One does not need to integrate to obtain $u'(x)$ Applying the corollary, we have

$$u'(x) = \tan(\arctan x)\ D \arctan x - \tan(\arcsin x)\ D \arcsin x$$

$$= x \cdot \frac{1}{1+x^2} - \frac{x}{\sqrt{1-x^2}} \cdot \frac{1}{\sqrt{1-x^2}} = \frac{-2x^3}{1-x^4}$$

$u'(x) = 0$ when $x = 0$, $u'(x) > 0$ when $x < 0$ and $u'(x) < 0$ when $x > 0$ if x is near 0. $u(0)$ is a maximum.

In evaluating a definite integral examination of the party of the integrand may reduce the work a great deal as a result of the properties stated in the following corollary:

Corollary. If $e(x)$ and $\omega(x)$ are even and odd functions respectively, then

$$1.\ \int_{-a}^{a} e(x)dx = 2\int_{0}^{a} e(x)dx, \qquad 2.\ \int_{-a}^{a} \omega(x)dx = 0$$

Proof. Since a primitive without arbitrary constant of an even (odd) function is an odd (even) function, setting

$$e(x)dx = \omega_1(x) + c, \quad \omega(x)dx = e_1(x) + c$$

we have

$$1.\ \int_{-a}^{a} e(x)dx = \omega_1(x)\Big|_{-a}^{a} = \omega_1(a) - \omega_1(-a) = 2\omega_1(a)$$
$$= 2\Big[\omega_1(a) - \omega_1(0)\Big] = 2\int_{0}^{a} e(x)dx,$$

$$2.\ \int_{-a}^{a} \omega(x)dx = e_1(x)\Big|_{-a}^{a} = e_1(a) - e_1(-a) = 0.$$

Example. Evaluate the following

a) $A = \int_{-\pi}^{\pi} x \sin x\, dx$, b) $B = \int_{-\pi}^{\pi} \arcsin^3 x\, dx$

Solution.

a) The integrand $x \sin x$ is an even function and

$$\int_{-\pi}^{\pi} x \sin x\, dx = 2\int_{0}^{\pi} x \sin x\, dx$$

$$u = x, \quad dv = \sin x\, dx$$
$$du = dx, \quad v = -\cos x$$

$$A = 2\Big[-x\cos x\Big]_0^{\pi} + 2\int_0^{\pi} \cos x\, dx$$
$$= 2\Big[-(-\pi - 0)\Big] + 2\int_0^{\pi} \cos x\, dx = 2\pi + 2\sin x\Big|_0^{\pi} = 2\pi.$$

b) Since the integrand is an odd function, than $B = 0$.

C. EVALUATION OF SOME LIMITS BY DEFINITE INTEGRALS:

Evaluation of the limit of a sum

$$a_1(n) + \ldots + a_n(n)$$

which is in the form or which can be put in the form of a RIEMANN sum of a function in an interval can be done by the use of a definite integral applying the following corollary:

<u>Corollary</u>. If $f(x) \varepsilon C(a, b)$, and $(b-a)/n = h$, then

$$\lim_{n \to \infty} \frac{b-a}{n} \left[f(a+h) + f(a+2h) + \ldots + f(a+nh)\right] = \int_a^b f(x)dx.$$

<u>Example 1</u>. Evaluate

$$S = \lim_{n \to \infty} \frac{\operatorname{Sin}\frac{\pi}{n} + \operatorname{Sin}\frac{2\pi}{n} + \ldots + \operatorname{Sin}\frac{n\pi}{n}}{n}$$

<u>Solution</u>. The given expression is the RIEMANN sum for sinx in $(0, \pi)$ Then

$$S = \frac{1}{\pi} \lim_{n \to \infty} \frac{\pi}{n}\left[\sin\frac{\pi}{n} + \ldots + \sin\frac{n\pi}{n}\right] = (\text{where } \frac{\pi}{n} \doteq \frac{b-a}{n})$$

$$= \frac{1}{\pi} \int_0^{\pi} \sin x \, dx = \frac{1}{\pi} \left[-\cos x\right]_0^{\pi} = 2/\pi.$$

Solution can also be obtained by taking $f(x) = \sin \pi x$ instead of sinx, and the interval $(0, 1)$:

$$S = \lim_{n \to \infty} \frac{\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \ldots + \sin\frac{n\pi}{n}}{n} = \int_0^1 \sin \pi x \, dx = 2/\pi.$$

<u>Example 2</u>. Evaluate

$$S = \lim_{n \to \infty} \frac{\sqrt{1} + \sqrt{2} + \ldots + \sqrt{n}}{n^{3/2}}$$

<u>Solution</u>. Writing the denominator as $n\sqrt{n}$, and combining $\sqrt{n}$ with numerator, we have

$$S = \lim_{n \to \infty} \frac{1}{n}\left[\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \ldots + \sqrt{\frac{n}{n}}\right]$$

so that $f(x) = x$ and the interval is $(0, 1)$:

$$S = \int_0^1 \sqrt{x}\, dx = \frac{x^{3/2}}{3/2}\Big|_0^1 = 2/3.$$

EXERCISES (5. 2)

16. Find the smallest interval in which $\int_a^b f(x)dx$ lies for the following cases:

a) In $[-1, 6]$ $m = -2$, $M=5$, b) In $[-2/3, 7/2]$ $m = 1/2$, $M=4/3$

where $m = \min f(x)$; $M = \max f(x)$ in the given closed interval.

17. Find the area under the parabola $y = x^2$ above x-axis in $[1, 2]$ by using RIEMANN sum for a regular partition and finding the limit.

18. Evaluate the following integral by use of geometric interpretation of the definite integral.

a) $\int_0^3 (2x + 1)dx$ b) $\int_0^2 x\, dx + \int_2^{2\sqrt{2}} \sqrt{8-x^2}\, dx$

c) $\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx$ d) $\frac{b}{a} \int_0^a \sqrt{a^2 - x^2}\, dx$

19. Show the following inequality without integration:

a) $\int_0^{\pi/4} \cos x\, dx \geq \int_0^{\pi/4} \sin x\, dx$, b) $\int_0^{\pi/3} \tan^2 x\, dx \leq \int_0^{\pi/3} \sec^2 x\, dx$

c) $\int_2^5 \frac{x^2+1}{x}\, dx > \int_2^5 \frac{x+1}{x}\, dx$ d) $\int_{-2/3}^{-1/3} (x^3+2x)dx > \int_{-2/3}^{-1/3} (x^2+4x)dx$

20. Same question for

a) $0 < \int_0^2 \frac{x^2}{x+2}\, dx < 2$ b) $0 < \int_0^{\pi/3} \tan x\, dx < \frac{\pi}{3}\sqrt{3}$

c) $\frac{\pi}{2} < \int_0^{\pi/2} (\sin x + \cos x)dx < \frac{\pi}{2}\sqrt{2}$, d) $\frac{9}{20} < \int_{3/2}^{9/5} |x| [\![x]\!]\, dx < \frac{37}{50}$

21. Let $H(x) = \int_x^b f(t)dt$, $a \leq x \leq b$ with $f(x) \in C[a, b]$
Show that $H(x) \in C[a, b]$.

22. Find c or the equation satisfied by c as required by the MVT for the following integrals:

a) $\int_{-1}^{3} (x^3 - 2x)dx$ b) $\int_0^9 (x^{3/2} + 3x^{1/2})dx$

c) $\int_0^3 \sqrt{1 + x}\, dx$ d) $\int_0^{\sqrt{8}} x\sqrt{1 + x^2}\, dx$

23. Find the average value of the given function on the given interval:

a) $y = \sin x$, $[0, \pi]$ b) $y = \sin x$, $[0, \pi/2]$

c) $y = \cos^2 x$, $[0, \pi]$ d) $y = \cos^2 x$, $[0, \pi/2]$

24. For the following function given by integrals, find and identify the critical points:

a) $\phi(x) = \int_5^{x^2+x} \frac{t}{t+1}\, dt$ b) $\phi(x) = \int_x^{x^2} \frac{t+1}{t^2}\, dt$

25. Evaluate (without integration)

a) $\frac{d}{dx} \int_{\sin x}^{\cos x} \sqrt{1-t^2}\, dt$ b) $\frac{d}{dx} \int_{1/2}^{\cos x} \sec(\arcsin t)dt$

26. Evaluate the given integral by the given substitution:

a) $\int_0^1 \sqrt{1 + x^2}\, x\, dx$, $1+x^2=u^2$ b) $\int_{-1}^{1} \frac{dx}{x\sqrt{x^2 - 1}}$, $x = \frac{1}{t}$

27. Evaluate by parts:

a) $\int_0^{\pi/2} x \sin x\, dx$ b) $\int_0^{\pi/2} x^2 \cos x\, dx$

28. Check the integrands for evennes, oddness, and then evaluate the integrals:

a) $\int_{-3}^{3} \sin x\, dx$ b) $\int_{-2}^{2} (3x^2 - 5)dx$

29. Same question for:

a) $\int_{-\pi}^{\pi} x^3 \cos 2x\, dx$ b) $\int_{-\pi/2}^{\pi/2} x^2 \sin^3 x\, dx$

c) $\int_0^2 \cos^2 t \sin^5 t \, dt$ d) $\int_0^{2\pi} \tan\theta \sec^2\theta \, d\theta$

30. Evaluate by a definite integral

a) $\lim_{n \to \infty} \left[\frac{1}{2n+1} + \frac{1}{2n+3} + \ldots + \frac{1}{4n+1} \right]$

b) $\lim_{n \to \infty} \frac{1^7 + 2^7 + \ldots + n^7}{n^8}$

c) $\lim_{n \to \infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{p+n} \right]$, $p \in N^+$

d) $\lim_{n \to \infty} \left(\frac{1^2}{2^3 + n^3} + \frac{2^2}{4^3 + n^3} + \ldots + \frac{n^2}{(2n)^3 + n^3} \right)$

ANSWERS TO EVEN NUMBERED EXERCISES

16. a) $(-14, 35)$, b) $(25/12, 50/9)$

18. a) 12, b) π, c) $\frac{1}{2} \pi a^2$, d) $\frac{1}{4} \pi ab$

22. a) $c^3 - 2c - 3 = 0$, b) $c^3 + 6c^2 + 9c - 84^2/25 = 0$

c) $115/81$, d) $\sqrt{(-3 + 347)/6}$

24. a) At $x = 0$ and $x = -1$ min; at $x = -1/2$ max,

b) At $x = 1$ max.

26. a) $(2\sqrt{2} - 1)/3$, b) $-\pi$.

28. a) 0, b) -4

30. a) 1/8, b) $(\ln 3)/12$, c) $(\ln 2)/2$, d) $\ln p$.

5. 3. Area and Numerical Evaluation of Definite Integrals

A. AREA OF A PLANE REGION

As a first application of definite integral we formulate the area of a plane region bounded by certain number of curves.

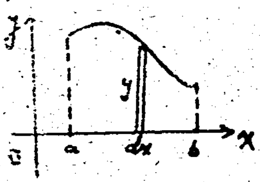
<u>Theorem</u>. The area $|R|$ of the plane region R bounded by the curve $y = f(x)$, x-axis and the vertical lines $x = a$, $x = b$, is given by

$$|R| = \int_a^b |f(x)|dx \quad (f(x) \in C(a, b))$$

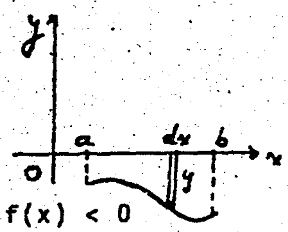
<u>Proof</u>. The statement is trivially true if $f(x) > 0$ on (a, b), since the RIEMANN sum

$$\sum_{i=1}^{n} f(t_i)\Delta x_i$$

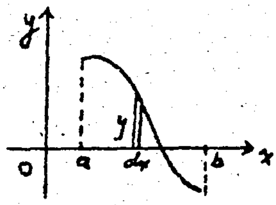
is an approximation of the area under the curve and the limit is the area $|R|$. (See left Fig.)



$f(x) > 0$



$f(x) < 0$



$f(x) >< 0$

If $f(x) < 0$ on (a, b), we have

$$|R| = \int_a^b (-f(x))dx = \int_a^b |f(x)|dx$$

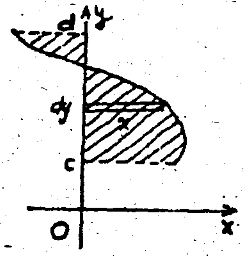
If $f(x)$ is positive and negative on (a, b), say positive on (a, x_0) and negative on (x_0, b), then one gets

$$|R| = \int_a^{x_0} f(x)dx + \int_{x_0}^b (-f(x))dx$$

$$= \int_a^{x_0} |f(x)|dx + \int_{x_0}^b |f(x)|dx = \int_a^b |f(x)|dx. \blacksquare$$

<u>Corollary</u>. The area of a plane region bounded by the curve $x = f(y)$, the y-axis and the horizontal lines $y = c$, $y = d$ is given by

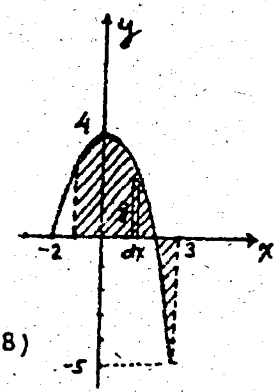
$$|R| = \int_c^d |f(y)|\,dy.$$



In evaluating an area it will be useful to sketch the region in the first step and also draw an elementary area as a horizontal or vertical strip of width dy or dx respectively.

Example 1. Find the area of the plane region bounded by the parabola $y = 4 - x^2$, y-axis and vertical lines $x = -1$, $x = 3$.

Solution. The parabola intersects x-axis at $x = 2$ and $x = -2$, and y-axis at $y = 4$. The region is then the shaded one. Hence



$$|R| = \int_{-1}^{3} |4 - x^2|\,dx$$

$$= \int_{-1}^{2} (4 - x^2)\,dx + \int_{2}^{3} (x^2 - 4)\,dx$$

$$= \left[4x - \frac{x^3}{3}\right]_{-1}^{2} + \left[\frac{x^3}{3} - 4x\right]_{2}^{3}$$

$$= (8 - \tfrac{8}{3}) - (-4 \quad \tfrac{1}{3}) + (9 - 12) - (\tfrac{8}{3} - 8)$$

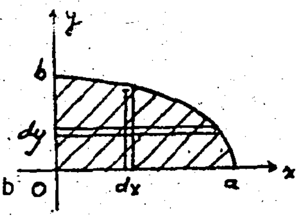
$$= 17 - \frac{16}{3} - \frac{1}{3} = 34/3.$$

Example 2. Find the area of a quarter an ellipse with semi major axis a and semi major axis b.

Solution 1. The standard equation of the ellipse (center at the origin) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Taking a vertical strip as elementary area (or differential of the area)



$$dA = y\,dx = b\sqrt{1 - \frac{x^2}{a^2}}\,dx = \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

and integrating from 0 to a we get

$$A = \frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx = \frac{b}{a}\cdot\frac{\pi}{4}a^2 = \frac{\pi}{4}ab$$

If instead of a vertical strip, a horizontal one were taken, we have similarly

$$A = \frac{a}{b}\int_0^b \sqrt{b^2 - y^2}\,dy = \frac{a}{b}\cdot\frac{\pi}{4}b^2 = \frac{\pi}{4}ab.$$

Note: The above integrals are evaluated by the use geometric interpretation as area of a quarter of circle.

Solution 2. If parametric equations

$$x = a\cos\theta, \quad y = b\sin\theta$$

of the ellipse are used, we have

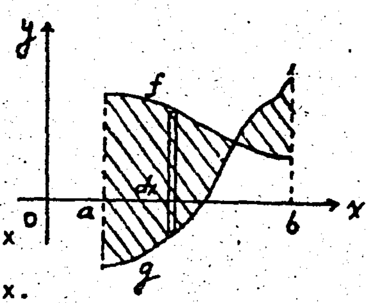
$$A = \int_{\pi/2}^{0} b\sin\theta\, d(a\cos\theta) = -ab\int_{\pi/2}^{0}\sin^2\theta\, d\theta$$

$$= -ab\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_{\pi/2}^{0} = \frac{\pi}{4}ab$$

Theorem. The area R of a plane region R bounded by the curves of $y = f(x)$, $y = g(x)$ and the vertical line $x = a$, $x = b$ is given by

$$R = \int_a^b |f(x) - g(x)|\,dx \qquad (a < b)$$

Proof. Consider a vertical strip as elementary area (or differential of area) of width dx and ending on the given curves. Then



$$dA = |f(x) - g(x)|\,dx$$

and

$$A = |R| = \int_a^b |f(x) - g(x)|\,dx.$$

Corollary. The area of a plane region bounded by the curves of $x = f(y)$, $x = g(y)$, and the horizontal line $y = c$, $y = d$ is given by

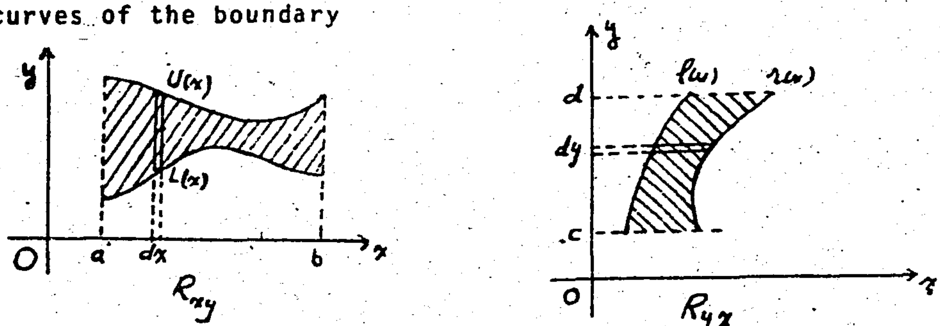
$$|R| = \int_c^d |f(y) - g(y)|dy \qquad (c < d).$$

The regions defined by

$$R_{xy} = \{(x, y): a \leqslant x \leqslant b,\ L(x) \leqslant y \leqslant U(x)\} = [a,\ b;\ L(x), U(x)]$$

$$R_{yx} = \{(x, y): c \leqslant y \leqslant d,\ \ell(y) \leqslant x \leqslant r(y) = [c,\ d;\ \ell(y),\ r(y)]$$

are called *normal regions* in xy-plane where $L(x)$, $U(x)$ are the lower and upper curves, and $\ell(y)$, $r(y)$ left and right curves of the boundary



Normal regions in xy-plane

For the areas we have

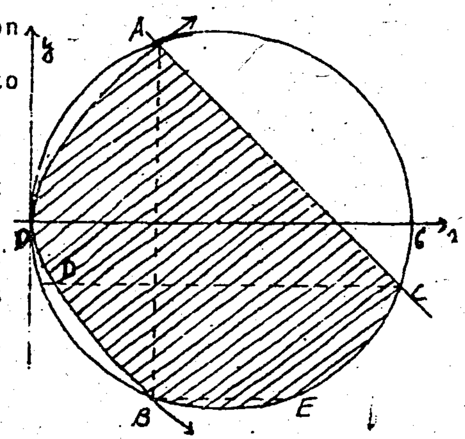
$$|R_{xy}| = \int_a^b (U(x) - L(x))dx,$$

$$|R_{yx}| = \int_c^d (r(y) - \ell(y))dy.$$

Any plane region can be split up into some number of normal regions any two of which have no common area.

<u>Example</u>. Find the area of the region inside the parabola $y^2 = 4x$ and circle $x^2 + y^2 - 6x = 0$ and under the line $y = -x \quad 2(1 + \sqrt{2})$.

<u>Solution</u>. The figure shows the sketching of the region. From the simultaneous solutions we have $A(2,\ 2\sqrt{2})$, $B(2,\ -2\sqrt{2})$, $C(3 + 2\sqrt{2},\ -1)$.



Observe that the shaded region is not normal. If it is split up into R_{xy} normal regions we get (at least) two such regions (AODB, ABEC). If it is split up into R_{yx} normal regions we get (at least) three such regions (AODC, DBEC, BFE).

It is reasonable to use the first splitting for this problem since the number of subregions is less than that of the other case. But in some problems such a selection may arise difficulty in integration.

Then our regions AODB, and ABEC are respectively:

$$R^1_{xy} = \{(x,\ y):\ 0 \le x \le 2,\ -2\sqrt{x} < y < 2\sqrt{x}\} = [0,\ 2;\ -2\sqrt{x},\ 2\sqrt{x})$$

$$R^2_{xy} = \{(x,\ y):\ 2 \le x \le 3 + 2\sqrt{2},\ -\sqrt{6x - x^2} \le y \le -x + 2(1 + \sqrt{2})$$

$$= [2,\ 3 + 2\sqrt{2};\ \ -\sqrt{6x - x^2},\ -x + 2(1 + \sqrt{2}))$$

$$|A| = |R^1_{xy}| + |R^2_{xy}|$$

$$= \int_0^2 (2\sqrt{x} - (-2\sqrt{x}))dx + \int_2^{3+2\sqrt{2}} -x + 2(1 + \sqrt{2}) - (-\sqrt{6x-x^2}))dx$$

$$= \frac{16}{3}\sqrt{2} + \frac{7}{2} + \int_2^{3+2\sqrt{2}} \sqrt{6x - x^2}\ dx$$

Writing $6x - x^2 = 9 - (x - 3)^2$ and setting $x - 3 = 3\sin t$ we have

$$\int \sqrt{6x-x^2}dx = \int \sqrt{9 - 9\sin^2 t}\ \ 3\cos t\ dt$$

$$= 9\int \cos^2 t\ dt = \frac{9}{2}\left(t + \frac{\sin 2t}{2}\right) + c$$

$$\alpha = \int_2^{3+2\sqrt{2}} \sqrt{6x-x^2}dx = \frac{9}{2}\arcsin\frac{2\sqrt{2}}{3} + \arcsin\frac{1}{3} + \frac{4\sqrt{2}}{9}$$

$$A = \frac{16}{3}\sqrt{2} + \frac{7}{2} + \alpha.$$

B. APPROXIMATION OF DEFINITE INTEGRALS BY NUMERICAL EVALUATION

In one of the following cases numerical evaluation of a definite integral may be needed:

1. One may not be able to evaluate by any method,
2. Evaluation by the method is possible, but numerical value is desired,
3. The integrand is given emprically in a tabular form.

We give three methods for evaluating numerically a given definite integral $\int_a^b f(x)dx$ for $f(x) > 0$.

<u>Rectangular rule</u>:

When we use a regular partion $x_0\ (= a)$, $x_1, \dots, x_{n-1}$, $x_n (= b)$ with $h = (b-a)/n$ of $[a, b]$ in a RIEMANN sum we have

$$\int_a^b f(x)dx = h[f(x_0)\ \ f(x_1)\ \ \dots\ \ f(x_{n-1})],$$

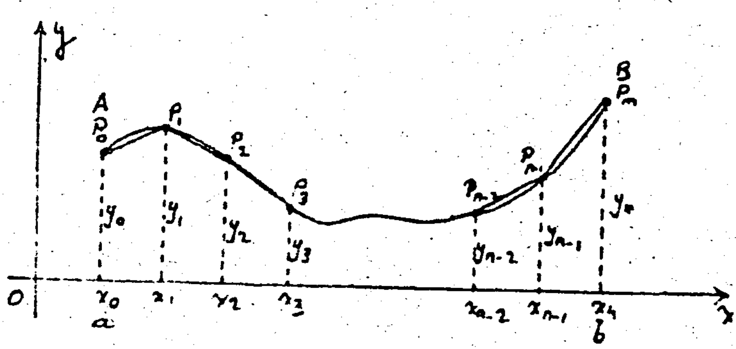
or

$$\int_a^b f(x)dx = h[f(x_1)\ \ f(x_2)\ \ \dots\ \ f(x_n)]$$

where $h = (b-a)/n$.

<u>Trapezoidal rule</u>:

For the same regular partion, errect perpendiculars to x-axis at x_i to meet the curve at $P_0, P_1, \dots, P_{n-1}, P_n$.



Then one replaces the arcs of the curves corresponding to sub-intervals by their respective chords $(P_0 P_1), \ldots, (P_{n-1} P_n)$ so that the area under the curve which is integral $\int_a^b f(x)dx$ will be approximated by the sum of the areas of the trapezoids thus formed.

Setting $y_i = f(x_i)$ we have

$$A_i = \frac{h}{2} (y_{i-1} + y_i),$$

as the areas of the trapezoids whose sum from 1 to n gives the required approximation for the integral:

$$\int_a^b f(x)dx \simeq \frac{h}{2} \left[y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n\right]$$

where $h = (b-a)/n$. It is the arithmetic mean of the two approximations given in Rectangular rule.

<u>SIMPSIN's rule</u>:

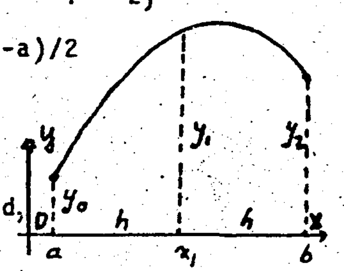
This approximation differs from the trapezoid rule in that the arcs are replaced by parabolic arcs instead of chords and regular partition is done for an *even number* n.

The rule is obtained by the use of the following lemma:

<u>Lemma</u> (SARRUS) Let $y = f(x) = \alpha x^2 = \beta x + \gamma$ be a parabola defined on (a, b), and let (a, b) be subdivided regularly into two subintervals by $x_0 (= a)$, x_1, $x_2 (= b)$. Then

$$A = \int_a^b (\alpha x^2 + \beta x + \gamma)dx = \frac{h}{3} \left[y_0 + 4y_1 + y_2\right]$$

where $y_i = f(x_i)$, $i = 0, 1, 2$, and $h = (b-a)/2$



<u>Proof</u>. The integral does not change if the parabola is translated parallel to the x-axis. If under translation x_1 is moved to the origin the interval of integration

will be $-h, h$ so that

$$A = \int_{-h}^{h} \left[\alpha x^2 + \beta x + \gamma\right] dx = \left(\frac{\alpha}{3} x^3 + \frac{\beta}{2} x^2 + \gamma x\right]_{-h}^{h}$$

$$= \frac{2}{3} \alpha h^3 + 2\gamma h = \frac{h}{3} (2\alpha h^2 + 6\gamma).$$

Since,

$$\begin{aligned} y_0 &= \alpha h^2 - \beta h + \gamma \\ 4y_1 &= 4\gamma \\ y_2 &= \alpha h^2 + \beta h + \gamma \\ \hline y_0 + 4y_1 + y_2 &= 2\alpha h^2 + 6\gamma \end{aligned}$$

we have our result.

Now partitioning $[a, b]$ regularly for an even number n and applying the above lemma for consective pairs of strips and adding the results of each pair, we have

$$\frac{h}{3} \left[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \ldots + (y_{n-2} + 4y_{n-1} + y_n)\right],$$

and

$$\int_a^b f(x)dx \cong \frac{h}{3} \left[y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n\right],$$

where $h = (b-a)/n$ and n *is an even* number.

Observe that the coefficients of y_i are 1 for $i = 0$ and $i = n$; for others, 4 for odd i and 2 for even i.

Example 1. Evaluate the definite integral

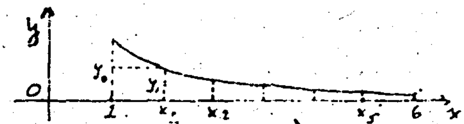
$$A = \int_1^3 \frac{dx}{x} = (\ln x)_1^3 = \ln 3$$

approximately (numerically) using the three rules, taking $n = 6$.

Solution. We have $h = \frac{3-1}{6} = \frac{1}{3}$ and

x_i	1	$\frac{4}{3}$	$\frac{5}{3}$	2	$\frac{7}{3}$	$\frac{8}{3}$	3
y_i	1	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{1}{2}$	$\frac{3}{7}$	$\frac{3}{8}$	$\frac{1}{3}$

1. By rectangular rule:



$$A \cong h\left(\frac{3}{4} + \frac{3}{5} + \frac{1}{2} + \frac{3}{7} + \frac{3}{8} + \frac{1}{3}\right)$$

$$= \frac{1}{3}\left(0{,}750 + 0{,}600 + 0{,}500 + 0{,}429 + 0{,}375 + 0{,}333\right)$$

$$= \frac{1}{3} \times 2{,}987 = \underline{0{,}996}. \quad \text{(lower sum)}$$

$$A = h\left[1{,}000 + 0{,}750 + 0{,}600 + 0{,}500 + 0{,}429 + 0{,}375\right]$$

$$= \frac{1}{3} \times 3{,}654 = \underline{1{,}218} \quad \text{(upper sum)}$$

The average of these two results is $\underline{1{,}107}$

2. $$A = \frac{h}{2}\left[1 + 2\text{x}0{,}750 + 2\text{x}0{,}600 + 2\text{x}0{,}500 + 2\text{x}0{,}429 + 2\text{x}0{,}375 + 0{,}333\right]$$

$$= \frac{1}{6} \times 6{,}641 = \underline{1{,}107}$$

3. By Simpson's rule: It is applicable since n is even.

$$A = \frac{h}{3}\left[1 + 4\text{x}0{,}750 + 2\text{x}0{,}600 + 4\text{x}0{,}500 + 2\text{x}0{,}429 + 4\text{x}0.375 + 0.333\right]$$

$$= \frac{1}{9} \times 9{,}791 = \underline{1{,}088}. \quad \text{Then } \ln 3 \cong 1{,}088.$$

In the same way $\ln 2$ can be computed and one gets

$$\ln 2 = \int_1^2 \frac{dx}{x} \cong \underline{0{,}69}$$

<u>Example 2</u>. For the function given in tabular form

x_i	0	1/4	1/2	3/4	1
$f(x_i)$	1	17/16	5/4	25/16	2

evaluate the definite integral

$$B = \int_0^1 f(x)dx$$

approximately by SIMPSON's rule with n necessarily equal to 2 or 4.

<u>Solution</u>. Taking $n = 4$, we have $h = 1/4$, and

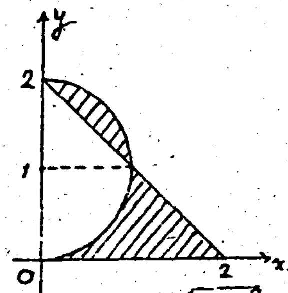
$$B = \left(\frac{1}{12} \quad 1{,}000 + 4 \times \frac{17}{16} + 2 \times \frac{5}{4} + 4 \times \frac{25}{16} + 2\right)$$

$$= \frac{1}{12} \times 16{,}000 = \underline{1{,}333}$$

<u>Note</u>: When a function is given in tabular form and $x_i - x_{i-1}$

is not constant for all i, by interpolation or extrapolation we can obtain a regular partition.

EXERCISES (5. 3)



31. Write the shaded region as union of R_{xy} normal regions, and also as union of R_{yx} regions.

32. Sketch the normal region R_{xy} and write it as R_{yx} region:

    a) $(1,\ 4;\ -1,\ 3-x)$, b) $(-1,\ 0;\ x+1,\ \sqrt{1-x^2})$

33. Compute the area of the region enclosed by the given curves:

    a) $y = x^{1/3}$, $y = x^{1/5}$ b) $y = -x^2$, $y = x^4 - 20$

    c) $y = \sqrt{x}$, $y = \frac{1}{4x}$, and normal to $y = \sqrt{x}$ at $(1,\ 1)$.

    d) $y = x^3 - x$ and the tangent to it at $x = -1$.

34. Same question for:

    a) $y^2 = 4x$, $y^2 = 12-8x$ b) $y = x^3$, $y^2 = 8x$

    c) $y = x^2$, $y = 4-2x-x^2$ d) $y = x^2$, $x - y + 6 = 0$, $x = 2$

35. Evaluate the areas of the regions bounded by:

    a) $y = 3x^2-2x$, $x=0$, $x=3$ b) $y = 4x$, $y^2 = 8x - 4$

    c) $y = x - 2$, $x = y^2$ d) $y = \sin x$, $y = \cos x$ over $(0;\pi)$

    e) $\sqrt{x} + \sqrt{y}\ \sqrt{a}$ $(a > 0)$, $x = 0$, $y = 0$

36. Compute the area of the regions enclosed by the parametric curves:

$$\left.\begin{array}{l} x = t + \frac{1}{t} \\ y = t - \frac{1}{t} \end{array}\right| \quad \text{and} \quad x = 4$$

37. Same question for:

$$x = 13 \cos t - 2, \; y = 5 \sin t + 3$$

38. Compute the area enclosed by the curve $y = x^3$, its tangent at $x = 1$ and its normal at $x = -1$.

39. Compute the area of the region bounded by the curves $y = x^2$, $y^2 = 20 - 2x$ and x-axis..

40. Compute the given integrals approximately:

a) $\int_{0,1}^{0,2} x^2 dx$ b) $\int_{0,2}^{0,3} (1-x^2) dx$ c) $\int_{1/2}^{3/2} \frac{2x}{2x+1} dx$

41. Evaluate $\int_{-7}^{7} (2x^3 + x^2 + 3x + 2) dx$ by the use of SARRUS's rule.

42. Evaluate $\int_{1}^{5} \frac{dx}{x}$ by SIMPSON's rule by taking

a) $n = 4$, b) $n = 8$

43. Evaluate $\int_{1}^{7} f(x) dx$ if $f(x)$ is given emprically as

x	1	2	3	4	5	6	7
f(x)	7	5	2	3	3	4	5

a) by the trapezoidal rule $(n = 6)$

b) by the SIMPSON's rule $(n = 6)$

44. Compute the following integral using the trapezoidal rule for $n = 4$.

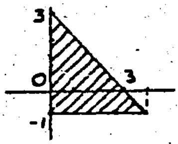
a) $\int_{0}^{1} \frac{dx}{1+x^2}$, b) $\int_{0}^{\pi} \sin x \, dx$ c) $\int_{0}^{2} \frac{x}{x^2 + 2} dx$ d) $\int_{1}^{2} \sqrt{1+x^2} \, dx$

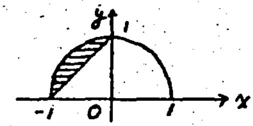
45. Same question as in Exercise 43 for

x	0	2	3	5	6
f(x)	1	5	7	8	9

(<u>Hint</u>: Obtain a regular partition introducing the point $x = 1$ and $x = 4$ and finding corresponding values by interpolation).

ANSWERS TO EVEN NUMBERED EXERCISES

32. a) 

b) 

34. a) 4, b) 4/3, c) 9, d) 56/3

36. $4\sqrt{5} - 2 \ln (2 + \sqrt{5})$

38. 1

40. By trapezoidal rule for $n = 3$, a) 0,00235; $0,002\overline{3}$
  b) 0,09364; $0,093\overline{6}$, c) 0,65; $1 - \frac{1}{2} \ln 2 \cong 0,655$

42. a) 1,622, b) 1,609

44. a) 5323/6800, b) $\frac{\pi}{4}(\sqrt{2} + 1)$, c) 329/612,
  d) $\frac{1}{8}\left(\sqrt{2} + \frac{\sqrt{41}}{2} + \frac{\sqrt{52}}{2} + \frac{\sqrt{65}}{2} + \sqrt{5}\right)$

5. 4. IMPROPER INTEGRALS

So far we have discussed the definite integrals of fnuctions that are continuous on a closed interval. Now we consider the cases where the integrand is discontinuous at finite number of points on the interval, or the continuous integrand is defined on an unbounded interval. In the former case the definite integral is called an *improper integral of the first kind*, and in the second an imprcper integral of the second kind.

Improper integral of the first kind:

This was the case where the integrand has a discontinuity at a point on a clcsed interval $[a, b]$. The following are of this kind:

$$\int_{-1/2}^{5/2} x[\![x]\!]\, dx, \quad \int_0^{\pi/2} \frac{dx}{\sin x}, \quad \int_1^2 \frac{dx}{x-2} \quad \int_{-1}^{1} \frac{dx}{x^2}$$

where the first integral has finite jump discontinuities at 0, 1, 2, the remaining ones have infinite jump discontinuities at a (= 0), b (= 2) and at the interior point $x = 0$ respectively

If the integrand has a finite jump discontinuity at a point, say x_0, the integral is defined as

$$\int_a^b f(x)dx = \int_a^{x_0-} f(x)dx + \int_{x_{0+}}^b f(x)dx$$

<u>Example</u>. Evaluate the improper integral

$$A = \int_{-1/2}^{5/2} x[x]\ dx.$$

<u>Solution</u>. The integrand having finite jump discontinuities at 0, 1 and 2 we have

$$A = \int_{-1/2}^{0-} x[x]\ dx + \int_{0+}^{1-} x[x]\ dx + \int_{1+}^{2-} x[x]\ dx + \int_{2+}^{5/2} x[x]dx$$

$$= \int_{-\frac{1}{2}}^{0} x.-1\ dx + \int_0^1 x.0\ dx + \int_1^2 x.1\ dx + \int_2^{5/2} x.2\ dx$$

$$= -\left.\frac{x^2}{2}\right]_{-1/2}^{0} + 0 + \left.\frac{x^2}{2}\right]_1^2 + \left.x^2\right]_2^{5/2}$$

$$= \left(\frac{1}{8}\right) + \left(2 - \frac{1}{2}\right) + \left(\frac{25}{4} - 4\right) = 15/4.$$

The improper integrals where the integrand $f(x)$ has infinite jump discontinuities on a, b are defined by

1. $|f(a)| = \infty \Rightarrow \int_a^b f(x)dx = \lim_{\alpha \to a^+} \int_\alpha^b f(x)dx$

2. $|f(b)| = \infty \Rightarrow \int_a^b f(x)dx = \lim_{\beta \to b^-} \int_a^\beta f(x)dx$

3. $|f(x_0)| = \infty, \quad a < x_0 < b$

$$\Rightarrow \int_a^b f(x)dx = \int_a^{x_0-} f(x)dx + \int_{x_{0+}}^b f(x)dx.$$

If limit exists in each case the improper integral is said to be *convergent*, otherwise *divergent*.

Example. Test convergence of the improper integral

$$I = \int_a^b \frac{dx}{(x-a)^p}$$

Solution. The case $p = 1$ is related to logarithmic function that will be taken up in the next Chapter and in that case we have divergence.

If $p \neq 1$, we have

$$I = \lim_{\alpha \to a+} \int_\alpha^b (x-a)^{-p}dx = \frac{1}{1-p} \lim_{\alpha \to a+} \Big[(x-a)^{1-p}\Big]_\alpha^b$$

$$= \frac{(b-a)^{1-p}}{1-p} - \lim_{\alpha \to a+} (\alpha - a)^{1-p}$$

If the exponent $1-p$ is positive the limit is zero and there is convergence for $p < 1$. Otherwise limit does not exist and there is divergence for $p > 1$.

Thus we have obtained the following which we call *p-test*, namely

$$\int_a^b \frac{dx}{(x-a)^p} \quad \text{is} \quad \begin{cases} \text{convergent when} & p < 1 \\ \text{divergent when} & p > 1 \end{cases}$$

The same test holds for the other end b.

According to this test the improper integrals

$$\int_2^3 \frac{dx}{\sqrt[3]{x-2}}, \qquad \int_0^1 \frac{dx}{x^{2/3}}, \qquad \int_0^1 \frac{dx}{\sqrt{1-x}}$$

are convergent, while the following ones are divergent:

$$\int_{-1}^2 \frac{dx}{x+1}, \qquad \int_0^3 \frac{dx}{x\sqrt{x}}, \qquad \int_0^2 \frac{dx}{(x-2)^{3/2}}$$

Comparison test: Let $\int_a^b f(x)dx$ be an improper integral of the first kind, and let $g(x)$ be such that

$$0 \leq |f(x)| < g(x) \text{ for } x\varepsilon(a, b)$$

Then,

$\int_a^b f(x)dx$ converges if $\int_a^b g(x)dx$ converges,

but there is no information when the latter diverges.

Proof. The result follows from the inequalities

$$0 \leq \int_a^b f(x)dx \leq \int_a^b |f(x)|dx \leq \int_a^b g(x)dx.$$

Example 2. Test $\int_0^{\pi/2} \frac{\cos x}{\sqrt{x}}\, dx$ for convergence.

Solution.

$$\left|\frac{\cos x}{\sqrt{x}}\right| \leq \frac{1}{\sqrt{x}} \Rightarrow \left|\int_0^{\pi/2} \frac{\cos x}{\sqrt{x}}\, dx\right| \leq \int_0^{\pi/2} \frac{dx}{\sqrt{x}} \quad (p = 1/2 < 1)$$

Since the last integral converges, the given integral is convergent.

Improper integrals of the second kind:

This is the case where the continuous integrand is defined on an unbounded interval:

1. $\int_a^{\infty} f(x)dx = \lim_{b \to \infty} \int_a^b f(x)dx$
2. $\int_{-\infty}^{b} f(x)dx = \lim_{a \to -\infty} \int_a^b f(x)dx$
3. $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{t} f(x)dx + \int_t^{\infty} f(x)dx$ or

$$= \lim_{\substack{a \to -\infty \\ b \to \infty}} \int_a^b f(x)dx.$$

If limit exists the improper integral is said to be convergent, otherwise divergent.

Example 3. Test the convergence of

$$\int_1^{\infty} \frac{dx}{1+x^2}$$

Solution.

$$\int_1^\infty \frac{dx}{1+x^2} = \lim_{b \to \infty} \int_1^b \frac{dx}{1+x^2} = \lim_{b \to \infty} (\arctan b - \arctan 1)$$

$$= (\lim_{b \to \infty} \arctan b) - \frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{2}. \text{ (convergent.)}$$

EXERCISES (5. 4)

46. Evaluate the following definite integrals:

a) $\int_{-2}^{3} f(x)dx$ where $f(x) = \begin{cases} -x & \text{when } x < 0 \\ x^2+1 & \text{when } x > 0 \end{cases}$

b) $\int_{-3}^{5} |x^2 - 4| dx$

47. Evaluate $\int_{-1}^{2} f(x)dx$ where

$$f(x) = \begin{cases} x^3 + 2x & \text{if } -2 \leq x \leq 2 \\ 3 & \text{if } -3 \leq x < -2 \end{cases}$$

48. Evaluate

a) $\int_{-3/2}^{2} x[\![x]\!] dx$

49. Evaluate when convergent:

a) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ b) $\int_0^1 \frac{dx}{\sqrt[3]{x^2}}$ c) $\int_0^\infty \frac{\arctan^2 x}{x^2+1}$

50. Test the improper integrals for convergence:

a) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ b) $\int_0^1 \frac{dx}{x^{2/3}}$ c) $\int_0^\pi \frac{\cos^2 x}{\sqrt{x}} dx$ d) $\int_2^\infty \frac{dx}{\sqrt{x}}$

e) $\int_1^\infty \frac{dx}{(x+2)\sqrt{x}}$ f) $\int_0^\infty \frac{x\, dx}{(x^2+1)^3}$ g) $\int_1^\infty \frac{\sin x\, dx}{x^{3/2}}$ h) $\int_0^\infty \frac{\cos x\, dx}{1+x^2}$

51. Same question for:

a) $\int_1^3 \frac{x^2-4}{x-2} dx$ b) $\int_1^\infty \frac{dx}{1+x^2}$ c) $\int_{-\infty}^{2} \frac{dx}{(x+1)^2}$ d) $\int_{-\infty}^{\infty} \sin \frac{1}{x}\, d(\frac{1}{x})$

52. Discuss the convergence of the following improper integrals, and find the value when convergent:

a) $\int_0^3 \frac{dx}{x^2}$ b) $\int_0^6 \frac{dx}{(6-x)^2}$ c) $\int_2^\infty \frac{dx}{x^{5/2}}$ d) $\int_{-\infty}^\infty \frac{dx}{(x+1)^2}$

53. Test $\int_0^\infty \frac{\sqrt{1+\sin x}}{\cos x}\,dx$ for convergence.

54. Prove: $\int_a^\infty \frac{dx}{x^p}$ is $\begin{cases} \text{convergent when} & p > 1 \\ \text{divergent when} & p \leq 1 \end{cases}$ $(a > 0)$

55. Discuss the following improper integrals for convergence, and find values when convergent:

a) $\int_{-1}^1 \frac{dx}{\sqrt{|x|}}$ b) $\int_{-1}^4 \frac{dx}{(2x+1)^2}$

c) $\int_0^\infty \frac{x\,dx}{(x+1)^3}$ d) $\int_0^{\sqrt{2}} \frac{x\,dx}{(x^2-1)^{4/5}}$

ANSWERS TO EVEN NUMBERED EXERCISES

46. a) 14, b) 36

48. 13/4.

50. a) conv, $\pi/2$, b) conv., 3, c) conv., d) div., e) conv., f) conv., g) conv., h) conv.,

52. a) Div., b) div., c) conv., $\sqrt{2}/6$, d) div.

A SUMMARY
(CHAPTER 5)

5. 1. $F(x) = \int f(x)dx \Rightarrow DF(x) = f(x)$, $F(x)$ is *a primitive* or *indefinite integral* of $f(x)$.

Properties:

$\int du(x) = u(x) + c$, $D\int f(x)dx = f(x)$

$\int (f(x)+g(x))dx = \int f(x)dx + \int g(x)dx$,

$\int \lambda f(x)dx = \lambda \int f(x)dx$ (λ is a const)

Methods of integration:

$\int f(x)dx = \int f(u(t))\, u'(t)dt$ (substitution or change of var.)

$\int u dx = uv - \int v du$ (by parts)

5. 2. If $f \varepsilon C[a, b]$ and $x_0 (= a)$, $x_1, \ldots, x_n (=b)$ is a partiation then $\sum_{i=1}^{n} f(t_i)\Delta x_i$ ($x_i = x_i - x_{i-1}$, $t_i \varepsilon [x_{i-1}, x_i]$) is a RIEMANN sum,

$$\int_a^b f(x)dx = \lim_{\substack{n \to \infty \\ \eta \to 0}} \sum_{i=1}^{n} f(t_i)\Delta x_i \quad (\eta = \max\Delta x_i)$$ is the

the RIEMANN integral.

MVT for integral: If $f \varepsilon C[a, b]$, then for some $c \varepsilon (a, b)$

$$\int_a^b f(x)dx = (b-a)\, f(c)$$

Average of $f(x)$ on $[a, b]$: $\overline{y} = \overline{f(x)} = \dfrac{\int_a^b f(x)dx}{b - a}$

Differentiation: $$\frac{d}{dx} \int_{a(x)}^{b(x)} f(f)dt = f(b(x))\, \frac{d\, b(x)}{dx} - f(a(x))\, \frac{d\, a(x)}{dx}$$

Evaluation of a limit by integral: If $h = (b-a)/n$,

$$\lim_{n \to \infty} \frac{h}{n} \left[f(a+h) + f(a+2h) + \ldots + f(a+nh)\right] = \int_a^b f(x)dx.$$

5. 3. Areas of normal regions:

$R_{xy} = [a, b;\ y = L(x),\ y = U(x)]$ $\quad |R_{xy}| = \int_a^b |U(x)-L(x)|dx$

$R_{yx} = [c, d;\ x = \ell(y),\ x = r(y)]$ $\quad |R_{yx}| = \int_c^d |r(y)-\ell(y)|dy$

Approximate integration of $\int_a^b f(x)dx$ for $h = \dfrac{b-a}{n}$:

Rectangular rule: $h[y_0 + y_1 + \ldots + y_{n-1}]$ or $h[y_1 + y_2 + \ldots + y_n]$

Trapezoidal rule: $\frac{h}{2}[y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n]$

SIMPSON's rule: $\frac{h}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n]$ (even n)

MISCELLANEOUS EXERCISES

56. Find a primitive that vanishes at $x = -2$ of the function $(x^2 + 1)/x^2$.

57. Determine the function $F(x)$ with the following conditions $F(0) = 1$, $F(1) = 7/12$ and $F''(x) = x^2 - 3x$.

58. Prove or disprove that a primitive function of a positive function defined in an interval has at most one root. What if we replace the word "positive" by "non negative", by "negative"?

59. Evaluate the following integral using integration by parts

a) $\int x^2 \sin x \, dx$ b) $\int \cos t \, d \sin t$

60. Evaluate the following indefinite integrals:

a) $\int f(x)\, dg(x) + g(x)\, df(x)$, b) $\int \arcsin x \dfrac{dx}{\sqrt{1 - x^2}}$

c) $\int \arctan^{-2} x \dfrac{dx}{1+x^2}$ d) $\int f'(u)\, u'(t)\, t'(x) dx$

61. Find the polynomial function of the second degree with $(-1, 3)$ as a maximum point.

62. Show that the following equalities are correct:

a) $\int f(x) \cos x \, dx = f(x) \sin x + f'(x) \cos x - \int f''(x) \cos x \, dx$

b) $\int \arcsin x \sin x \, dx = -\arcsin x \cos x + \int \dfrac{\cos x}{\sqrt{1 - x^2}} dx$

c) $\int \dfrac{x^2}{x^2+1} dx = x - \arctan x$ (by ordinary division)

d) $\int \tan^n x \, dx = \dfrac{1}{n-1} \tan^{n-1} x - \tan^{n-2} x \, dx$ $(n \in N_2)$

63. Evaluate the following indefinite integrals by the use of given substitution:

a) $\int \sqrt{1 + x} \, dx$, $1+x = u^2$ b) $\int \dfrac{dx}{\sqrt{1 - 4x^2}}$, $2x = \sin u$

c) $\int \dfrac{(x + 1)^2}{(x - 1)^4}$, $x - 1 = t$ d) $\int x\sqrt{x + 1} \, dx$, $x + 1 = t^2$

64. Same question for

a) $\int \frac{x^2}{(x^3 + 1)^2}\,dx$, $x^3 + 1 = u$

b) $\int \frac{(\sqrt{x} + 1)^5}{2\sqrt{x}}\,dx$, $x = t^2$

c) $\int \frac{dx}{x\sqrt{x^2 - a^2}}$, $x = 1/t$

65. Evaluate $\int \frac{\sqrt{1 + x^4}}{1 - x^4}\,dx$ (EULER) <u>Hint</u>: Set $x = \frac{\sqrt{1 + t^2} + \sqrt{1 - t^2}}{t\sqrt{2}}$

66. If $F(x)$ is a primitive of $f(x)$, find a primitive of

a) $g(x) = f(x^n)x^{n-1}$ with $n \in Z$ b) $g(x) = F^n(x)\, f(x)$ with $n \in Z$, $n \neq 1$.

67. If $F(x)$, $G(x)$ are primitives of $f(x)$, $g(x)$ respectively, find a primitive of $f(x)\,G(x) + F(x)\,g(x)$

68. If $x^3 + 2x = \int_0^x f(y)dy$ what is $f(4)$?

69. If $\int_0^{g(x)} (y+1)^4 dy = \frac{1}{5}(x^{10} - 1)$, what is $g(2)$?

70. Prove the following:

a) $\int_a^b f(x)dx = \int_{-b}^{-a} f(x)dx$ if $f(x)$ is even

b) $\int_a^b f(x)dx = -\int_{-b}^{-a} f(x)dx$ if $f(x)$ is odd.

71. Evaluate the following definite integral using the given substitution:

a) $\int_0^9 \sqrt{1 + \sqrt{x}}\,dx$, $1+\sqrt{x} = t^2$ b) $\int_0^1 \frac{(x^{1/3} - 1)^6}{x^{2/3}}\,dx$, $x = t^3$

72. The base of a solid is the region $x^2 + y^2 < 1$. Each crossection perpendicular to the x-axis is a square, compute the volume of the solid.

73. The figure represents a parabola with F as focus and d as directrix. If Δ, Δ' are the areas of the shaded regions for arbitrary points P_1, P_2 on the parabola prove

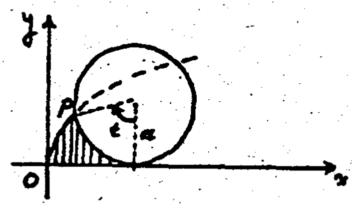
$$\Delta = 2\,\Delta'$$

74. Prove
$$\int_0^1 \sqrt[q]{1 - x^p}\, dx = \int_0^1 \sqrt[p]{1 - x^q}\, dx$$
without integration. $(p,\ q > 0)$

75. The parametric equations of the cycloid generated as the locus of a fixed point P of a circle of radius a rolling on x-axis without sliding are:

$$x = a(\theta - \sin\theta)$$
$$y = a(1 - \cos\theta)$$

Show that the area of the shaded region is equal to ax.



76. Given:
$$f(x) = \int_0^x \sin^2 t \cos^2 t \, dt$$
find the slope at the origin, and points of inflections.

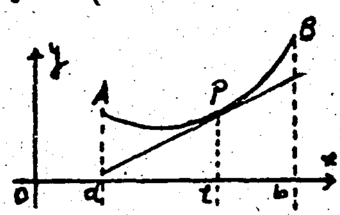
77. Prove that if
$$\int_a^b \left[A\, f(x) + B\, g(x)\right]^2 dx$$
exists (A, B are constants), then
$$\int_a^b f^2(x)dx \int_a^b g^2(x)dx \geq \int_a^b f(x)\, g(x)dx$$

78. If $f(x),\ g(x) \varepsilon C[a, b]$, $g(x) > 0$ and decreasing, then prove that there exists $c \varepsilon (a, b)$ such that
$$\int_a^b f(x)\, g(x)dx = g(c) \int_a^b f(x)dx$$

79. Sketch the normal regions R_{yx} and write them as R_{xy} normal regions:

a) $R_{yx} = \{0, 3; -1, -1-y^2\}$, b) $R_{yx} = \{0, 3; y^2/3, y+1\}$

80. Let $y = f(x)$ be a concave up function defined on the closed interval $[a, b]$. Locate the point P on the curve such that the area



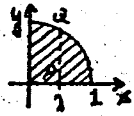
bounded by $y = f(x)$, the tangent line at P and the vertical lines $x = a$, $x = b$ be an extremum.

81. If $a < b < c$, show that the area bounded by the curves of $y = (x-2a)^2$, $y = (x-2b)^2$ and $y = (x-2c)^2$ is equal to $2(c - a)(c - b)(b - a)$.

82. Find the area under the curve of $f(x) \geq 0$ over the given interval:

a) $f(x) = x\sqrt{2x^2 + 1}$, $[0, 2]$, b) $f(x) = \dfrac{\sin 2x}{\cos^2 2x}$, $[0, \pi/6]$

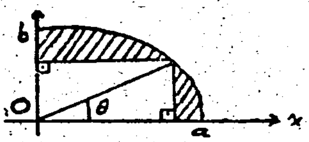
83. Find a relation for λ such that the vertical segment (PQ) bisects the shaded area under the quarter of unit circle.



84 Find the area of the region bounded by the curve $y = x^3-6x^2$, the tangent line at its inflection point and the lines $x=0, x=6$.

85. Given the parabola $y = x^2$ and a point $P = (t, t^2)$ on it, find $t > 0$ such that the area between the parabola and the normal line at P be minimum.

86. The curve is a quarter of an ellipse. Determine $\tan\theta$ for which the shaded areas be equal to each other.



87. Compute the area of the region enclosed by the curve of $y = (x^3-8)/x^2$, and its tangent line at $x = 2$ and the joining the points $(-2, -4)$, $(2, 0)$.

88. Compute the area of the region defined by $y-x^3 > 0$, $y^2-32x < 0$ and $6x + y - 7 > 0$.

89. Compute the area of the region bounded by

a) $y = (x-1)/x^3$, $x=20$, x-axis b) $y = 1/\sqrt{x+3}$, $x=0$, $x=1$, $x=6$

c) $y = x/\sqrt{x^2 + 9}$, $y = 0$, $x = 4$

90. Evaluate by the use of definite integral:

a) $\lim_{n \to \infty} \left(\frac{n}{n^2 + 1^2} + \frac{n}{n^2 + 2^2} + \dots + \frac{n}{n^2 + n^2} \right)$

b) $\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \cos(\frac{k}{n} \frac{\pi}{2})$

91. Find the unbounded area between $y = 1/(x+1)^2$, $x=0$ and x-axis.

92. Test for convergence the following improper integrals:

a) $\int_0^1 \frac{dx}{x+\sqrt{x}}$ b) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ c) $\int_{-1}^1 \frac{dx}{(1-4x)^2}$

93. By comparison, test the for convergence:

a) $\int_0^\infty \frac{\sin x}{x^2} dx$ b) $\int_1^\infty \frac{x+1}{x} dx$ c) $\int_1^\infty \frac{1}{\sqrt{x}} dx$ d) $\int_1^\infty \frac{dx}{x+\sqrt{x}}$

94. Evaluate $\int_0^\infty \frac{\mathrm{Arctan}^2 x}{x^2+1} dx$

95. Discuss the convergence of the following improper integrals; if convergent find the value:

a) $\int_1^\infty \frac{dx}{1+x^3}$ b) $\int_0^1 \frac{dx}{x+\sqrt{x}}$

c) $\int_0^2 \frac{x+1}{(x^2-4)^{4/3}} dx$ d) $\int_{1/4}^1 \frac{dx}{(x-1)^2}$

96. Evaluate the following definite integral.

a) $\int_{-1}^3 \left[\!\left[\frac{x}{2} + 1 \right]\!\right] dx$ b) $\int_{-1}^1 \sin(\pi x)\, dx$

97. Evaluate

a) $\int_{-2}^5 \frac{x^3+x^2-4x-4}{x+1} dx$ b) $\int_{-3}^4 f(x)dx$ where $f(x) = \begin{cases} x & \text{if } x \le -2 \\ x^2-3 & \text{if } x > -2 \end{cases}$

98. Given the function in a table find by interpolation (or extrapolation) obtain the value of the function for a regular partition and apply the trapezoidal and the SIMPSON's rule to compute the definite integral $\int_0^{1,2} y\, dx$

x	0	0,2	0,4	0,6	0,8	1,0
y	2	5	8	12	17	21

99. Compute approximately the given definite integral by the use of lower sum and upper sum for the regular partition taking $n = 4$ and give maximum error:

a) $\int_0^2 \frac{dx}{1+x^2}$ b) $\int_1^2 (x^2 - 3x)dx$

100. Compute approximately the following definite integral by trapezoidal and SIMPSON's rule taking $n = 4$.

a) $\int_1^2 \frac{x+1}{6-x^2}\,dx$ b) $\int_1^2 \frac{6-x^2}{x+1}\,dx$

ANSWERS TO EVEN NUMBERED EXERCISES

56. $x - 1/x + 3/2$

60. a) $f(x)\,g(x) + c$, b) $\frac{1}{2}\arcsin^2 x + c$

c) $1/\arctan x + c$, d) $f(u(f(x))) + c$

64. a) $-\frac{1}{3(x^2+1)} + c$, b) $\frac{1}{6}(\sqrt{x}+1)^6 + c$, c) $-\frac{1}{a}\arcsin\frac{a}{x} + c$

66. a) $\frac{1}{n}F(x^n) + c$, b) $(F(x))^{n+1}/(n+1) + c$

68. 50

72. 16/3

76. Slope is 0, points of inflections at $x = 0$, $x = \pi/4$.

80. $t = (a + b)/2$

82. a) 7/3, b) 1/2

84. 1048

86. b/a

88. 533/16

90. a) 1/2, b) $2/\pi$

92. a) Div., b) conv., $\pi/2$, c) conv. $-2/15$.

94. $\pi^3/24$ 96. a) 4, b) 0 98. 153/2, 229/3.

100. a) 3,1018, b) 3,582

CHAPTER 6

LOGARITHMIC AND ITS RELATED FUNCTIONS

In this Chapter we define one of the important functions, namel the natural logarithmic function $\ln x$ and from which we deduce some other important functions, namely the exponential functions, hyperbolic and inverse hyperbolic functions. This way we complete the discussion of all elementary functions that are encountered in calculus.

6. I. LOGARITHMIC AND EXPONENTIAL FUNCTIONS

A. THE NATURAL LOGARITHMIC FUNCTION AND ITS INVERSE

Consider the positive function $y = f(x) = 1/x$ defined on the open interval $(0, \infty)$. Then the natural logarithmic function, denoted by $\ln x$, is defined by

$$\ln x = \int_1^x \frac{dt}{t}, \quad (x > 0)$$

As a result of the Fundamental Theorem of calculus we get the derivative of $\ln x$:

$$\frac{d}{dx} \ln x = \frac{1}{x} \quad (> 0),$$

and consequently

1. $\frac{d}{dx} \ln u(x) = \frac{1}{u(x)} \frac{du}{dx} = \frac{u'}{u}$ (by chain rule)

2. $\frac{d}{dx} \ln |x| = \frac{1}{|x|} \cdot \frac{|x|}{x} = \frac{1}{x}$ (since $D|x| = |x|/x$)

3. $\int \frac{dx}{x} = \ln|x| + c$ (From 2)

<u>Example</u>. Differentiate $\ln \sin x$

<u>Solution</u>. Applying chain rule we have

$$D \ln \sin x = \frac{1}{\sin x} \cdot D \sin x = \frac{\cos x}{\sin x} = \cot x$$

<u>Properties</u>. If $a, b \in R^+$, then

1. $\ln 1 = 0$

2. $\ln \frac{1}{a} = -\ln a$

3. $\ell n(ab) = \ell n\ a + \ell n\ b$ 4. $\ell n \frac{a}{b} = \ell n\ a - \ell n\ b$

5. $\ell n\ a^n = n\ \ell n\ a, \ (n \varepsilon \mathbb{Z})$ 6. $\ell n\ a^r = r\ \ell n\ a, \ (r \varepsilon Q)$

<u>Proof</u>.

1. $\ell n\ 1 = \int_1^1 \frac{dt}{t} = 0,$

2. $\ell n \frac{1}{a} = \int_1^{1/a} \frac{dt}{t}$ (Set $t = 1/s$)

$$= \int_1^a s(-ds/s^2) = -\int_1^a \frac{ds}{s} = -\ell n\ a$$

3. $\ell n(ab) = \int_1^{ab} \frac{dt}{t} = \int_1^a \frac{dt}{t} + \int_a^{ab} \frac{dt}{t} = \ell n\ a + \int_a^{ab} dt/t$

Setting $t = as$ with $t = a \Rightarrow s = 1$, and $t = ab \Rightarrow s = b$,

$$\ell n(ab) = \ell n\ a + \int_1^b \frac{ads}{as} = \ell n\ a + \ell n\ b.$$

4. $\ell n \frac{a}{b} = \ell n(a \cdot \frac{1}{b}) = \ell n\ a + \ell n \frac{1}{b} = \ell n\ a - \ell n\ b.$

5. $\ell n\ a^n = n\ \ell n\ a$ can be proved by induction when $n \varepsilon N$. If $n \varepsilon Z^-$, setting $n = -m$ with $m \varepsilon N$, we have

$$\ell n\ a^n = \ell n\ a^{-m} = \ell n \frac{1}{a^m} = -\ell n\ a^m = (-m)\ell n\ a = n\ \ell n\ a.$$

6. Setting $r = p/q$ with $p,\ q \varepsilon \mathbb{Z},\ q \neq 0$ and $u = a^{p/q}$, we have $u^q = a^p \Rightarrow q\ \ell n\ u = p\ \ell n\ a \Rightarrow$

$$\ell n\ u = \frac{p}{q}\ \ell n\ a \Rightarrow \ell n\ a^r = r\ \ell n\ a.$$

<u>Graph of $y = \ell n\ x$:</u>

The domain of $\ell n\ x$ is by definition the interval $(0, \infty)$. Since $D\ \ell n\ x = 1/x > 0$, the curve is an increasing one. From $D^2\ \ell n\ x = -\frac{1}{x^2} < 0$, it follows that the curve is concave downward at every point of $(0, \infty)$.

To show that $\ell n\ x$ increases indefinitely when $x \to \infty$ observe that $2^n \to \infty$ when $n \to \infty$, and

$$\lim_{n \to \infty} \ell n\ 2^n = \lim_{n \to \infty} (n\ \ell n\ 2) = \infty$$

showing also non existence of HA.

Non existence of oblique asymptote $y = ax + b$ is seen from

$$a = \lim_{x \to \infty} \frac{y}{x} = \lim_{x \to \infty} \frac{\ln x}{x} = \left(\frac{\infty}{\infty}\right) = \lim_{x \to \infty} \frac{1/x}{1} = 0$$

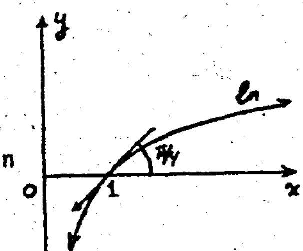
$$b = \lim_{x \to \infty} (y - ax) = \lim_{x \to \infty} \ln x = \infty.$$

These also show non existence of HA.

As to a vertical asymptote, it will suffice to examine it at $x = 0$, since differentiability of $\ln x$ implies continuity at every point of $(0, \infty)$:

$$\lim_{x \to 0^+} \ln x = \lim_{n \to \infty} \ln 2^{-n} = \lim_{n \to \infty} (-n)\ln 2 = -\infty$$

Hence y-axis is the only vertical asymptote.



The graph passes through the point (1, 0) having slope 1 there, and the function

$$\ln : \mathbb{R} \to \mathbb{R}, \quad y = \ln x$$

is one to one onto and consequently we have the corollary:

<u>Corollary</u>. For $a, b > 0$,

1. $\ln a = \ln b \Leftrightarrow a = b$,
2. $\ln a < \ln b \Leftrightarrow a < b$

<u>The number e</u>: Since $\ln : \mathbb{R}^+ \to \mathbb{R}$ is onto, there is a positive number e such that $\ln e = 1$, and since

$$\ln 2 = \int_1^2 \frac{dx}{x} \cong 0{,}69, \quad \ln 3 = \int_1^3 \frac{dx}{x} \cong 1{,}01$$

from computations in §5. 3 B Example 1, it follows that

$$\ln 2 < 1 < \ln 3 \Rightarrow 2 < e < 3.$$

Indeed the accurate value of e up to five decimal places is

$$e \cong 2{,}71828$$

This number e will be called the *base* of the natural (NAPIERian) logarithm.

Example. Show that the tangent line to $y = \ln x$ at the point $(e, 1)$ passes through the origin.

Solution. $y' = 1/x \Rightarrow y'(e) = 1/e \Rightarrow$

$$y - 1 = \frac{1}{e}(x - e) \Rightarrow y = x/e.$$

The exponential function e^x as inverse of $\ln x$

Since $y = \ln x$ is an increasing function on $(0, \infty)$, it admits an inverse function denoted by $y = \exp x$, so that

$$y = \exp x \Leftrightarrow x = \ln y$$

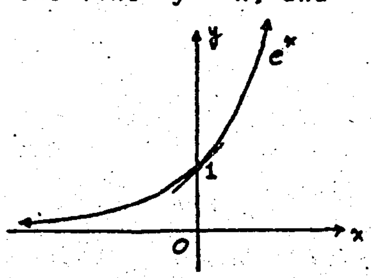
where the second holds true if y is taken e^x for any rational x by the property $\ln e^r = r$ for $a = e$, since $\ln e = 1$.

Since $y = \exp x = e^x$ holds true for all $x \in Q$ we assume it to be true for all $x \in R$ without proof. Thus

$$y = e^x \Leftrightarrow x = \ln y \quad \text{or} \quad y = \ln x \Leftrightarrow x = e^y.$$

The number e is also called the *base* of the exponential function e^x.

The graph of $y = e^x = \exp x$ is then the symmetric of the graph of $y = \ln x$ with respect to the line $y = x$, and passes through the point $(0, 1)$ having slope 1 there.



It is an increasing, concave up and positive function on $(-\infty, \infty)$ with the x-axis as horizontal asymptote.

Properties of e^x:

1. $e^0 = 1$
2. $e^{\ln u} = u$
3. $e^u e^v = e^{u+v}$
4. $(e^u)^v = e^{uv} = (e^v)^u$

Proof. In view of $y = e^x \Leftrightarrow x = \ln y$, these equalities are equivalent to the equalities $0 = \ln 1$, $\ln u = \ln u$, $u+v = u+v$ and $uv = uv$ respectively.

Theorem. $\frac{d}{dx} e^x = e^x$, $\int e^x dx = e^x + c$

Proof. Let $y = e^x$. Then $\ln y = x$ holds, implying $\frac{1}{y} \cdot y' = 1$ or $y' = y = e^x$.

Example. Evaluate the following

a) $A = (2 - 3D + D^2)e^{x^2}$ b) $B = \lim_{x \to \infty} x\, e^{-x}$ c) $C = \int_0^\infty x\, e^{-x} dx$

Solution.

a) $A = 2e^{x^2} - 3(De^{x^2}) + D^2 e^{x^2} = 2e^{x^2} - 3(2xe^{x^2}) + D(2xe^{x^2})$
$= 2e^{x^2} - 6xe^{x^2} + (2e^{x^2} + 4x^2 e^{x^2}) = (4x^2 - 6x + 4)e^{x^2}$

b) $B = \lim_{x \to \infty} xe^{-x} = \lim_{x \to \infty} \frac{x}{e^x} = \left(\frac{\infty}{\infty}\right) = \lim_{x \to \infty} \frac{1}{e^x} = 0$

c) With $u = x$, $dv = e^{-x}$; $du = dx$, $v = -e^{-x}$, we have

$$\int_0^\infty xe^{-x}dx = -\left[xe^{-x}\right]_0^\infty + \int_0^\infty e^{-x}dx = -(0-0) - \left[e^{-x}\right]_0^\infty = 1 \quad \text{from (b)}$$

B. The Exponential Function a^x and its Inverse

If "a" is any positive real number, instead of e, we have the *exponential function* $y = a^x$ with base "a" which, in view of the identity $a^x = e^{\ln a^x}$, is defined by

$$y = a^x = e^{x \ln a}$$

coinciding with e^x when $a = e$.

Corollary. $D\, a^x = D\, e^{x \ln a} = e^{x \ln a} \ln a = a^x \ln a$.

Since the derivative of a^x is positive (negative) when $a > 1$ $(a < 1)$, this function is increasing (decreasing) on $(-\infty, \infty)$, and admits an inverse function denoted by

$$y = \log_a x$$

so that we have

$$y = a^x \Leftrightarrow x = \log_a y \quad \text{or} \quad y = \log_a x \Leftrightarrow x = a^y$$

The number "a" is the *base* of the exponential function a^x also written $exp_a x$, and the *base* of the logarithmic function $y = \log_a x$.

<u>Corollary</u>. $D \log_a x = \dfrac{1}{x \ln a}$

<u>Proof</u>. Setting $y = \log_a x$, we have $x = a^y$ implying $1 = (a^y \ln a)y' \Rightarrow 1 = (x \ln a)y' \Rightarrow y' = 1/(x \ln a)$.

We get the graphs of a^x and $\log_a x$ for various values of "a"

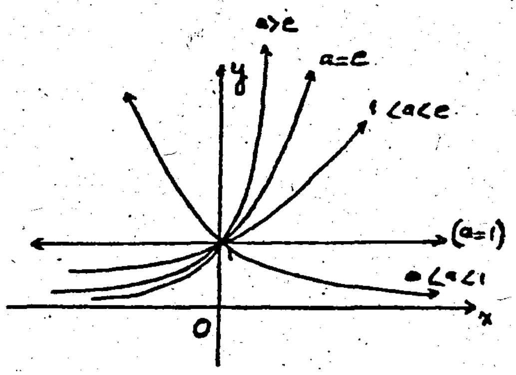


Figure representing graphs of a^x for various values of the base "a"

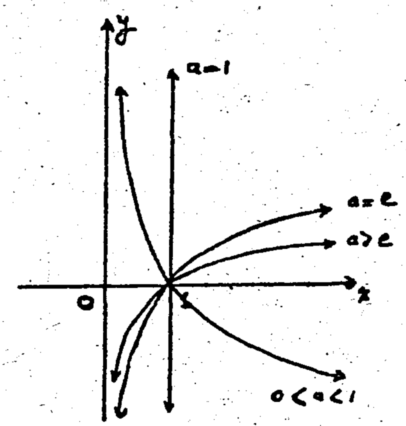


Figure representing graphs of $\log_a x$ for various values of the bases "a"

From the previous Section on logarithmic and exponential functions we have the following properties:

1. $a^0 = 1 \quad (a > 0)$
2. $a^{-u} = \dfrac{1}{a^u}$

1. $\log_c 1 = 0 \quad (c > 0)$
2. $\log_c \dfrac{1}{u} = -\log_c u \; (= \text{colog}_c u)$

3. $a^{\log_a u} = u$ 3. $\log_a a^{\alpha} = \alpha$

4. $a^u a^v = a^{u+v}$ 4. $\log_c ab = \log_c a + \log_c b$

5. $(a^u)^v = a^{uv} = (a^v)^u$ 5. $\log_c \frac{a}{b} = \log_c a - \log_c b$

6. $a^x = b^{x \log_b a}$ 6. $\log_a c \log_c b = \log_a b$

(chain rule for logarithm)

where the equality (6.) for exponential function is verifiable by taking logarithms of each side in the base b, and the last one is derived below.

C. Change of Base

Setting, for $a, b > 0$,

$$a = b^{\alpha}, \quad b = a^{\beta} \quad \text{or} \quad \alpha = \log_b a, \quad \beta = \log_a b$$

we have, from

$$a = b^{\alpha} = (a^{\beta})^{\alpha} = a^{\alpha\beta} \Rightarrow \alpha\beta = 1,$$

the identity

$$\log_a b \log_b a = 1 \qquad \text{(a)}$$

Now we solve the following problem:

Problem. Knowing the logarithms of numbers x and b in the base "a", find the logarithm of x in the base "b".

Solution. Starting with the identity

$$x = a^{\log_a x}$$

and taking the logarithms of both sides in the base b and using (a) we have

$$\log_b x = \log_a x \, \log_b a = \log_a x : \log_a b.$$

which is the desired result.

If we adopt the notation

$$\log_a x = \frac{x}{a} \, ,$$

the above transforming equality takes the form

$$\underset{b}{\overset{x}{\sim}} = \underset{a}{\overset{x}{\sim}} \; \underset{b}{\overset{a}{\sim}}$$

which we may call the *chain rule* for change of bases.

The most commonly used bases are the numbers 10 and e, and $\log = \log_{10}$, $\ln = \log_e$ are called the *common logarithm*, *NAPIER'ian (natural) logarithms* respectively.

By the above chain rule we have

$$\underset{10}{\overset{x}{\sim}} = \underset{e}{\overset{x}{\sim}} \; \underset{10}{\overset{e}{\sim}} \Rightarrow \log x = \ln x \log e$$

$$\underset{e}{\overset{x}{\sim}} = \underset{10}{\overset{x}{\sim}} \; \underset{e}{\overset{10}{\sim}} \Rightarrow \ln x = \log x \ln 10$$

where setting

$$M = \log e \cong 0{,}4343, \quad \frac{1}{M} \cong 2{,}3026$$

we get

$$\log x = M \ln x, \qquad \ln x = \frac{1}{M} \log x.$$

and the relation (a) gives

$$\ln b = \frac{1}{\log_b e}.$$

D. Some Examples on the Solution of Equation and Inequations

Example 1. Find the solutions of the following equations:

a) $3^x - 2.3^{x-1} - 3^{x-3} = \frac{8}{3}$ b) $2^x + 4^x - 8^{x-1} - 8^{x-2} = 0$

c) $\log(x-2) + \log(x+3) = \log(x+30)$,

d) $\log_2(x+2) + 2\log_4(x+8) = \log_2 40$.

Solution.

a) $3^x - 2.3^{x-1} - 3^{x-3} = 3^x - 2.3^x.3^{-1} - 3^x.3^{-3}$

$$= 3^x\left(1 - \frac{2}{3} - \frac{1}{27}\right) = \frac{8}{27}\,3^x \Rightarrow \frac{8}{27}\,3^x = \frac{8}{3}$$

$$\Rightarrow \quad 3^x = 9 \Rightarrow x = 2.$$

b) $2^x + 4^x - 8^{x-1} - 8^{x-2} = 2^x + 2^{2x} - 2^{3(x-1)} - 2^{3(x-2)} - 2^{3(x-2)} = 0$

$u + u^2 - u^3 \cdot 2^{-3} - u^3 \cdot 2^{-6} = 0$ (with $u = 2^x$)

$-(\frac{1}{8} + \frac{1}{64})u^3 + u^2 + u = 0$

$\Rightarrow \quad u(u - \frac{5}{64} u^2 + u + 1) = 0 \qquad u_1 = 0, \quad u_2 = -\frac{8}{9}, \quad u_3 = 8$

$2^x = 0, \ 2^x = -\frac{8}{9}, \quad 2^x = 8 \qquad x = 3$ since the first two have no solutions.

c) $\log(x-2) + \log(x+3) = \log(x+30)$, $(x-2, \ x+3, \ x+30 > 0)$

$\Rightarrow \quad \log(x-2)(x+3) = \log(x+30)$

$\Rightarrow \quad x^2 + x - 6 = x + 30 \ \Rightarrow \ x^2 = 36 \ \Rightarrow \ x_1 = -6, \ x_2 = 6,$

and $x = 6$ is the only solution.

d) $\log_2(x+2) + 2 \log_4(x+8) = \log_2 40$, $(x+2 > 0, \quad x+8 > 0)$

$\log_2(x+2) + 2 \cdot \frac{1}{2} \log_2(x+8) = \log_2 40$

(from $\log_4 a = \log_2 a \cdot \log_4 2 = \frac{1}{2} \log_2 a$)

$\Rightarrow \quad \log_2(x+2)(x+8) = \log_2 40$

$\Rightarrow \quad x^2 + 10x + 16 = 40 \ \Rightarrow \ x_1 = -12, \quad x_2 = 2.$

$x = 2$ is the only solution.

Example 2. Solve the following

a) $2^x - 4^y = 4, \quad 4^x + 2^y = 66$ for x and y.

b) $\log(x+y) + \text{lof}(x-y) = 1, \ \log x + \log y = \log \frac{\sqrt{21}}{2}$.

c) $\log_2 x - \log_8(x - 1) < 1$.

Solution.

a) $2^x - 4^y = 4, \quad 4^x + 2^y = 66$

$\Rightarrow \quad 2^x - 2^{2y} = 4, \quad 2^{2x} + 2^y = 66$

$\Rightarrow \quad u - v^2 = 4, \quad u^2 + v = 66$ (with $u = 2^x > 0$, $v = 2^y > 0$)

$\Rightarrow \quad (v^2 + 4)^2 + v = 66 \ \Rightarrow \ v^4 + 8v^2 + v - 50 = 0$

$\Rightarrow \quad (v-2)(v^3+2v^2+12v+25) = 0 \Rightarrow$ $v=2$ as the only positive root.

$u = 4 + v^2 = 8$. Then

$u = 8, \quad v = 2 \ \Rightarrow \ x = 3, \ y = 1.$

b) $\log(x+y) + \log(x-y) = 1, \quad \log x + \log y = \log(\sqrt{21}/2)$

$$x^2 - y^2 = 10, \quad xy = \sqrt{21}/2.$$

$$(x^2 + y^2)^2 = 100 + 21 = 121 \Rightarrow x^2 + y^2 = 11$$

$$x^2 - y^2 = 10,\ x^2 + y^2 = 11 \Rightarrow 2x^2 = 21,\quad 2y^2 = 1$$

$$\Rightarrow x = \pm\sqrt{21/2},\quad y = \pm\sqrt{1/2} \Rightarrow x = \sqrt{21/2},\quad y = \sqrt{1/2}$$

c) $\log_2 x - \log_8(x-1) < 1 \Rightarrow \log_2 x - \frac{1}{3}\log_2(x-1) < 1$

$$\Rightarrow \log_2 \frac{x}{\sqrt[3]{x-1}} < \log_2 2 \Rightarrow 0 < \frac{x}{\sqrt[3]{x-1}} < 2$$

$$\Rightarrow \frac{x^3}{x-1} < 8 \Rightarrow \frac{x^3 - 8x + 8}{x - 1} < 0.$$

$x^3 - 8x + 8 = 0$ has a single real root α which is in $(-4, -3)$. Then the solution set is the interval $(\alpha, 1)$.

Example 3. Sketch the relation:

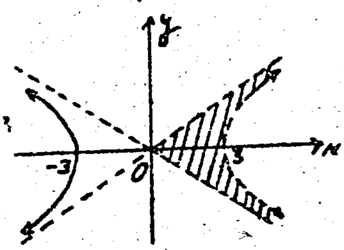
$$\log_6(2x - 3y) + \log_6(2x + 3y) < 2$$

Solution.

$$\log_6(4x^2 - 9y^2) < \log_6 6^2 \Rightarrow 4x^2 - 9y^2 < 36$$

$$\Rightarrow \frac{x^2}{9} - \frac{y^2}{4} < 1 \quad \text{and} \quad 2x - 3y > 0, \quad 2x + 3y > 0.$$

Since $\frac{x^2}{9} - \frac{y^2}{4} = 1$ is a hyperbola and (0, 0) satisfies the given inequality, the graph is the shaded region:



E. Functions $a^{v(x)}$, $\log_a u(x)$

These are the first generalizations of the exponential function a^x and logarithmic function $\log_a x$. By a change of basis they are identical with $e^{v(x)\ln a}$ and. $(\log_a e)\ln u(x)$ respectively.

The domain of $a^{v(x)}$ is certainly the domain of $v(x)$ and that of $\log_a u(x)$ is the set $\{x: u(x) > 0\}$.

Example. Find the domains of

a) $f(x) = 3^{\frac{x^2}{x+1}}$ b) $g(x) = \ln \frac{x-1}{x+2}$

Solution.

a) $D_f = \mathbb{R} - \{-1\}$ b) $D_g = \mathbb{R} - [-2, 1] = (-\infty,-2) \cup (1,\infty)$

Theorem.

1. $\frac{d}{dx} a^{v(x)} = a^{v(x)} \cdot v'(x) \ln a$

2. $\frac{d}{dx} \log_a u(x) = \frac{u'(x)}{u(x)} \log_a e$

Proof.

1. $a^{v(x)} = e^{v(x) \ln a} \Rightarrow$

$D\, a^{v(x)} = D\, e^{v(x) \ln a} = e^{v(x) \ln a} v'(x) \ln a = a^{v(x)} v'(x) \ln a.$

2. $\log_a u(x) = (\ln u(x)) \log_a e \Rightarrow$

$D \log_a u(x) = D \ln u(x) \log_a e = \frac{u'(x)}{u(x)} \log_a e.$ ∎

where the ratio $u'(x)/u(x)$ is called the *logarithmic derivative* of $u(x)$.

Corollary. The logarithmic derivative of a product (ratio) of two functions is the sum (difference) of logarithmic derivatives of the functions.

Proof. Let $y = u(x)\, v(x)$. Then $y' = u'v + uv'$, and

$$\frac{y'}{y} = \frac{u'v + uv'}{uv} = \frac{u'}{u} + \frac{v'}{v}.$$

For $y = \frac{u(x)}{v(x)}$, we have $yv = u$ and

$$\frac{y'}{y} + \frac{v'}{v} = \frac{u'}{u} \Rightarrow \frac{y'}{y} = \frac{u'}{u} - \frac{v'}{v}. \quad ∎$$

More generally $\frac{d}{dx} \prod_{i=1}^{n} u_i(x)^{\alpha_i} = \sum_{i=1}^{n} \alpha_i \frac{u_i'}{u_i}$.

To differentiate a function in the form of product or ratio, involving power and roots, logarithmic differentiation is very useful one.

Example. Differentiate $y = \sqrt[3]{\dfrac{(x^2+1)(x-1)^2}{x-2}}$

Solution. We have

$$y = (x^2+1)^{1/3}(x-1)^{2/3}(x-2)^{-1/3}$$

which can be differentiated directly. But for such a function the process of logarithmic differentiation is more easier with the necessary assumption $y > 0$:

$$\frac{y'}{y} = \frac{1}{3}\,\frac{2x}{x^2+1} + \frac{2}{3}\,\frac{1}{x-1} - \frac{1}{3}\,\frac{1}{x-2}$$

which gives y'.

Example. Sketch the curves of

a) $f(x) = \ln\dfrac{x-1}{x+2}$,   b) $g(x) = 3^{\frac{x^2}{x+1}}$

Solution.

a) $D_f = (-\infty, -2)U(1, \infty)$

We make a table of variation of the function $u(x) = \dfrac{x-1}{x+2}$ and then obtain values of $f(x)$ corresponding to some values of $u(x)$ given in the table

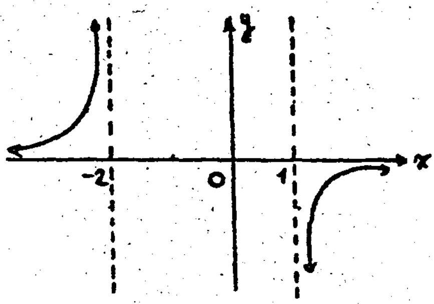
x	$-\infty$		-2	1		∞
u(x) (> 0)	1	↗	∞	0	↗	1
$\ln u$	0	↗	∞ |	| $-\infty$	↗	0

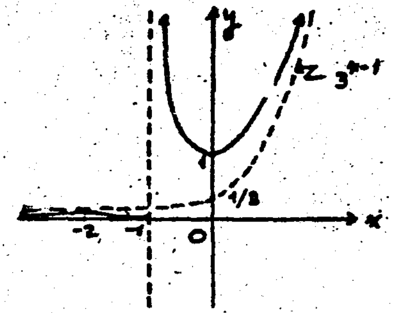
b) $D_g = D_v = \mathbb{R} - \{-1\}$

Proceeding as above we arrange a table of variation for the exponent (power) function $v(x)$, and then obtain values of $g(x)$ corresponding to the ones obtained for $v(x)$.

x	$-\infty$		-2		-1		0		∞
v(x)	$-\infty$	↗	-4	↘	$-\infty$ | ∞	↘	0	↗	∞
3^v	0	↗	3^{-4}	↘	0 | ∞	↘	1	↗	∞

The graphs are:





Note. $v(x)$ has the inclined asymptote $y = x - 1$ implying that $y = 3^{x-1}$ is the curvilinear asymptote of $g(x)$.

F. The General Exponential Function $u(x)^{v(x)}$ and Indeterminate Forms 0^0, ∞^0, 1^∞

The function $f(x) = u(x)^{v(x)}$ is an exponential function with a variable base. The particular cases of constancy of $u(x)$ or $v(x)$ were examined previously.

The domain of the function $f(x) = u(x)^{v(x)}$ is certainly

$$D_f = \{x: u(x) > 0\} \cap D_v .$$

$y = x^x$ $(x > 0)$ is of this type, and $y = x^{x^x}$ is defined to be $x^{(x^x)}$.

Example. Find the domains of

a) $F(x) = x^{\frac{x+1}{x}}$ b) $G(x) = (x^2-4x+3)^x$, c) $H(x) = (1+x)^{1/x}$

Answer.

a) $D_f = \mathbb{R}^+$, b) $D_G = (-\infty, 1) \cup (3, \infty)$, c) $D_H = (-1, 0) \cup (0, \infty)$

Corollary. $\frac{d}{dx} u(x)^{v(x)} = u^v \left[v' \ell n\, u + v \frac{u'}{u} \right]$

Proof. Since $u^v = e^{v\, \ell n\, u}$, we have

$$D\, u^v = D\, e^{v\, \ell n\, u} = e^{v\, \ell n\, u}\left[v' \ell n\, u + v \frac{u'}{u} \right] = u^v \left[v' \ell n\, u + v \frac{u'}{u} \right]$$

or

$$y = u^v \Rightarrow \ln y = v \ln u \Rightarrow \frac{y'}{y} = v' \ln u + v \frac{u'}{u}$$

which is the logarithmic derivative of u^v giving the required result. ■

If u^v is differentiable in its domain of definition, it is a continuous function in the same domain.

Example. Find the critical point(s) of $y = x^x$.

Solution. $y = x^x \Rightarrow \ln y = x \ln x \Rightarrow$

$y'/y = \ln x + 1 \Rightarrow y' = x^x (1 + \ln x)$.

$y' = 0, \; y \neq 0 \Rightarrow 1 + \ln x = 0 \Rightarrow \ln x = -1 \Rightarrow x = e^{-1} = 1/e$.

The indeterminate forms: 0^0, ∞^0, 1^∞.

Some functions $y = u(x)^{v(x)}$ lead to the indeterminate forms 0^0, ∞^0, 1^∞ when $x \to a$ or $x \to \infty$, and they are reducible to the familiar indeterminate form $0.\infty$ observed from $\ln y = v(x) \ln u(x)$.

So the problem of evaluating lim y is reduced to that of lim ln y, and from the continuity of ln y we have

$$\lim(\ln y) = \ln(\lim y)$$

Example 1. Prove

a) $\lim\limits_{x \to 0^+} x^x = 1$ b) $\lim\limits_{x \to \infty} (1 + \frac{1}{x})^x = e$

Solution.

a) $y = x^x \Rightarrow \ln y = x \ln x$

$$\ln(\lim_{x \to 0} y) = \lim_{x \to 0} (x \ln x) = (0. \infty)$$

$$= \lim_{x \to 0} \frac{\ln x}{1/x} = \lim_{x \to 0} \frac{1/x}{-1/x^2} = 0 = \ln 1$$

Then

$$\ln (\lim_{x \to 0} y) = \ln 1 \Rightarrow \lim_{x \to 0} y = 1.$$

b) $y = (1 + \frac{1}{x})^x \Rightarrow \ell n\, y = x\, \ell n\, (1 + \frac{1}{x})$

$$\ell n(\lim_{x \to \infty} y) = \lim_{x \to \infty} \left[x\, \ell n\, (1 + \frac{1}{x})\right] = (\infty.\ 0)$$

$$= \lim_{x \to \infty} \frac{\ell n(1 + \frac{1}{x})}{1/x} = \lim_{x \to \infty} \frac{\frac{-1/x^2}{1 + 1/x}}{-1/x^2} = 1 = \ell n\ e$$

$$\Rightarrow \lim_{x \to \infty} y = e.$$

Example 2. Evaluate $\lim_{x \to 0} (\cos 2x)^{1/x^2}$

Solution. $y = (\cos 2x)^{1/x^2} \Rightarrow \ell n\, y = \frac{1}{x^2}\, \ell n \cos 2x$

$$\ell n(\lim_{x \to 0} y) = \lim_{x \to 0} \frac{\ell n \cos 2x}{x^2} = \left[\frac{0}{0}\right] = \lim_{x \to 0} \frac{\frac{-2 \sin 2x}{\cos 2x}}{2x}$$

$$= -2 \lim_{x \to 0} \left(\frac{\sin 2x}{2x}\ \frac{1}{\cos 2x}\right) = -2 = \ell n\ e^{-2}$$

$$\Rightarrow \lim_{x \to 0} y = e^{-2}$$

Sketching. The procedure for sketching the curve of $y = u(x)^{v(x)}$ is the same as that given for the case $u(x)$ is a constant function. One determines first the domain, and makes a table of variation for $u(x)$ and $v(x)$ and get the values or limits of y corresponding to the specific values obtained for x in the table.

Example. Sketch the curves of

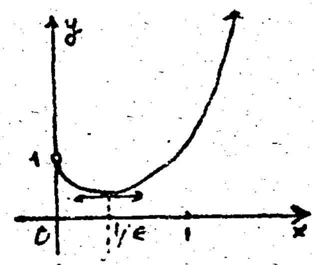
a) $y = f(x) = x^x$ b) $f(x) = (1 + x)^{1/x}$

Solution.

a) $D_f = (0, \infty)$

$y' = x^x(1 + \ell n\ x) = 0 \Rightarrow x = 1/e$

x	0		1/e		1		∞
u = x v = x	0		1/e		1		∞
y'	−∞	−	0	+	1	+	∞
x^x	(0^0)	↘	$(1/e)^{1/e}$	↗	1	↗	∞

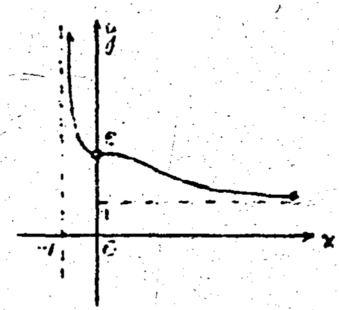


b) $D_f = D_u\ D_v = (-1,\infty)\ \ ((-\infty,\ 0)\cup(0,\ \infty)) = (-1,\ 0)\cup(0,\ \infty).$

$y = (1+x)^{1/x} \qquad \ell n\ y = \frac{1}{x}\ \ell n(1+x) \qquad \frac{y'}{y} = -\frac{1}{x^2}\ \ell n(1+x) + \frac{1}{x}\ \frac{1}{1+x} = 0$

$\ell n(1+x) = \frac{x}{1+x} \qquad x = 0.$

x	−1		0		∞
u = 1 + x	0		1		∞
v = 1/x	−1		∞ −∞		0
y'	∞	−	(∞−∞) 0		
$(1+x)^{1/x}$	∞	↘	(1^∞) e	↘	(∞^0) 1



EXERCISES (6. I)

1. Simplify the following

   a) $e^0,\ e^{\ell n\ 1},\ e^{\ell n\ 2},\ e^{-\ell n\ 3},\ e^{\ell n\ x^2},\ e^{-\ell n\ x^2}$

   b) $\ell n\ e^x,\quad \ell n\ e^{-2},\quad \ell n\ \exp\sqrt{x},\quad \exp\ \ell n\sqrt{x}$

2. Prove by induction:

   $p(n):\ \ell n\ a^n = n\ \ell n\ a\ (a > 0),\quad n \in N.$

3. Find the domains of definition of the following functions.

   a) $y = e^{\frac{x}{x+1}}$ b) $e^{-\sqrt{\frac{x-2}{x+3}}}$ c) $y = e^{\sqrt{\sin\ x}}$ d) $y = e^{\tan\ x}$

4. Same question for:

   a) $y = \ell n(1+x^2)$ b) $y = \ell n\ \ell n(x-1)$ c) $y = \ell n\ \ell n\ \ell n(x-1)$

   d) $y = \ell n\ \arctan\ x$ e) $y = \arctan\ \ell n\ x$ f) $y = \arcsin\ \ell n\ x$

5. Find

a) $\frac{d}{dt}$ ℓn arctan t   b) $\frac{d}{dt}$ arctan ℓn t

6. Prove

a) $a^m \geqslant a^n \Rightarrow m < n$ when $a > 1$

b) $a^m \geqslant a^n \Rightarrow m < n$ when $a < 1$

$(m, n \in \mathbb{N})$

7. Given $y = \exp \int P(x)dx$, express Dy, D^2y in terms of $P(x)$.

8. Find the intervals of concavity of $y = e^{2x} - 6e^x + 4x$

9. Determine the constant λ for which $(D^3 - 2D^2 - D + 2)e^{\lambda x} = 0$

10. Evaluate

a) $\lim_{x \to 0} \left[\frac{1}{x(1+x)} - \frac{\ell n(1+x)}{x^2}\right]$ b) $\lim_{x \to \infty} \left(x \ \ell n \frac{x - a}{x + a}\right)$

c) $\lim_{x \to 0} \left[\frac{\pi x - 1}{x^2} + \frac{2\pi}{x(e^{2x} - 1)}\right]$,   d) $\lim_{x \to a} \left(\ell n(2 - \frac{x}{a}) \cot \frac{\pi x}{a}\right)$

11. Evaluate

a) $\lim_{x \to 0} \frac{\ell n(x+2) - 2e^x}{\ell n(x+1)}$ b) $\lim_{x \to 1} \frac{e^{x-1} - x}{\ell n(x^2+1) - \ell n 2}$

c) $\lim_{x \to \pi/2} \frac{\ell n \sin x}{\sin x - 1}$ d) $\lim_{x \to \pi/2} \frac{\ell n \tan x}{\sec x}$

12. Sketch the graphs of

a) $y = 2 \ \ell n \ x$ b) $y = \ell n \ x^2$ c) $y = \ell n|x|$

d) $|y| = \ell n \ x$ e) $|y| = \ell n|x|$

13. Sketch the graphs of:

a) $y = \ell n \tan x$ b) $y = \ell n \ (x+1)$ c) $y = \exp x^2$ d) $y = e^{\sin x}$

14. Find the equation of the tangent and normal to the curve of the given function at the given point:

a) $y = \ell n \frac{x}{x+1}$, $(-2, \ell n \ 2)$ b) $y = e^{\frac{x^2}{x+1}}$, $(0, 1)$

15. Verify the following equalities:

a) $\int \sec x \, dx = \ell n|\sec x + \tan x| + c = \ell n \tan(\frac{x}{2} + \frac{\pi}{4}) + c$

b) $\int \csc x \, dx = -\ln|\csc x + \cot x| + c = \ln \tan \frac{x}{2} + c$

c) $\int \sec^3 x \, dx = \frac{1}{2} \ln|\sec x \tan x| + \sec x \tan x + c.$

16. Evaluate the following indefinite integrals:

a) $\int \frac{dx}{x \ln x}$ b) $\int \ln x \, dx$ c) $\int \frac{1}{x-1} \ln\ln(x-1) dx$

d) $\int \sec^2 \ln x \frac{dx}{x}$ e) $\int \frac{x \, dx}{\cos^2 x}$ (use integration by parts) f) $\int x^3 \ln\sqrt{x} \, dx$

17. Evaluate

a) $\int u e^{u^2} du$ b) $\int e^{\sin\theta} \cos\theta \, d\theta$ c) $\int_e^{e^2} \ln^2 x \frac{dx}{x}$ d) $\int_1^9 \frac{dt}{t+\sqrt{t}}$

18. Sketch the graph of the function

$$f(x) = \int_1^x e^{-t^2} dt$$

19. Evaluate

a) $\int x e^{x^2} dx$ b) $\int x e^x dx$ (by parts)

c) $\int \sin x \, e^x dx$ d) $\int \exp \ln \frac{x+1}{x^2+2x} dx$

20. Find the function with the given conditions:

a) $f'(x) = \frac{1}{3} f(x)$, $f(0)=2$, b) $f'(x) = 5f(x)$, $f(2) = e$

21. Find the area of the region enclosed by:

a) $y = e^x$, $x = 0$, $x = \ln 5$, $y=0$ b) $y = \ln x$, $y = -x+e+1$, $y=0$

22. Find the area of the unbounded region between $y = e^x/\left[1+(e^x)^2\right]$ and x-axis, if finite.

23. Suppose a radioactive material has the half life of 1 year. How long it takes for 10 gr of this material to decay to 1 gr?

24. Find the family of curves such that y-intercept of any tangent line is constant.

25. Find the family of curves having subnormal of constant length.

26. Find the domain of definition and derivative of the following functions:

a) $y = \log_6(x + 1)$ b) $y = \log_5 \frac{x+2}{x-1}$

27. Sketch the graphs of:

a) $y = 9^{\frac{x+2}{x^2}}$ b) $y = 5^{\sqrt{\frac{2x}{x-1}}}$

28. Evaluate:

a) $\int \frac{3^{\sqrt{x}}}{2\sqrt{x}}\, dx$ b) $\int 5^x \sin 5^x\, dx$

29. a) $\log_{10}3 \cong 0,477$ and $\ln 10 \cong 2,301$, $\ln 3 = ?$

b) $\log_a x = y \Rightarrow \log_{a^2} x = ?$

c) $a^{\log_{a^2} x} = ?$ d) $\log_a x = y \Rightarrow \log_a y = ?$

30. Solve the following equations:

a) $\log_2 x + 2\log_4 x = 8$, b) $\log_3 x^2 + \log_9 x^4 = 8$

31. Same question for:

a) $5.5^x - 2.5^{x+1} + 3.5^{x+2} - 5^{x+3} = -11$ b) $3^x + 9^x - 6 = 0$

32. Solve the equations:

a) $\ln(x-1) + \ln(x+4) = \ln 14$ b) $\ln(x+2) + \ln\sqrt{x+2} = 6$

33. Solve the following systems of equations:

a) $\ln x + \ln y = 8$
$\ln x^2 - \ln y = 4$

b) $2^x - 4^y = 0$
$4^x + 2^y = 18$

34. Solve the inequations:

a) $\log_{10}(x^2 - 5x + 4) < 1$, b) $\log_2(x + 2) + \log_4(x + 8) < 2$

35. Solve the system for x and y:

$2^x 3^y = 5^{2x-1} + 5^{2x+1}$, $2^y 3^x = 5^{3y-2} + 5^{3y+2}$

36. Solve the inequations:

a) $e^{x/(x+2)} < 1$ b) $3^x - 9^x > 1$

37. Sketch the graph of the relation:

a) $3^x \, 9^y < 1$,   b) $4^{x+y} \, 2^x < 2^6$

38. Evaluate

a) $\int (2^x + x^2 + 2^2)dx$ b) $\int 2^x \, 3^x \, dx$

39. Find the domain of definition and derivative of the following functions:

a) $y = x^{\frac{3x}{x+1}}$ b) $y = (\ln x)^{\sqrt{\frac{4}{x-4}}}$

40. Evaluate

a) $\lim_{x \to \infty} (1 + \frac{a}{x})^x$ b) $\lim_{x \to 0} (1 + \frac{a}{x})^x$ c) $\lim_{x \to 3} \frac{\ln(x - 2)}{2(x - 3)}$

d) $\lim_{x \to 0^+} (\frac{e^x - e^{-x}}{2})^{1/x}$ e) $\lim_{x \to 0} x^{1/\ln x}$

41. Evaluate

a) $\lim_{x \to \infty} (\frac{1}{x})^{\sin \frac{1}{x}}$ b) $\lim_{x \to 1} (\arccos x)^{\ln x}$

c) $\lim_{x \to 0} \ln x^{\arcsin x}$ d) $\lim_{x \to 0} (\frac{1}{x})^{\arctan x}$

42. Find

a) $D \, x^{x^x}$ b) $D^2 \, x^x$ c) $D \, 2^{\sin x}$ d) $D^2 \, 2^{\sin x}$ e) $D \, 2^x$

f) $D^2 \, 2^x$

43. Sketch:

a) $y = x^{-x}$ b) $y = \sqrt[x]{x}$

44. Evaluate

a) $\lim_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$ b) $\lim_{x \to 0} \frac{(x+a)^x - a^x}{x^2}$

45. Sketch the curve of $y = x^{x^2 - 4}$

ANSWERS TO EVEN NUMBERED EXERCISES

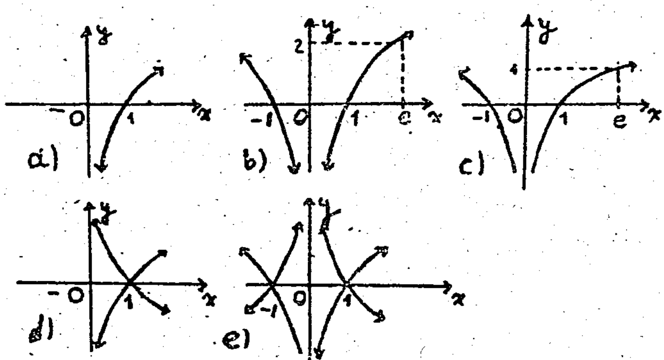
4. a) R, b) $(2, \infty)$, c) $(e+1, \infty)$, d) $(0, \infty)$
   e) $(0, \infty)$, f) $[e^{-1}, e]$

6. Hint: Take logarithm of both sides

8. $(3/2, \infty)$ upward, $(-\infty, 3/2)$ downward

10. a) $-1/2$, b) $-2a$, c) no limit, d) $-1/\pi$.

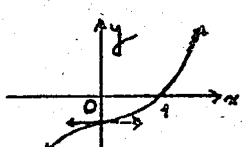
12.



14. a) $-x + 2y = 1 + \ln 2$; $2x+y = -4+\ln 2$, b) $y = 1$, $x = 0$

16. a) $\ln|\ln x| + c$, b) $x \ln x - x + c$, c) n $n(x-1)-1$
   c) $(\ln \ln(x-1)-1)\ln(x-1) + c$ d) $\tan \ln x + c$
   e) $x \tan x + \ln|\cos x| + c$ f) $x^4(8 \ln\sqrt{x} - 1)/32 + c$

18.



20. a) $f(x) = 2e^{x/3}$, b) $f(x) = e^{5x-9}$

22. $\pi/2$.

24. If y-intercept is k, $y = mx + k$.

26. a) $(-1, \infty)$; $y' = \dfrac{1}{(x+1)\ln 6}$, b) $(-\infty,-2)\cup(1,\infty)$; $y'' = -\dfrac{3}{(x-1)^2} \cdot \log_5 e$

28. a) $3^{\sqrt{x}} \log_3 e + c$, b) $-\cos 5^x \cdot \log_5 e + c$

30. a) 16, b) 9

32. a) 3, b) $e^4 - 2$

34. a) $(-1, 6)$, b) $(-2, -5 + \sqrt{17})$

36. a) $(-2, 0)$, b) No solution.

38. a) $2^x \log_2 e + \frac{x^3}{3} + 4x + c$, b) $6^x \log_6 e + c$

40. a) e^a, b) 1 c) 1/2, d) 1 e) e

42. a) $x^{x^x} \cdot x^x ((\ln x + 1)\ln x + 1/x)$, b) $x^x((\ln x + 1)^2 + 1/x)$

c) $2^{\sin x} \cos x \ln 2$, d) $2^{\sin x}\left[\cos^2 x \ln 2 - \sin x\right]\ln 2$

e) $2^x \ln 2$, f) $2^x \ln^2 2$

44. a) 1, b) 1/a, c) e/2

6. 2. HYPERBOLIC FUNCTIONS

A. DEFINITIONS AND IDENTIFIES:

We define below six functions involving the exponential functions e^{Θ} and $e^{-\Theta}$ having similar properties as those of trigonometric ones. As the trigonometric functions are called circular functions by their relations to the unit circle, $x^2 + y^2 = 1$, the new ones are called *hyperbolic functions* by their relations to the unit hyperbola[1] of equation $x^2 - y^2 = 1$.

1. $\text{Cosh}\Theta = \frac{e^{\Theta} + e^{-\Theta}}{2}$ (cosine hyperbolic Θ), even

2. $\text{Sinh}\Theta = \frac{e^{\Theta} - e^{-\Theta}}{2}$ (sine hyperbolic Θ), odd

(1) The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is called an *equilateral hyperbola* if $a = b$ and *unit hyperbola* if $a = b = 1$.

3. $\text{Tanh}\Theta = \dfrac{\sinh\Theta}{\cosh\Theta}$ (tangent hyperbolic Θ), odd

4. $\text{Coth}\Theta = \dfrac{\cosh\Theta}{\sinh\Theta}$ (cotangent hyperbolic Θ), odd

5. $\text{Sech}\Theta = \dfrac{1}{\cosh\Theta}$ (secant hyperbolic Θ), even

6. $\text{Cosh}\Theta = \dfrac{1}{\sinh\Theta}$ (cosecant hyperbolic Θ), odd

where Θ is called *argument* and a unit of it is referred to as *hyperbolic radian*. Geometric meaning of the argument is postponed until the integration of such functions and we establish its analogy with the angle Θ for trigonometric functions.

For simplicity the first three of above functions will be denoted by <u>Ch Θ</u>, <u>Sh Θ</u>, and <u>Th Θ</u>.

From 1 and 2, we have the relations

$$\left.\begin{aligned} e^{\Theta} &= \text{Ch}\Theta + \text{Sh}\,\Theta \\ e^{-\Theta} &= \text{Ch}\Theta - \text{Sh}\,\Theta \end{aligned}\right\} \quad (*)$$

<u>Some identifies</u>:

$$\left.\begin{aligned} \text{Ch}(a+b) &= \text{Ch a Ch b} + \text{Sh a Sh b}, \\ \text{Sh}(a+b) &= \text{Sh a Ch b} + \text{Ch a Sh b}, \\ \text{Th}(a+b) &= \frac{\text{Th a} + \text{Th b}}{1 + \text{Th a Th b}} \end{aligned}\right\} \qquad (1)$$

<u>Proof</u>. We prove the first one by the use of (*)

$$2\text{Ch}(a+b) = e^{a+b} + e^{-a-b} = e^a e^b + e^{-a} e^{-b}$$

$$= (\text{Cha} + \text{Sha})(\text{Chb} + \text{Shb}) + (\text{Cha} - \text{Sha})(\text{Chb} - \text{Shb})$$

$$= 2\ \text{Cha Chb} + 2\ \text{Sha Shb}.$$

The second one can be proved similarly, and the third one is a consequence of $\text{Th}\,\Theta = \text{Sh}\,\Theta/\text{Ch}\,\Theta$.

The reader may obtain formulas for the same functions with argument a-b considering evenness or oddness of the related functions.

Taking b a in (1) we have

$$\left.\begin{aligned} Ch\ 2a &= Ch^2 a + Sh^2 a \\ Sh\ 2a &= 2\ Sh\ a\ Ch\ a \\ Th\ 2a &= \frac{2\ Th\ a}{1 + Th^2 a} \end{aligned}\right\} \quad \textit{double argument formulas} \quad (2)$$

and taking $b = -a$

$$\left.1 = Ch^2 a - Sh^2 a \quad (Ch\ 0 = 1)\right\} \quad \textit{The fundamental identity}$$

$$Sh\ 0 = 0 \qquad (3)$$

$$Th\ 0 = 0$$

By the use of the fundamental identity, we have

$$\begin{aligned} Ch\ 2a &= Ch^2 a - Sh^2 a \\ &= 2\ Ch^2 a - 1 = 1 + 2\ Sh^2 a \end{aligned} \qquad (3')$$

which when solved for $Ch^2 a$ and $Sh^2 a$, and $2a$ is replaced by a gives

$$\left.\begin{aligned} Ch^2 \frac{a}{2} &= \frac{Cha + 1}{2} \\ Sh^2 \frac{a}{2} &= \frac{Cha - 1}{2} \\ Th^2 \frac{a}{2} &= \frac{Cha - 1}{Cha + 1} \end{aligned}\right\} \quad \textit{The half-argument formulas} \quad (4)$$

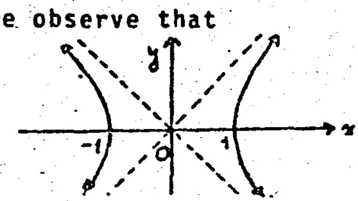
Starting with the fundamental identity $Ch^2 a - Sh^2 a = 1$, dividing it by $Ch^2 a$ or $Sh^2 a$ we obtain

$$\left.\begin{aligned} 1 - Tanh^2 a &= Sech^2 a \\ Coth^2 a - 1 &= Csch^2 a \end{aligned}\right\} \qquad (5)$$

From the fundamental identity, we observe that

$$x = Ch\ \Theta$$
$$y = Sh\ \Theta$$

are the parametric equations of the unit hyperbola $x^2 - y^2 = 1$.



Graph of unit hyperbola

One can also obtain formulas for $\mathrm{Ch}\ p \pm \mathrm{Ch}\ q$ and $\mathrm{Sh}\ p \pm \mathrm{Sh}\ q$ in product form, from $\mathrm{Ch}(p+q)$, $\mathrm{Ch}(p-q)$, $\mathrm{Sh}(p+q)$ and $\mathrm{Sh}(p-q)$:

$$\left.\begin{aligned} \mathrm{Ch}\ p + \mathrm{Ch}\ q &= 2\ \mathrm{Ch}\ \frac{p+q}{2}\ \mathrm{Ch}\ \frac{p-q}{2} \\ \mathrm{Ch}\ p - \mathrm{Ch}\ q &= 2\ \mathrm{Sh}\ \frac{p+q}{2}\ \mathrm{Sh}\ \frac{p-q}{2} \\ \mathrm{Sh}\ p + \mathrm{Sh}\ q &= 2\ \mathrm{Sh}\ \frac{p+q}{2}\ \mathrm{Ch}\ \frac{p-q}{2} \\ \mathrm{Sh}\ p - \mathrm{Sh}\ q &= 2\ \mathrm{Ch}\ \frac{p+q}{2}\ \mathrm{Sh}\ \frac{p-q}{2} \end{aligned}\right\} \qquad (6)$$

Now we obtain the multiple argument formulas (7) by the use of

$$e^{\Theta} = \mathrm{Ch}\ \Theta + \mathrm{Sh}\ \Theta, \qquad e^{-\Theta} = \mathrm{Ch}\ \Theta - \mathrm{Sh}\ \Theta$$

and $e^{n\Theta} = (e^{\Theta})^n$ for $n \in \mathbb{N}$, as

$$\mathrm{Ch}\ n\Theta + \mathrm{Sh}\ n\Theta = (\mathrm{Ch}\ \Theta + \mathrm{Sh}\ \Theta)^n$$

$$\mathrm{Ch}\ n\Theta - \mathrm{Sh}\ n\Theta = (\mathrm{Ch}\ \Theta - \mathrm{Sh}\ \Theta)^n$$

implying

$$\left.\begin{aligned} \mathrm{Ch}\ n\Theta &= \frac{1}{2}\left[(\mathrm{Ch}\ \Theta + \mathrm{Sh}\ \Theta)^n + (\mathrm{Ch}\ \Theta - \mathrm{Sh}\ \Theta)^n\right] \\ \mathrm{Sh}\ n\Theta &= \frac{1}{2}\left[(\mathrm{Ch}\ \Theta - \mathrm{Sh}\ \Theta)^n - (\mathrm{Ch}\ \Theta - \mathrm{Sh}\ \Theta)^n\right] \end{aligned}\right\} \qquad (7)$$

<u>Example 1</u>. Find the values of other hyperbolic functions at $\Theta = \alpha$ when $\mathrm{Sh}\alpha = 5/3$.

<u>Solution</u>. From the fundamental identity $\mathrm{Ch}^2\alpha - \mathrm{Sh}^2\alpha = 1$, we have

$$\mathrm{Ch}^2\alpha = 1 + (5/3)^2 = 34/9 \qquad \mathrm{Ch}\alpha = \sqrt{34}/3, \text{ since } \mathrm{Ch}\Theta > 0.$$

Then

$$\mathrm{Tanh}\alpha = 5/\sqrt{34}, \ \mathrm{Coth}\alpha = \sqrt{34}/5, \ \mathrm{Sech}\alpha = 3/\sqrt{34}, \ \mathrm{Csch}\alpha = 3/5.$$

<u>Example 2</u>. Find the value of

$$F(\alpha) = \frac{\mathrm{Ch}\ 3\alpha - 2\ \mathrm{Sh}\alpha}{\mathrm{Th}\alpha} \text{ for } \alpha = \ln \frac{5}{2}$$

Solution.

$$2\text{Sh } \ell n \frac{5}{2} = e^{\ell n \frac{5}{2}} - e^{-\ell n \frac{5}{2}} = \frac{5}{2} - \frac{2}{5} = \frac{21}{10}.$$

$$\text{CH}(3\ \ell n \frac{5}{2}) = \text{Ch } \ell n \frac{125}{8} = \frac{1}{2}\left(\frac{125}{8} + \frac{8}{125}\right) = \frac{15689}{2000}$$

$$\text{Ch } \ell n \frac{5}{2} = \frac{1}{2}\left(\frac{5}{2} + \frac{2}{5}\right) = \frac{29}{20},\quad \text{Sh } \ell n \frac{5}{2} = \frac{21}{20} \Rightarrow \text{Th } \ell n \frac{5}{2} = \frac{21}{29},$$

$$F(\ell n \frac{5}{2}) = \frac{29}{21}\ \frac{15689 - 4200}{2000} = \frac{29}{21}\ \frac{11489}{2000}$$

Example 3. Write the sum (product) as product (sum) form:

a) Ch 4 + Ch 12 b) Sh ℓn 3 . Sh ℓn 6

Solution.

a) Ch 4 + Ch 12 = 2 Ch 8 Ch 4

b) $\text{Sh } \ell n\ 3 - \text{Sh } \ell n\ 6 = \frac{1}{2}\ (\text{Ch}(\ell n\ 3 + \ell n\ 6) - \text{Ch}(\ell n\ 6 - \ell n\ 3)$

$\Rightarrow \frac{1}{2}\ (\text{Ch } \ell n\ 18 - \text{Ch } \ell n\ 2)$

Example 4. Express Ch 3Θ, Sh 3Θ interms of Ch Θ and Sh Θ.

Solution.

$$\text{Ch } 3\Theta + \text{Sh } 3\Theta = (\text{Ch}\Theta + \text{Sh}\Theta)^3 = \text{Ch}^3\Theta + 3\text{Ch}^2\Theta\ \text{Sh}\Theta + 3\ \text{Ch}\Theta\ \text{Sh}^2\Theta + \text{Sh}^3\Theta$$

$$\text{Ch } 3\Theta = \text{Ch}^3\Theta + 3\ \text{Ch}\,\Theta\ \text{Sh}^2\Theta.$$

$$\text{Sh } 3\Theta = 3\ \text{Ch}^2\Theta\ \ \text{Sh}\,\Theta + \text{Sh}^3\Theta.$$

B. DERIVATIVES, INTEGRALS AND GRAPHS

We write down the hyperbolic functions in argument x together with their domains and ranges:

Functions	Domain	Range[1]
$\cosh x = \frac{e^x + e^{-x}}{2}$	$(-\infty, \infty)$	$(1, \infty)$

(1) More conveniently obtainable from the graphs.

Functions	Domain	Range
$\text{Sinh } x = \frac{e^x + e^{-x}}{2}$	$(-\infty, \infty)$	$(-\infty, \infty)$
$\text{Tanh } x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$	$(-\infty, \infty)$	$(-1, 1)$
$\text{Coth } x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$	$(-\infty, 0)U(0, \infty)$	$(-\infty, -1)U(1, \infty)$
$\text{Sech } x = \frac{1}{e^x + e^{-x}}$	$(-\infty, \infty)$	$(0, 1)$
$\text{Csch } x = \frac{1}{e^x - e^{-x}}$	$(-\infty, 0)U(0, \infty)$	$(-\infty, 0)U(0, \infty)$

Derivatives and integrals:

1. $D \text{ Sinh } x = \text{Cosh } x$
2. $D \text{ Cosh } x = \text{Sinh } x$
3. $D \text{ Tanh } x = \text{Sech}^2 x$
4. $D \text{ Coth } x = -\text{Csch}^2 x$
5. $D \text{ Sech } x = -\text{Sech } x \text{ Tanh } x$
6. $D \text{ Csch } x = -\text{Csch } x \text{ Coth } x$

1. $\int \text{Cosh } x\, dx = \text{Sinh } x + c$
2. $\int \text{Sinh } x\, dx = \text{Cosh } x + c$
3. $\int \text{Sech}^2 x\, dx = \text{Tanh } x + c$
4. $\int \text{Csch}^2 x\, dx = -\text{Coth } x + c$
5. $\int \text{Sech } x \text{ Tanh } x\, dx = -\text{Sech } x + c$
6. $\int \text{Csch } x \text{ Coth } x\, dx = -\text{Csch } x + c$

Proof. The proofs of 1. and 2. are direct from their exponential expressions. The others are proved as in proofs for trigonometric functions.

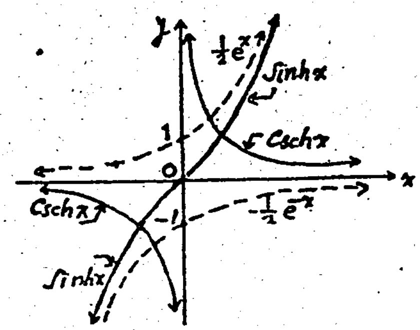
Graphs:

1. $y = \text{Sinh } x = \frac{e^x - e^{-x}}{2} = \frac{1}{2} e^x - \frac{1}{2} e^{-x}$

Since $-\frac{1}{2} e^{-x} \to 0$ as $x \to \infty$, and $\frac{1}{2} e^x \to 0$ as $x \to -\infty$, the curves $y = \frac{1}{2} e^x$ and $y = \frac{-1}{2} e^{-x}$ which are symmetric with respect to the origin are curvilinear asymptotes.

$y' = \text{Ch } x = \frac{e^x + e^{-x}}{2} > 0$ shows that Sh x is increasing and having slope $y'(0) = \text{Ch } 0 = 1$ at the origin. From its oddness the graph is symmetric with respect to the origin.

Since $\text{Csch } x = \frac{1}{\text{Sh } x}$, the graph of Csch x is obtained by taking the reciprocals of the ordinates of Sh x.



Graphs of Sinh x and Csch x

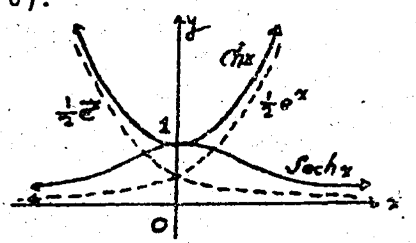
2. $y = \text{Cosh } x = \frac{e^x + e^{-x}}{2} = \frac{1}{2} e^x + \frac{1}{2} e^{-x}$

This time the curves $y = \frac{1}{2} e^x$ and $y = \frac{1}{2} e^{-x}$ which are symmetric with respect to y-axis, are curvilinear asymptotes.

$$y' = \text{Sh } x = \frac{e^x - e^{-x}}{2} = \frac{e^{2x} - 1}{2e^x} = 0 \Rightarrow e^{2x} = 1 \Rightarrow x = 0$$

The function has critical point at the origin and increasing (decreasing) when $x > 0$ $(x < 0)$.

The graphs of Cosh x and Sech x (= 1/cosh x) are shown in the Fig.



Graphs of Cosh x and Sech x

3. $y = \text{Tanh } x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

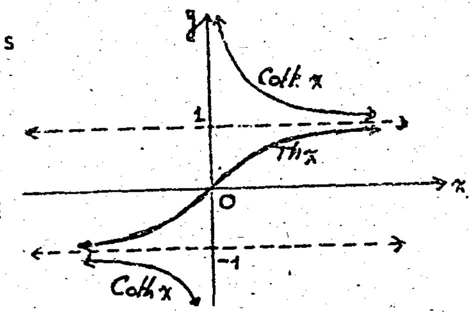
It is an odd function. We draw the curve for $x > 0$ and complete the curve by symmetry with respect to the origin.

The curve is increasing since $y' = \text{Sech}^2 x > 0$, and has slope 1 at the origin.

That the horizontal lines $y = 1$ and $y = -1$ are asymptotes is seen from

$$\lim_{x \to \infty} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = 1 \quad \text{and}$$

$$\lim_{x \to -\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = -1$$



Graphs of Tanh x and Coth x

Example 1. Given $y = \text{Ch}\, x$ and $y = 5e^x$

a) Find their acute angle of intersection,

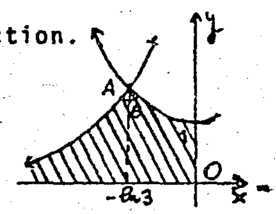
b) Find the area of the region enclosed by y-axis, negative x-axis and these curves.

Solution.

a) Equating y's we have $\frac{1}{2}(e^x + e^{-x}) = 5e^x$ or

$$9e^x = e^{-x} \Rightarrow 9e^{2x} = 1 \Rightarrow e^{2x} = 1/9 \Rightarrow e^x = 1/3 \Rightarrow x = -\ln 3$$

and $A = (-\ln 3, 5/3)$ is the point of intersection.



$D\,\text{Ch}\, x = \text{Sh}\, x \Rightarrow m_1 = -4/3$

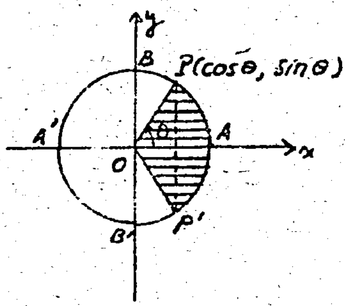
$D\, 5e^x = 5\, e^x \Rightarrow m_2 = 5/3$

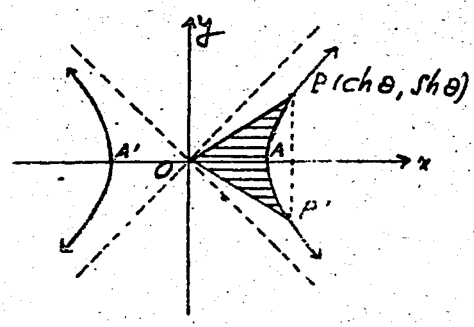
$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = 27/11 \Rightarrow \theta = \arctan 27/11.$$

$$\text{b) } A = \int_{-\infty}^{-\ln 3} 5\, e^x\, dx + \int_{-\ln 3}^{0} \text{Ch}\, x\, dx = 3.$$

Interpretation of positive arc and positive Arg as areas

Below we have drawn a unit circle and a unit hyperbola as locus of points $P(\cos\theta, \sin\theta)$ in the first and $P(\text{Ch}\theta, \text{Sh}\theta)$ in the second:





We are going to show that Θ as arc (angle) on the unit circle represents the area of the shaded segment of circle bouned by the line segment (OP), (OP') and the arc P'AP, and Θ as argument in hyperbolic functions represents the area of the shaded region bounded by (OP), (OP') and the arc of hyperbola P'AP.

a) $|OP'AP| = \frac{2\Theta}{2\pi}\ \pi r^2\ (r = 1) = \Theta$

b) $|OP'AP| = 2|POH| - 2\ |PAH| = \text{Ch}\ \Theta\ \text{Sh}\ \Theta - 2\int_A^P y\, dx$

where, having

$$2\int_A^P y\, dx = 2\int_0^\Theta \text{Sh}\ t\ d\text{Ch}t = 2\int_0^\Theta \text{Sh}^2 t\ dt$$

$$= 2\int_0^\Theta (\text{Ch}\ 2t - 1)dt = \frac{1}{2}\ \text{Sh}\ 2\Theta - \Theta,$$

we get

$$|OP'AP| = \text{Ch}\Theta\ \text{Sh}\ \Theta - \frac{1}{2}\ \text{Sh}\ 2\Theta + \Theta = \Theta$$

C. Inverse Hyperbolic Functions.

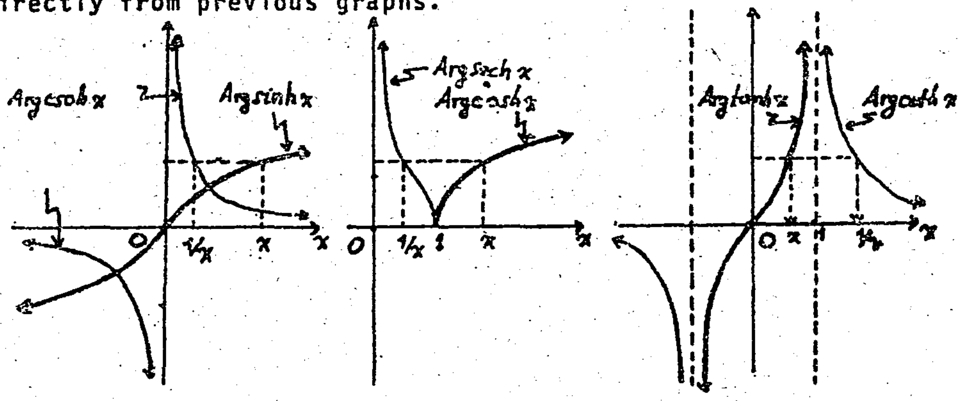
Observing from the graphs of hyperbolic functions th all these, except Ch x and Sech x, are monotone increasin or monotone decreasing on their domain, while[1] Ch x (Sech

(1) Since Ch x (Sech x) is monotone decreasing (increasing) in $(-\infty, 0)$ also inverse in that interval and the graphs is symmetric of the gi one w.r. to x-axis.

is monotone increasing (decreasing) in $(0, \infty)$. Hence they all have inverse functions at these intervals.

We use prefix "Arg" to name inverse functions.

Below we show the graphs of inverse functions obtained directly from previous graphs.



Graphs of inverse hyperbolic functions

In each graph a horizontal line is drown cutting the two curves at points whose abscissas are certainly reciprocal of each other. This means that .

$$\text{Argsinh } x = \text{Argcsch } \frac{1}{x}$$

$$\text{Argcosh } x = \text{Argsech } \frac{1}{x}$$

$$\text{Argtanh } x = \text{Argcoth } \frac{1}{x}$$

<u>Logarithmic expressions of inverse hyperbolic functions.</u>

<u>Theorem.</u>

1. $\text{Argsinh } x = \ln(x + \sqrt{x^2 + 1}), \quad x \in R$
2. $\text{Argcosh } x = \ln(x + \sqrt{x^2 - 1}), \quad x \geq 1$
3. $\text{Argtanh } x = \frac{1}{2} \ln \frac{1 + x}{1 - x}, \quad |x| < 1$
4. $\text{Argcoth } x = \text{Argtanh } \frac{1}{x} = \frac{1}{2} \ln \frac{x + 1}{x - 1}, \quad |x| > 1$

5. Argsech x = Argcosh $\frac{1}{x}$ = $\ln \frac{1 + \sqrt{1 - x^2}}{x}$, $0 < x < 1$

6. Argcsch x = Argsinh $\frac{1}{x}$ = $\ln(\frac{1}{x} + \frac{\sqrt{1 + x^2}}{|x|})$, $x \neq 0$

Proof.

1. $y = \text{Argsinh } x \Rightarrow x = \text{Sinh } y = \frac{1}{2}(e^y - e^{-y})$

$\Rightarrow e^y - e^{-y} - 2x = 0 \Rightarrow e^{2y} - 2x\, e^y - 1 = 0$

$\Rightarrow e^y = x \pm \sqrt{x^2 + 1}$, $(e^y > 0)$ $\quad e^y = x + \sqrt{x^2 + 1}$

$\Rightarrow y = \ln(x + \sqrt{x^2 + 1})$.

2. Proved similarly.

3. $y = \text{Argtanh } x \Rightarrow x = \text{Tanh } y = \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{e^{2y} - 1}{e^{2y} + 1}$

$e^{2y} - 1 = x\, e^{2y} + x \Rightarrow (1 - x)e^{2y} = 1 + x$

$e^{2y} = \frac{1 + x}{1 - x}$, $(e^{2y} > 0) \Rightarrow 2y = \ln \frac{1 + x}{1 - x}$, $|x| < 1$.

The proofs of the other three can be proved similarly.

Derivatives and Integrals

Theorem.

1. $D \text{ Argsinh } x = \frac{1}{\sqrt{x^2 + 1}}$

2. $D \text{ Argcosh } x = \frac{1}{\sqrt{x^2 - 1}}$

3. $D \text{ Argtanh } x = \frac{1}{1 - x^2}$, $|x| < 1$

4. $D \text{ Argcoth } x = \frac{1}{1 - x^2}$, $|x| > 1$

5. $D \text{ Argsech } x = -\frac{1}{x\sqrt{1 - x^2}}$

6. $D \text{ Argcsch } x = -\frac{1}{|x|\sqrt{1 + x^2}}$

Proof.

1. $D \text{ Argsinh } x = D \ln(x + \sqrt{x^2 + 1}) = \frac{1 + \frac{x}{\sqrt{x^2 + 1}}}{x + \sqrt{x^2 + 1}} = \frac{1}{\sqrt{x^2 + 1}}$

The others are proved similarly.

Corollary. (Integrals leading to inverse hyperbolic function).

$$\int \frac{dx}{x^2+1} = \text{Argsinh } x + c = \ln(x + \sqrt{x^2+1}) + c$$

$$\int \frac{dx}{x^2-1} = \text{Argcosh } x + c = \ln(x + \sqrt{x^2-1}) + c$$

$$\int \frac{dx}{1-x^2} = \begin{cases} \text{Argtanh } x + c, & |x| < 1 \\ \text{Argcoth } x + c, & |x| > 1 \end{cases} = \frac{1}{2} \ln\left|\frac{1+x}{1-x}\right| + c$$

$$\int \frac{dx}{x\sqrt{1+x^2}} = -\text{Argcsch } |x| + c = -\text{Argsinh } \frac{1}{|x|} + c$$

$$\int \frac{dx}{x\sqrt{1-x^2}} = -\text{Argsech } |x| + c = -\text{Argcosh } \frac{1}{|x|} + c$$

Example. Find the derivatives of the following functions at $x = \pi/3$.

a) $f(x) = \text{Argcoth } (\text{Sec } x)$ b) $g(x) = \text{Argsinh } (\tan x)$

Solution.

a) $f'(x) = \frac{1}{1-\text{Sec}^2 x} D \text{ Sec } x = \frac{1}{-\tan^2 x} \cdot \text{Sec}x \tan x = -\csc x$

$f'(\pi/3) = -\csc \pi/3 = -2/\sqrt{3}$.

b) $g'(x) = \frac{1}{\sqrt{1+\tan^2 x}} \cdot D \tan x = \frac{1}{\text{Sec } x} \cdot \text{Sec}^2 x = \text{Sec } x$

$g'(\pi/3) = \text{Sec } \pi/3 = 2$.

Example 2. Evaluate

a) $A = \int \frac{dx}{\sqrt{9+x^2}}$ b) $B = \int_2^3 \frac{dx}{1-x^2}$

Solution.

a) Setting $x = 3t$, $dx = 3\,dt$,

$$A = \int \frac{3\,dt}{3\sqrt{1+t^2}} = \ln(t + \sqrt{t^2+1}) + c_1$$

$$= \ln\left(\frac{x}{3} + \sqrt{\frac{x^2}{9}+1}\right) + c_1 = \ln(x + \sqrt{x^2+9}) + c$$

b) $B = \int_2^3 \frac{dx}{1-x^2} = \frac{1}{2} \ln \left|\frac{1+x}{1-x}\right|_2^3 = \frac{1}{2} \ln \frac{4}{2} - \ln \frac{3}{1}$

$= \frac{1}{2} (\ln 2 - \ln 3) = \ln \sqrt{2/3}.$

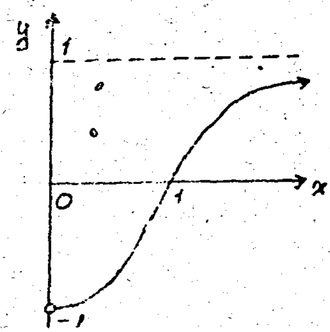
<u>Example 3</u>. Sketch: $y = \text{Tanh} (\ln x)$

<u>Solution</u>.

$$y \quad \text{Tanh}(\ln x) = \frac{e^{\ln x} - e^{-\ln x}}{e^{\ln x} + e^{-\ln x}} = \frac{x - \frac{1}{x}}{x + \frac{1}{x}} = \frac{x^2 - 1}{x^2 + 1}, \quad (x > 0).$$

$$y' = \frac{4x}{(x^2+1)^2} \begin{cases} > 0 & \text{if } x > 0 \\ = 0 & \text{if } x = 0 \\ < 0 & \text{if } x < 0 \end{cases}, \quad y(0) = -1$$

$\lim\limits_{|x| \to \infty} \frac{x^2 - 1}{x^2 + 1} = 1 \Rightarrow y = 1$ is a horizontal asymptote



EXERCISES (6, 2)

46. Compute the values of six hyperbolic functions at $x = \ln 7$.

47. Express Sh 4x, Ch 4x in terms of Sh x and Ch x.

48. Find the domain of definition and derivative of the following functions.

a) $y = \text{Sh}(x^2 + 2x)$ b) $y = \text{Sech}(x^2 - 2x)$

49. Find $\lim\limits_{x \to 0} \frac{\ln \text{Ch } x}{\text{Th } x}$

50. Show that the following functions are constant:

a) $y = \text{Ch } \ln x + \text{Sh } \ln x - x$ b) $y = \ln \sqrt{\frac{1 + \text{Th } x}{1 - \text{Th } x}} - x$

c) $y = (\text{Cos } x)^{1/\ln \cos x} + \sin x$

51. Evaluate

a) Ch(n Argth x) b) Sh(n Argth x) c) Th(n Argth x)

in terms of x and tell in which case these functions are rational.

52. Evaluate

a) D Argcoth (sec x) b) D Argtanh (Cos x)

53. Sketch the graph of the following functions:

a) $y = \text{Sinh } (x^2 + 2x)$ b) $y = \text{Sech } (x^2 - 2x)$

54. Same question for

a) $y = \text{Cosh } (\ln x)$ b) $y = \text{Tanh } (\ln x)$

55. Prove that a line intersects the curve of

a) y = Ch x at two points at most,

b) y = Sh x at three points at most,

c) y = Th x at three points at most.

56. Verify the identity

$$4 \text{ Ch}^4x + \text{Ch}^3x \text{ Sh } x + \text{Ch } x \text{ Sh}^3x + \text{Sh}^4x = e^{4x} + 3$$

57. Differentiate

a) $y = \text{Argsh } (a \text{ Sh } x + b \text{ Ch } x)$ with $a^2 - b^2 = 1$

b) $y = \text{Argch } (a \text{ Ch } x + b \text{ Sh } x)$ with $a^2 - b^2 = 1$

58. Solve the equation:

a) $2 \text{ Ch}^2x + 5 \text{ Ch } x \text{ Sh } x + 3 \text{ Sh}^2x = 7$

b) $32 \text{ Ch } x \text{ Sh } x - 6 (\text{Ch } x + \text{Sh } x) - 6 = 0$

59. Same question for

a) $2 \text{ Argsh } x + \text{Argth } \frac{x}{2} = \text{Argch } 3.$

b) $\text{Argsh } x + \ln(a + \sqrt{a^2 - 1}) = \text{Argsh } a + \ln (x + \sqrt{x^2 - 1}).$

60. Same question for:

a) $(\text{Ch } x + \text{Sh } x)^{-\text{Argch } x} = (\text{ch } x - \text{Sh } x)^{\text{Argsh}(x - 2)}$

b) $\text{Ch } 7x + \text{Ch } 5x + \text{Ch } 3x = \text{Ch } 6x + \text{Ch } 4x + \text{Ch } 2x$

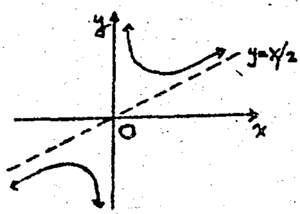
ANSWERS TO EVEN NUMBERED EXERCISES

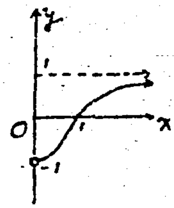
6. 24/7, 25/7, 24/25, 7/25, 7/24, 25/24.

48. a) $\mathbb{R}$, $(2x + 2)\, \text{Ch}(x^2 + 2)$,

b) $\mathbb{R}$, $-(2x - 2)\, \text{Sech}(x^2 - 2x)\, \text{Th}(x^2 - 2x)$

52. a) $-\csc x$, b) $\csc x$

54. a) 

b) 

58. a) $\ln \sqrt{3}$ b) 0

60. a) 5/4 b) 0

A Summary

5. 1 $\ln x = \int_1^x \frac{dt}{t}$ $(x > 0)$, $\ln 1 = 0$, $\ln e = 1$

$\ln ab = \ln a + \ln b$, $\ln \frac{a}{b} = \ln a - \ln b$

$\log_b x = \log_a x \log_b a$ (change of base)

$\frac{d}{dx} a^{u(x)} = a^u \frac{du}{dx} \ln u$, $\frac{d}{dx} \log_a u(x) = \frac{1}{u} \frac{du}{dx} \log e$

$\frac{d}{dx} u(x)^{v(x)} = u^v \left(\frac{dv}{dx} \ln u + \frac{v}{u} \frac{du}{dx} \right)$

$y = uv \ldots w \Rightarrow \frac{y'}{y} = \frac{u'}{u} + \frac{v'}{v} + \ldots + \frac{w'}{w}$ (logarithmic derivative

6.2 $\text{Ch } x = \frac{1}{2}(e^{x}+e^{-x})$, $\text{Sh } x = \frac{1}{2}(e^{x}-e^{-x})$, $\text{Th } x = \text{Sh } x/\text{Ch } x$

$\text{Ch}(\alpha \pm \beta) = \text{Ch}\alpha \text{ ch}\beta \pm \text{Sh}\alpha \text{ Sh}\beta$, $\text{Ch } 2\alpha = \text{Ch}^2\alpha + \text{Sh}^2\alpha$

$\text{Sh}(\alpha \pm \beta) = \text{Sh}\alpha \text{ ch}\beta \pm \text{Ch}\alpha \text{ Sh}\beta$, $\text{Sh } 2\alpha = 2 \text{ Sh}\alpha \text{ Ch}\alpha$

$\text{Th}(\alpha \pm \beta) = \frac{\text{Th}\alpha \pm \text{Th}\beta}{1 \pm \text{Th}\alpha \ \text{Th}\beta}$, $\text{Th } 2\alpha = \frac{2 \text{ Th}\alpha}{1 + \text{Th}^2\alpha}$

$D \text{ Sinh } x = \text{Cosh } x$, $D \text{ csch } x = -\text{Csch } x \text{ Coth } x$

$D \text{ Cosh } x = \text{Sinh } x$, $D \text{ sech } x = -\text{Sech } x \text{ Tanh } x$

$D \text{ Tanh } x = \text{Sech}^2 x$, $D \text{ coth } x = -\text{csch}^2 x$

$\text{Argsinh } x = \ln(x + \sqrt{x^2+1})$, $x \in \mathbb{R}$

$\text{Argcosh } x = \ln(x + \sqrt{x^2-1})$, $x \geq 1$

$\text{Argtanh } x = \frac{1}{2} \ln \frac{1+x}{1-x}$, $x < 1$

$\text{Argcoth } x = \text{Argtanh } \frac{1}{x} = \frac{1}{2} \ln \frac{x+1}{x-1}$, $|x| > 1$

$\text{Argsech } x = \text{Argcosh } \frac{1}{x} = \ln \frac{1+\sqrt{1-x^2}}{x}$, $0 < x < 1$

$\text{Argcsch } x = \text{Argsinh } \frac{1}{x} = \ln \left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)$, $x \neq 0$

MISCELLANEOUS EXERCISES

61. Prove $\int_0^1 x^{\alpha} \ln^n x \, dx = \frac{n!}{(\alpha+1)^{n+1}}$ $(n \in N, \quad \alpha \neq -1)$

62. Prove

a) $x - \frac{1}{2}x^2 < \ln(1+x) < x$ for $x > 0$

b) $1 + x < e^x < (1+x)(1+x^2)$ for $x > 0$ near 0.

63. Evaluate

a) $\lim_{n \to \infty} \sqrt[n]{\frac{n(n+1) \ldots (n+m)}{n^n}}$, $m \in N_1$

b) $\lim_{n \to \infty} \left[\ln(n!) - n \ln \frac{n}{e} - \frac{1}{2} \ln n\right]$

64. Find the family of curves having normal lines passing through

the origin.

65. Find the function with the given condition:

a) $f'(x) = 3x\, f(x)$, $f(0) = e$, b) $f'(x) = \frac{1}{\sqrt{x}}\, f(x)$, $f(3) = 1$

66. Evaluate $\int x^n \ln x \, dx$ $(n \in \mathbb{N})$

67. Sketch the graph of the following functions:

a) $y = \ln \frac{x}{x+1}$ b) $y = \ln \sin x$

c) $y = \ln \frac{x^2 - 3x}{x + 1}$ d) $y = \ln \sqrt{\frac{x + 1}{x - 2}}$

68. Sketch the graph of

a) $y = x^2 e^{2x}$ b) $y = \frac{\ln x}{x}$

69. Same question for:

a) $y = \ln(1 + x^2) - \arctan x$, b) $y = \frac{x^2 \ln x - x}{x - 1}$

70. a) $y = \frac{x \ln^2 x}{1 + \ln x}$ b) $y = \ln \sqrt{\frac{1 - \cos x}{1 + \cos x}}$

71. Prove $1 - x^2 < e^{-x^2} < \frac{1}{1 + x^2}$

72. Differentiate $f(x) = 2 \frac{\arcsin x}{1 - x^2} + \ln \frac{1 - x}{1 + x}$

73. If $f(x) = \ln(\sqrt{x + a} + \sqrt{x - a})$ evaluate

a) $D\, f(x)$ b) $\int f(x) dx$

74. Evaluate

a) $\int \frac{1}{\ln x} dx$ b) $\int \ln(\ln x) dx$ c) $\int e^{x^2} dx$

75. Sketch:

a) $y = x \exp\left(\frac{1}{2} \ln x^2\right)$ b) $y = \ln \sin(n \arctan x)$, $(n \in \mathbb{N}^+)$

76. Show that the chord of $y = \log_a x$ with end points having abscissas x and $1/x$ is bisected by x-axis.

77. If $f(x)$, $g(x)$ and $h(x)$ are continuous functions then prove the implications:

a) $f(x)\, f(y) = f(x + y) \Rightarrow f(x) = a^x$

b) $g(x) + g(y) = g(xy) \Rightarrow g(x) = \log_a x$

c) $h(x+y) = \dfrac{h(x) + h(y)}{1+h(x)h(y)} \Rightarrow$? (Hint: Show that $\dfrac{1 + h(x)}{1 - h(x)}$ is exponential)

78. Solve the following equations:

a) $4^{2x} - 3.4^x + 2 = 0$ b) $2^x + 2^{-x} = \dfrac{15/4}{2^x - 2^{-x}}$

79. Solve the inequalities:

a) $\log_a(x^2 + 5x) - \log_a(x + 2) < 0$ b) $\log_a \sqrt{\dfrac{x + 1}{x - 1}} > 1$

80. Same question for:

a) $9^x - 3x + 2 > 0$ b) $4^{\frac{x^2}{x + 3}} < 2$

81. Sketch the graph of the relation:

a) $3^x . 9^y < 1$ b) $4^{x + y} . 2^x > 2^6$

82. Solve for x:

a) $\log_2(x + 2) \pm \log_8(x + 8) = \log_8 7$

b) $\log_{10}(x^2 + 5x - 26) + \log_{1/10} x = 2 - \log_{10} 9$

83. Solve for x and y:

a) $\log_2 x + \log_4 y = 5, \quad \log_4 x - \log_2 y = 1$

b) $2^{\frac{x + 2y}{3}} = 4, \quad 4^{x - y} = 2$

84. Solve for x:

a) $2^{x + 1} + \dfrac{5}{2^{x - 1}} + 1 = 0$

b) $\ln(x^2 + 3x + 4) - \ln(x^2 - 4x + 6) = \ln 7$

c) $\ln(x - 3) + \ln(x + 5) = \ln(2x + 1)$

85. Find the domain of definition and derivative:

a) $y = \log_8 [x^2]$ b) $y = \log_6 \log_6 (x + 2)$

86. Evaluate

a) $\int 2^x \cos 2^x \, dx$ b) $\int 6^x \sin x \, dx$

87. Sketch the graph of:

a) $y = \exp_2 \frac{x^2 - 4}{x + 1}$ b) $y = \exp_6 \sqrt[3]{\frac{x^2}{x-1}}$

88. Find the domain of definition of the following functions:

a) $y = 3^{\ln x}$ b) $y = 5^{[x]}$

c) $y = -2^x$ d) $y = \exp_7 \sqrt{\frac{x^2 - 4}{x + 1}}$

89. Evaluate

a) $\lim_{x \to 1} x^{\frac{1}{1 - x^2}}$ b) $\lim_{x \to \infty} (1 - 2x)^{3/x}$

c) $\lim_{x \to 0} (\frac{\sin x}{x})^{1/x^2}$ d) $\lim_{x \to 0} \frac{x - \text{Argth } x}{x - \sin x}$

90. Sketch the curve of $y = (x^2 - 4)^{1/x}$

91. Evaluate

a) $\lim_{x \to \infty} \left[\frac{\pi}{2} - \arctan x\right]^{1/x}$ b) $\lim_{x \to 0} (-\ln x)^x$

c) $\lim_{x \to 0} \left[\frac{\tan x}{x}\right]^{1/x^2}$ d) $\lim_{x \to \pi/2} (\tan x)^{\sin 2x}$

92. Evaluate

a) $\lim_{x \to \infty} (\frac{x + 1}{x - 1})^x$ b) $\lim_{x \to \pi/4} (\tan x)^{\tan 2x}$

c) $\lim_{x \to \infty} \left[\sin \frac{\pi x}{2x + 1}\right]^{x^2}$ d) $\lim_{x \to 0} (\cos x)^{\cot^2 x}$

e) $\lim_{x \to 0} (\cos x - x \sin x)^{1/x^2}$ f) $\lim_{x \to 1} (x-1) \ln \cos(\frac{\pi}{2} x)$

93. Evaluate

a) $\lim_{x \to 0} x^x$ b) $\lim_{x \to 0} x^{x^x}$ c) $\lim_{x \to 0} \frac{x^x - 1}{x^{x^x} \ln x}$

94. Evaluate

a) $\lim_{x \to \infty} \left[\sqrt{e^{2x}+2}-\sqrt{e^{2x}-1}\right]$ b) $\lim_{x \to 0} \frac{1}{x} \left[1 - (\frac{x}{x+1})^x\right]$

c) $\lim_{x \to \infty} \frac{x}{2+3x^2} \ln(2+3x)$ d) $\lim_{x \to 0} \frac{1-\cos(n \arcsin x)}{\ln(1 + x^2)}$

95. Sketch the graph of:

a) $y = a^x - x$ b) $y = a^x/x$ c) $y = x - \log_a x$

d) $y = (\log_a x)/x$ e) $y = x - \ln(\ln x)$ f) $y = [\ln(\ln x)]/x$

96. Evaluate $\lim_{x \to 0} \frac{(1 + \sin x)^{1/x} - e}{(1 + \tan x)^{1/x} - e}$

97. Sketch the graph of:

a) $y = x^{\frac{x}{x+1}}$ b) $y = (\frac{x}{x-1})^x$

98. Same question for:

a) $y = \frac{x^2 \ln x - x}{x - 1}$ b) $y = \frac{x + \ln x}{x}$

99. Setting $x = e^\Theta$ compute $\text{ch } n\Theta$ interms of $u = x + \frac{1}{x}$, and then obtain $\text{ch } n\Theta$ as a function of $\text{ch } \Theta$.

100. Compute the value of the other five hyperbolic function if $\text{Th } x = 2/3$.

101. Evaluate

$$(x^2D^2 + xD + 1)\ [\text{Sin}(\ln x) + \cos(\ln x)]$$

102. Evaluate

a) $\int x^4 \ln^4 x\, dx$ b) $\int x^2 \text{ch } x\, dx$

103. Sketch the graph of:

a) $y = \text{Coth}(x^2 - 4x)$ b) $y = \text{Sinh } 3^{x^2+4x}$

104. Same question for

a) $y = \text{Sh } \ln \frac{x}{x-3}$ b) $y = \text{Th } x^x$

105. Find the domain of definition and derivative of the following functions:

a) $y = \text{Sh } \ln \frac{x}{x-3}$ b) $y = \text{Th } x^x$

106. Evaluate $\lim_{x \to 0} \frac{\ln \text{Ch}(n \text{ Argth } x)}{x^2}$

107. Prove that $f(h) = n(x^{1/n} - 1)$ is a monotononic function of

n. For $x > 0$ deduce that $\lim_{n \to \infty} f(n)$ exists. Prove also that if $\phi(x) = \lim_{n \to \infty} n(x^{1/n} - 1)$, then $\phi(x)$ satisfies the functional relations

a) $\phi(xy) = \phi(x) + \phi(y)$ b) $\phi(x) = -\phi(1/x)$

108. Given n positive numbers $a_1, \ldots, a_n$, then the numbers A, G, Q, H defined by

$$A = \frac{a_1 + \ldots + a_n}{n}, \quad G = \sqrt[n]{a_1 \ldots a_n}, \quad Q = \sqrt{\frac{a_1^2 + \ldots + a_n^2}{n}},$$

$$\frac{n}{H} = \frac{1}{a_1} + \ldots + \frac{1}{a_n}$$

are called the *arithmetic mean, geometric mean, quadratic mean* (Root Mean Square, RMS), and *harmonic mean* of $a_1, \ldots, a_n$ respectively.

In general the number

$$g_\alpha = \left(\frac{a_1^\alpha + \ldots + a_n^\alpha}{n}\right)^{1/\alpha} \qquad (\alpha \neq 0)$$

is called the *mean of order* α of $a_1, \ldots, a_n$ and one has $A = g_1$, $Q = g_2$, and $H = g_{-1}$. Prove that

$$G = g_0 = \lim_{\alpha \to 0} g_\alpha .$$

109. Evaluate

a) $\lim_{x \to \infty} (x \ln x) \left[\left(\frac{\ln(x+1)}{\ln x}\right)^p - 1\right]$

b) $\lim_{x \to 0} \frac{\ln \operatorname{ch} x}{\operatorname{Th} x}$ c) $\lim_{x \to 0} \frac{1}{x} \left[\exp\left(\frac{1}{x} \ln \frac{\operatorname{ch} x}{\cos x}\right) - e\right]$

110. Solve for x:

a) $2 \operatorname{Argsh} x + \operatorname{Argth} \frac{1}{2} = \operatorname{Argch} 3$

b) $\operatorname{Argth} x + \ln(a + \sqrt{a^2 - 1}) = \operatorname{Argsh} a + \ln(x + \sqrt{x^2 - 1})$

111. Solve the system:

a) $\operatorname{Ch} x + \operatorname{Ch} y = a \operatorname{ch} \alpha$, $\operatorname{Sh} x + \operatorname{Sh} y = a \operatorname{sh} \alpha$

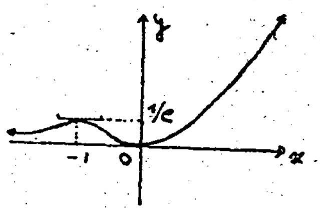
b) Argsh x = Argch y,   x ℓn x = 3 ℓn y.

ANSWERS TO EVEN NUMBERED EXERCISES

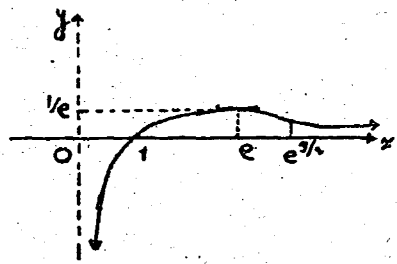
64. $x^2 + y^2 = c^2$

66. $\frac{x^{n+1}}{n+1} (\ln x - \frac{1}{n+1}) + c$

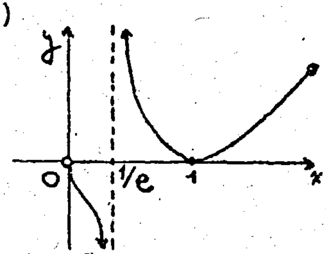
68. a)



b)



70. a)



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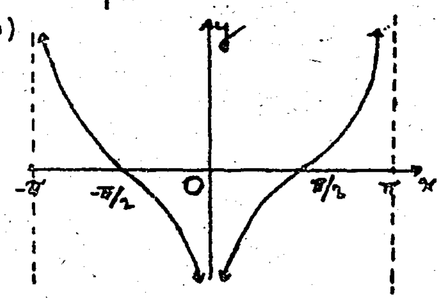


72. $2x(\arcsin x)/(1 - x^2)$

74. a), b), c) are impossible.

78. a) 0; 1/2, b) 1

80. a) $x \in R$, b) $x \in (-\infty, -3)$ or $x \in (-1, 3/2)$

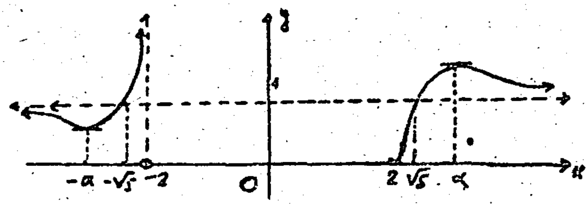
82. a) -1, b) 9

84. a) 1, b) 2; 19/6, c) 4

86. a) $(\sin 2^x) \log_2 e + c$, b) $\frac{6x}{1 + \ln^2 6} \left[(\ln 6) \sin x - \cos x\right] + c$

88. a) $\mathbb{R}^+$, b) $\mathbb{R}$, c) $\mathbb{R}$, d) $(-2, -1) \cup (2, \infty)$

90

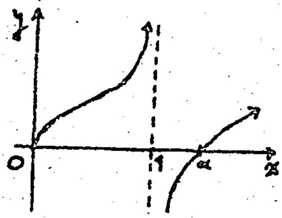


92. a) e^2, b) e^{-1}, c) $e^{-\pi^2/32}$, d) $e^{-1/2}$ e) $e^{-3/2}$, f) 1

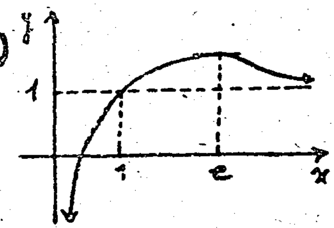
94. a) 0, b) No limit (∞), c) 0 d) $n^2/2$.

96. 1.

98. a)



b)

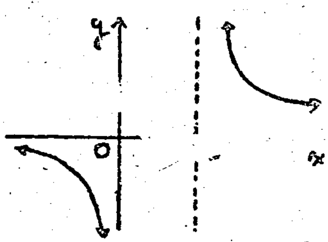


100. $2/\sqrt{5}$, $3/\sqrt{5}$, $\sqrt{5}/3$, $\sqrt{5}/2$, $3/2$.

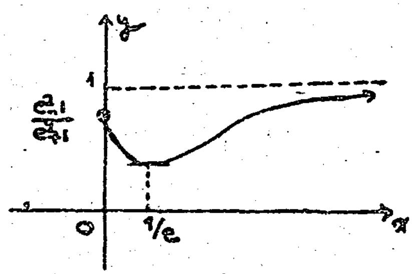
102. a) $\frac{x^5}{5}\left(\ell n^4 x - \frac{4}{5}\ell n^3 x + \frac{12}{25}\ell n\ x - \frac{12}{125}\right) + c$

b) $x^2 \mathrm{Sh}\ x - 2x\ \mathrm{Ch}\ x + 2\ \mathrm{Sh}\ x + c$

104. a)



b)



106. $n^2/2$.

110. a) $\pm\sqrt{22 + 6\sqrt{8}}/4$, b) a

CHAPTER 7

TECHNIQUES OF INTEGRATION

7.1 INTEGRATION OF RATIONAL FUNCTIONS BY PARTIAL FRACTIONS

As we shall see many integrals are reducible, by some suitable substitutions, to the integral of rational functions. Evaluation of such integrals is done by the use of a theorem stating the decomposition of a rational function into a sum of some number of simpler ones.

A. Definitions.

Let

$$R(x) = \frac{P(x)}{Q(x)} = \frac{a_0x^m + \dots + a_{m-1}x + a_m}{b_0x^n + \dots + b_1 x + b_n} \qquad (1)$$

be a rational function where $\deg P(x) = m$ and $\deg Q(x) = n$. For $n = 0$, the rational function being a polynomial, we suppose $n \geq 1$.

If $m \geq n$, then the fraction $R(x)$ is called *improper*, otherwise *proper*. We also suppose that $R(x)$ is a *reduced* one, that is, $P(x)$ and $Q(x)$ have no common factor.

A proper fraction having the form

$$\frac{A}{(x-a)^n} \quad \text{or} \quad \frac{Bx + C}{(x^2 + px + q)^n}$$

where A, B, C are constant and $x^2 + px + q$ has imaginary conjugate roots $(\Delta = p^2 - 4q < 0)$, is called a *partial fraction*. Thus, the proper fractions

$$\frac{2}{x-1}, \quad \frac{-4}{(x+2)^3}; \quad \frac{2x+3}{x^2+1}, \quad \frac{-5x}{(x^2+1)^3}, \quad \frac{-3}{x^2+x+1}, \quad \frac{x-3}{(x^2+x+1)^7}$$

are partial fractions, while the proper fractions

$$\frac{x}{(x+2)^3};\ \frac{x^2+3}{(x^2+1)^2},\ \frac{2x+1}{x^2-1},\ \frac{2x^3+1}{(x^2+x+1)^7}$$

are not.

When $R(x)$ is an improper one, we perform division of $P(x)$ by $Q(x)$ to have

$$P(x) = q(x)\ Q(x) + r(x)$$

where either $r(x) = 0$ or $\deg r(x) < \deg Q(x)$. If $r(x) = 0$, $R(x)$ is the polynomial $q(x)$. Otherwise we obtain

$$\frac{P(x)}{Q(x)} = q(x) + \frac{r(x)}{Q(x)}$$

which is sum of a polynomial and a proper fraction.

<u>Example</u>. Decompose the improper rational function

$$\frac{P(x)}{Q(x)} = \frac{2x^4+5x^3+2x^2+21}{x^3-2x+5}$$

into a polynomial and a proper fraction.

<u>Solution</u>. Ordinary division gives

$$2x^4+5x^3+2x^2+21 = (2x+5)(x^3-2x+5)+6x^2-4$$

or

$$\frac{P(x)}{Q(x)} = 2x+5+\frac{6x^2-4}{x^3-2x+5}$$

where the new-fraction is a proper one.

If an improper fraction $R(x) = P(x)/Q(x)$ is decomposed into

$$q(x) + \frac{r(x)}{Q(x)}$$

where $q(x)$ is a polynomial and $r(x)/Q(x)$ is a proper fraction, then the integral $\int R(x)dx$ is reduced to that of a polynomial and of a proper fraction $r(x)/Q(x)$. Thus

$$\int \frac{2x^4+5x^3+2x^2+23}{x^3-2x+5}\,dx = \int (2x+5)dx \qquad \frac{6x^2-4}{x^3-2x+5}\,dx$$

The evaluation of the second integral is based on the decomposition of a proper fraction into partial fractions which we discuss below:

B. Decomposition of a Proper Fraction into Partial Fractions.

Let $r(x)/Q(x)$ be a proper fraction. If $r(x) = k\,\frac{dQ(x)}{dx}$, then evaluation of $\int r(x)dx/Q(x)$ is immediate. So we suppose this is not the case.

The decomposition of a proper fraction $r(x)/Q(x)$ into partial fractions, that is, writing it as the sum of certain number of partial fractions is possible only when $Q(x)$ is written in the factored form:

$$Q(x) = k.(x-a)^{\alpha}\ (x-b)^{\beta} \ldots (x^2+px+q)^{\lambda}\ (x^2+rx+s)^{\mu} \ldots \quad (*)$$

with $\alpha, \beta, \ldots, \lambda, \mu, \ldots \varepsilon N_1$, which is theoretically possible, where $a, b, \ldots$ are real roots with multiplicities $\alpha, \beta, \ldots$ respectively, and x^2+px+p, $x^2+rx+s, \ldots$ have pairs of imaginary roots $(a \pm ib,\ b \neq 0)$ with multiplicities $\lambda, \mu, \ldots$ respectively, so that the second degree factors have negative discriminants.

Now we state the theorem on decomposition of a proper rational function into partial fractions:

<u>Theorem</u>. A proper rational function $r(x)/Q(x)$ with $Q(x)$ having the factorization (*) above can be written uniquely in the form

$$\left(\frac{A_1}{x-a} + \ldots + \frac{A_\alpha}{(x-a)^\alpha}\right) + \left(\frac{B_1}{x-b} + \ldots + \frac{B_\beta}{(x-b)^\beta}\right) + \ldots$$

$$+(\frac{C_1x+D_1}{x^2+px+q}+\dots+\frac{C_\lambda x+D_\lambda}{(x^2+px+q)^\lambda})+(\frac{E_1x+F_1}{x^2+rx+s}+\dots+\frac{E_\mu x+F_\mu}{(x^2+rx+s)^\mu})+\dots$$

of finite number of partial fractions with $A_\alpha \neq 0$, $B_\beta \neq 0, \dots,$ $C_\lambda x + D_\lambda \neq 0$, $E_\mu x + F_\mu \neq 0$.

Proof. See Appendix at the end of the book. ∎

By this theorem, in the decomposition, to a real root of multiplicity ν correspond ν partial fractions (some of which may be zero), and to a pair of conjugate imaginary roots of common multiplicity ν correspond ν partial fractions (some of which may be zero).

For instance

1. $$\frac{x^6-2x+5}{(x-1)x^3(x^2+1)^2} = \frac{A}{x-1} + (\frac{B_1}{x}+\frac{B_2}{x^2}+\frac{B_3}{x^3})+(\frac{C_1x+D_1}{x^2+1}+\frac{C_2x+D_2}{(x^2+1)^2})$$

2. $$\frac{3}{(x-1)^2} = \frac{A}{(x-1)^2} \quad (\Rightarrow A = 3, \text{ why?})$$

3. $$\frac{2x^2-7}{(x^2-x+1)^3} = \frac{A_1x+B_1}{x^2-x+1} + \frac{A_2x+B_2}{(x^2-x+1)^2} + \frac{A_3x+B_3}{(x^2-x+1)^3}$$

We remark that as in (2) above, the decomposition of a partial fraction consists of a single term which is the given fraction.

Example 1. Decompose the proper rational function

$$\frac{x^2+15}{(x-3)(x^2-2x-3)}$$

into partial fractions.

Solution. Since x^2-2x-3 has positive discriminant, it can be facroted: $x^2-2x-3=(x-3)(x+1)$.

Thus the given fraction is to be written as

$$\frac{x^2 + 15}{(x - 3)^2(x + 1)}$$

and according to the theorem we have the decomposition

$$\frac{x^2 + 15}{(x - 3)^2(x - 1)} = \frac{A}{x-3} + \frac{B}{(x - 3)^2} + \frac{C}{x+1}$$

which is an identity in x.

To determine the constants A, B, C one clears of denominators and sets the identity:

$$x^2 + 15 = A.(x - 3)(x + 1) + B(x + 1) + C.(x - 3)^2$$

The coefficients are usually obtained in two ways:

1) By arranging the right hand side in powers of x and equating the coefficients of like terms. This is called the method of *undetermined coefficients* leading to certain number of linear equations:

$$x^2 + 15 = (A + C)x^2 + (-2A + B - 6C)x - 3A + B + 5C$$

$$A + C = 1, \quad -2A + B - 6C = 0, \quad -3A + B + 5C = 15$$

The solution of this system is $A = 0$, $B = 6$, $C = 1$ and we have

$$\frac{x^2 + 15}{(x - 3)(x^2 - 2x - 3)} = \frac{6}{(x - 3)^2} + \frac{1}{x + 1}$$

2) By substitution:

Setting in the identity

$$x^2 + 15 = A(x - 3)(x + 1) + B(x + 1) + C(x - 3)^2$$

the values $x = 3$, $x = -1$ which make zero some terms we have

for $x = 3$: $\quad 24 = 0 + 4B + 0 \qquad B = 6$

for $x = -1$: $\quad 16 = 0 + 0 + 16C \qquad C = 1$

and then setting any other value, say $x = 0$, we have

$$15 = -3A + B + 9C \qquad A = 0$$

We see that the second method gives the coefficients quickly.

Example 2. Decompose the following into partial fractions.

$$\frac{-4x^2 + x - 1}{(x^2 - x)(x^2+1)^2}$$

Solution. It is a proper fraction with denominator

$$x(x - 1)(x^2 + 1)^2$$

Hence by the theorem it has the decomposition

$$\frac{-4x^2+x-1}{x(x-1)(x^2+1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx + D}{x^2 + 1} + \frac{Ex + F}{(x^2+1)^2}$$

which when cleared of denominators gives the identity

$$-4x^2+x-1 = \left[(A+B)x-A\right](x^2+1)^2+x(x-1)\left[(Cx+D)(x^2+1)+Fx+F\right]$$

Now,

$$\left.\begin{aligned} x &= 0: \quad -1 = -A \\ x &= 1: \quad -4 = B \end{aligned}\right\} \Rightarrow A = 1, \quad B = -1$$

Since $i = \sqrt{-1}$ $(i^2 = -1)$ is a root if $x^2 + 1 = 0$,

$$x = i: \ -4i^2 + i - 1 = 0 + i(i - 1)(0 + Ei + F)$$

$$\Rightarrow \quad 4 + i - 1 = (-1 - i)(F + iE)$$

$$3 + i = (E - F) - (E + F)i$$

implying $E - F = 3$, $E + F = -1$ $\Rightarrow$ $E = 1$, $F = -2$. Then the identity identity becomes

$$-4x^2 + x - 1 = -(x^2 + 1)^2 + x(x - 1)\left[(Cx + D)(x^2 + 1)+(x - 2)\right]$$

$$x = -1: \ -6 = -4 + 2\left[(-C + D)\,2 - 3\right] \quad \Rightarrow \quad C - D = -1$$

$$x = 2: \ -15 = -25 + 2(2C + D)5 \quad \Rightarrow \quad 2C + D = 1$$

Hence, $C = 0$, $D = 1$, and

$$\frac{-4x^2 + x - 1}{x(x - 1)(x^2 + 1)^2} = \frac{1}{x} - \frac{1}{x-1} + \frac{1}{x^2+1} + \frac{x-2}{(x^2+1)^2}$$

Therefore the integrals of proper rational functions are reducible to integrals of the form

$$A = \int \frac{1}{(x-a)^n}\,dx, \qquad B = \int \frac{ax+b}{(x^2+px+q)^n}\,dx$$

of which the first one (A) is easily integrable by the substitution $u = x - a$, $du = dx$.

The second one (B) is reducible to (A) if

$$ax + b = D(x^2 + px + q) = 2x + p;$$

otherwise, writing $ax + b$ as

$$ax + b = \frac{a}{2}(2x + p) + (b - \frac{ap}{2})$$

we have

$$B = \frac{a}{2}\int \frac{2x+p}{(x^2+px+q)^n}\,dx + (b - \frac{ap}{2})\int \frac{1}{(x^2+px+q)^n}\,dx$$

$$= \frac{a}{2}\int \frac{du}{u^n} + c\int \frac{1}{(x^2+px+q)^n}\,dx$$

where $u = x^2 + px + q$ in the first integral. In the second integral, writing

$$x^2 + px + q = (x + \frac{p}{2})^2 + (q - \frac{p}{2})^2.$$

$$= (x + \frac{p}{2})^2 + \left[\frac{\sqrt{-(p^2-4q)}}{2}\right]^2, \quad (\Delta = p^2 - 4q < 0)$$

and setting

$$x + \frac{p}{2} = \frac{\sqrt{-\Delta}}{2}\,u, \qquad dx = \frac{\sqrt{-\Delta}}{2}\,du$$

we have

$$x^2 + px + q = (\frac{\sqrt{-\Delta}}{2}\,u)^2 + (\frac{\sqrt{-\Delta}}{2})^2 = \frac{-\Delta}{4}(u^2+1)$$

and the integral reduces to

$$k\int \frac{1}{(u^2+1)^n}\,du$$

which, omitting k and replacing u by x, becomes

$$I_n = \int \frac{dx}{(x^2+1)^n}$$

so that

$$I_1 = \int \frac{dx}{x^2 + 1} = \arctan x + c$$

For values $n > 1$, one establishes a relation between I_n and I_{n-1} by parts method applied to I_{n-1} as follows:

$$I_{n-1} = \int \frac{dx}{(x^2 + 1)^{n-1}}$$

$$u = (x^2 + 1)^{-n+1}, \quad dv = dx$$

$$du = (1 - n)(x^2 + 1)^{-n}\, 2x\, dx, \quad v = x$$

$$I_{n-1} = x(x^2 + 1)^{-n+1} + 2(n-1) \int \frac{x^2}{(x^2 + 1)^n}\, dx$$

$$= \frac{-x}{(x^2 + 1)^{n-1}} + 2(n-1) \left(\int \frac{x^2 + 1}{(x^2 + 1)^n}\, dx - \int \frac{1}{(x^2+1)^n} dx \right)$$

$$= \frac{x}{(x^2 + 1)^{n-1}} + 2(n-1) \int \frac{dx}{(x^2 + 1)^{n-1}} - 2(n-1)\, I_n$$

$$= \frac{x}{(x^2 + 1)^{n-1}} + 2(n-1)\, I_{n-1} - 2(n-1)\, I_n$$

$$(2n - 1) I_n = 2(n - 1)\, I_{n-1} + \frac{x}{(x^2 + 1)^{n-1}}$$

$$I_n = \frac{2n - 2}{2n - 1} I_{n-1} + \frac{1}{2n - 1} \frac{x}{(x^2 + 1)^{n-1}}$$

Such a relation is called a *recurrence relation* or a *reduction formula* which permits to obtain I_2, I_3, ... seccussively. Thus

$$I_2 = \frac{2}{3} I_1 + \frac{1}{3} \frac{x}{x^2 + 1} = \frac{2}{3} \arctan x + \frac{1}{3} \frac{x}{x^2 + 1} + c,$$

$$I_3 = \frac{4}{5} I_2 + \frac{1}{5} \frac{x}{(x^2 + 1)^2} = \frac{4}{5} \left(\frac{2}{3} \arctan x + \frac{1}{3} \frac{x}{(x^2+1)} \right) + \frac{1}{5} \frac{x}{(x^2+1)} + c$$

$$= \frac{8}{15} \arctan x + \frac{4}{15} \frac{x}{x^2 + 1} + \frac{1}{5} \frac{x}{(x^2 + 1)^2} + c$$

This completes the discussion of integration of rational functions.

Example. Evaluate $I = \int \frac{(x+1)^2}{x(x^2+1)}\,dx$.

Solution. The integrand being a proper fraction we decompose it into partial fractions:

$$\frac{x^2+2x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$x^2 + 2x + 1 = A(x^2+1) + x(Bx+C)$$

$$\begin{array}{lll} x = 0: & 1 = A & \\ x = 1: & 4 = 2A + B + C & \Rightarrow \quad B + C = 2 \\ x = -1: & 0 = 2A + B - C & \quad B - C = -2 \end{array}$$

$$\Rightarrow \quad A = 1, \quad B = 0, \quad C = 2$$

$$I = \int \left(\frac{1}{x} + \frac{2}{x^2+1}\right)dx$$

$$= \ln x + 2\arctan x + C$$

EXERCISES (7. I)

1. Find the partial fractions of the given rational functions:

a) $\dfrac{3x^2 - 6x + 4}{(x-3)(x^2+x+1)}$

b) $\dfrac{4x^2 + 6x + 6}{(x+1)^2\,(x^2+2x+3)}$

c) $\dfrac{x^2 - 2x + 3}{(x^2 - x + 1)^2}$

d) $\dfrac{-x^3 + 5x^2 - x + 7}{(x^2 - 2x + 3)(x^2 + 2x + 2)}$

2. Reduce first to proper fraction if necessary, and decompose into partial fractions (don't evaluate the coefficients)

a) $\dfrac{x^6}{x^5+1}$ $\left[x^5+1=(x+1)(x^2-\frac{1-\sqrt{5}}{2}x+1)(x^2-\frac{1+\sqrt{5}}{2}x+1)\right]$

b) $\dfrac{x^3+2x+5}{(x+3)^2(x-1)^2}$ c) $\dfrac{x-x^3}{(1+x^2)^4(1+x^4)}$ d) $\dfrac{4x^3}{(x^4-1)^2}$

e) $\dfrac{x^4+1}{x^4+x^2+1}$, $\left[x^4+x^2+1=(x^2+x+1)(x^2-x+1)\right]$

f) $\dfrac{1}{(x^2-1)^4(x^2-2)^3}$, g) $\dfrac{1}{x^6-3x^4-x^3}$ $\left[Q(x)=x^3(x+1)^2(x-2)\right]$

3. Evaluate

a) $\int\dfrac{x^2-x+2}{x^4-8x+4}\,dx$ $\left[Q(x)=(x^2+\frac{10}{3}x+2)(x^2-\frac{10}{3}x+2)\right]$

b) $\int\dfrac{6x^2-5}{(2x^2+1)^2}\,dx$ c) $\int\dfrac{dx}{(x^2-1)(x^3-1)}$ d) $\int\dfrac{x^2-6x+5}{x^2-5x+6}dx$

4. Find a recurrence relation for

$$I_n=\int\frac{x^n\,dx}{1+x^2}\qquad (n\varepsilon N_2)$$

5. Evaluate

a) $\int\dfrac{x\,dx}{(x-1)(x^2+1)}$ b) $\int\dfrac{x\,dx}{(x-1)(x^2-1)}$

6. Evaluate

$$A=\int\frac{(x+1)dx}{x^3(x^2+1)^2}$$

7. Evaluate

a) $\int\dfrac{x^2+1}{x^3+3x+7}\,dx$ b) $\int\dfrac{x^2+2}{x^3+3x+4}\,dx$, $\left[Q(x)=(x+1)(x^2-x+4)\right]$

8. Evaluate

a) $\int\dfrac{x^2-x+2}{x^4-5x^2+4}\,dx$ $\left[Q(x)=(x^2-1)(x^2-4)\right]$

b) $\int \frac{x^2}{x^4 + x^2 - 2}\,dx$. $\left[Q(x) = (x^2 - 1)(x^2 + 2)\right]$

c) $\int \frac{3x^2 + x - 2}{(x - 1)^3(x^2 + 1)}\,dx$

9. Evaluate

a) $\int \frac{x\ dx}{x^3 - 3x + 2}\,dx$ $\left[Q(x) = (x - 1)^2(x + 2)\right]$

b) $\int \frac{x^2\ dx}{(x^2 + 1)^2}$, c) $\int \frac{dx}{x^6 - 1}$

d) $\int \frac{dx}{x^4 + 2x^2 + 2}$ $\left[Q(x) = (x^2 + \sqrt{\sqrt{8} - 2}\,x + \sqrt{2})\cdot (x^2 - \sqrt{\sqrt{8} - 2}\,x + \sqrt{2})\right]$

10. Evaluate

$$\int \frac{x + 1}{x^3(x^2 + 1)^2}\,dx$$

11. Evaluate

$$\int \frac{3x^2 + x - 2}{(x - 1)^3\ (x^2 + 1)}\,dx$$

12. Evaluate the following integrals:

a) $\int \frac{-4x^2 + x - 1}{x(x - 1)(x^2 + 1)^2}\,dx$ b) $\int \frac{dx}{4x^2 + 2x + 3}$

13. Evaluate $\int_0^\infty \frac{dx}{x^4 + 1}$; $\left[x^4 + 1 = (x^2 - \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)\right]$

14. Evaluate

a) $\int \frac{5x - 2}{x^2 - 4}\,dx$ b) $\int \frac{x^2 dx}{x^3 - 3x + 2}$ (See 9a)

c) $\int \frac{6x^2 - 5x - 9}{x^3 - 2x^2 - x + 2}\,dx$ $\left[Q(x) = (x - 1)(x^2 - x - 2)\right]$

d) $\int \frac{x^2 + 5x + 1}{x(x + 1)^2}\,dx$

15. Evaluate

a) $\int \frac{dx}{x(x-1)(x-2)}$ b) $\int \frac{dx}{(x-1)^2(x-2)}$

c) $\int \frac{dx}{x(x^2+1)}$ d) $\int \frac{dx}{x(x^2+1)^2}$

16. Evaluate the following integrals of rational functions:

a) $\int \frac{x^2-6x-2}{(2x+1)(x^2+1)}\,dx$ b) $\int \frac{dx}{4x^2-9}$

17. Same question for:

a) $\int \frac{3x^2+4}{2x^3+8x}\,dx$ b) $\int \frac{4x^2+x+1}{4x^2+1}\,dx$

18. Evaluate

a) $\int_0^2 \frac{1-x}{x^2+3x+2}\,dx$ b) $\int_1^4 \frac{2x^2+13x+18}{x(x+3)^2}\,dx$

19. Evaluate

a) $\int_0^2 \frac{2x^2+6x+2}{x^3+6x^2+11x+6)}dx$ $\left[Q(x) = (x+2)(x^2+4x+3)\right]$

b) $\int_1^2 \frac{2x^2+x+4}{x^3+4x^2}\,dx$

20. Find the area of the region bounded by the given curves:

$$y = \frac{x-1}{x^2-5x+6}, \quad x = 4, \quad x = 6, \quad y = 0$$

ANSWERS TO EVEN NUMBERED EXERCISES

2. a) $x + \frac{A}{x+1} + \frac{Bx+C}{x^2 - \frac{1-\sqrt{5}}{2}x + 1} + \frac{Dx+E}{x^2 - \frac{1+\sqrt{5}}{2}x + 1}$

b) $\frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$.

c) $\frac{Ax+B}{1+x^2} + \frac{Cx+D}{(1+x^2)^2} + \frac{Ex+F}{(1+x^2)^3} + \frac{Gx+H}{(1+x^2)^4} + \frac{Ix+J}{x^2-\sqrt{2}x+1} + \frac{Kx+L}{x^2+\sqrt{2}x+1}$

d) $\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2}$.

e) $\dfrac{Ax+B}{x^2+x+1} + \dfrac{Cx+D}{x^2-x+1}$,

f) $\sum_{i=1}^{4} \dfrac{A_i}{(x-1)^i} + \sum_{i=1}^{4} \dfrac{B_i}{(x+1)^i} + \sum_{i=1}^{3} \dfrac{C_i}{(x-\sqrt{2})^i} + \sum_{i=1}^{3} \dfrac{D_i}{(x+\sqrt{2})^i}$

g) $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{D}{x+1} + \dfrac{E}{(x+1)^2} + \dfrac{F}{x-2}$

4. $I_n = \dfrac{x^{n-1}}{n-1} - I_{n-2}$

6. $-\dfrac{2}{x} + \dfrac{1}{x^2} + \dfrac{1}{x^3} + \dfrac{2x-1}{x^2+1} + \dfrac{x-1}{(x^2+1)^2} + C$

8. a) $\dfrac{1}{3} \ell n \dfrac{(x+1)^2(x-2)}{(x-1)(x+2)^2} + C$; b) $\dfrac{1}{6} \ell n \dfrac{x-1}{x+1} + \dfrac{\sqrt{2}}{3} \arctan \dfrac{x}{\sqrt{2}} + C$

c) $\dfrac{3}{4} \ell n \dfrac{x^2+1}{(x-1)^2} - \arctan x - \dfrac{3x-4}{2(x-1)^2} + C.$

10. $-\dfrac{3x^3+2x^2+2x+1}{2x^2(x^2+1)} + \ell n \dfrac{x^2+1}{x^2} - \dfrac{3}{2} \arctan x + c$

12. a) $\ell n \dfrac{x}{x-1} - \dfrac{1}{6} \ell n (x^2+1) - \dfrac{1}{3} \arctan x - \dfrac{x+1}{x^2+1} + C$

b) $\dfrac{1}{\sqrt{11}} \arctan \dfrac{4x+1}{\sqrt{11}} + C.$

14. a) $\ell n (x-2)^2 |x+2|^3 + 6$, b) $\dfrac{5}{9} \ell n|x-1| + \dfrac{4}{9} \ell n|x+2| - \dfrac{1}{3(x-1)} + C$

c) $\ell n (x-1)^4 + \dfrac{1}{3} \ell n|x+1| + \dfrac{5}{3} (x-2) + C.$

d) $\ell n|x| - \dfrac{3}{x+1} + C.$

16. a) $\dfrac{1}{2} \ell n |2x+1| - 3 \arctan x + C$, b) $\dfrac{1}{12} \ell n \left|\dfrac{2x-3}{2x+3}\right| + C.$

18. a) $\ell n (9/8)$, b) $3/28 + \ell n 16$.

20. $\ell n (9/2)$

7. 2 INTEGRATION OF TRANSCENDENTAL FUNCTIONS BY VARIOUS TECHNIQUES

A. $\int R(\cos\Theta, \sin\Theta)d\Theta$, $\int R(\text{Ch}\Theta, \text{Sh}\Theta)d\Theta$

We treat here the cases where the integrand is a rational function of its arguments $\cos\Theta$, $\sin\Theta$ or $\text{Ch}\Theta$, $\text{Sh}\Theta$.

The following are of this type:

$$\int \cos^2\Theta \sin^3\Theta \, d\Theta, \quad \int \frac{\sin\Theta - 2\cos\Theta}{\cos\Theta + \sin\Theta} d\Theta, \quad \int \text{Th}^3\Theta \, d\Theta$$

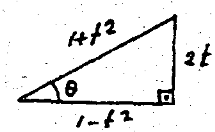
The general method is the *half-angle, half-argument substitution* respectively by which the integrand reduces to a rational function.

To show that this is the case, make in $R(\cos\Theta, \sin\Theta)$ the half-angle substitution:

$$\tan\frac{\Theta}{2} = t \Rightarrow \Theta = 2\arctan t \Rightarrow d\Theta = \frac{2dt}{1+t^2}$$

Then from

$$\tan\Theta = \frac{2\tan\frac{\Theta}{2}}{1-\tan^2\frac{\Theta}{2}} = \frac{2t}{1-t^2} \quad \text{and}$$



we get

$$\cos\Theta = \frac{1-t^2}{1+t^2}, \quad \sin\Theta = \frac{2t}{1+t^2}$$

(Observe similarity between the expressions for $d\Theta$ and $\sin\Theta$)

Then

$$\int R(\cos\Theta, \sin\Theta)d\Theta = \int R\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)\frac{2dt}{1+t^2} = \int R_1(t)dt$$

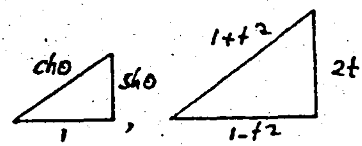
where $R_1(t)$ is a rational function of t.

For the hyperbolic case $R(\text{Ch}\Theta, \text{Sh}\Theta)$ the *half-argument substitution* is

$$\operatorname{Th}\frac{\Theta}{2} = t \Rightarrow \Theta = 2\operatorname{Arctanh} t \Rightarrow d\Theta = \frac{2dt}{1-t^2}$$

From

$$\operatorname{Th}\Theta = \frac{2\operatorname{Th}\frac{\Theta}{2}}{1+\operatorname{Th}^2\frac{\Theta}{2}} = \frac{2t}{1+t^2} \quad \text{and}$$



we get

$$\operatorname{Ch}\Theta = \frac{1+t^2}{1-t^2}, \quad \operatorname{Sh}\Theta = \frac{2t}{1-t^2}$$

(Observe similarity between the expression for $d\Theta$ and $\operatorname{Sh}\Theta$, and also between trigonometric and hyperbolic expressions).

As in trigonometric case the new integrand is a rational function of t.

Example 1. Evaluate $A = \int \frac{1}{2+\sin\Theta}\, d\Theta$

Solution. Since the integrand is a rational function of $\sin\Theta$, the half-angle substitution transform it into a rational function:

$$A = \int \frac{1}{2+\frac{2t}{1+t^2}} \cdot \frac{2\,dt}{1+t^2} = \int \frac{dt}{t^2+t+1},$$

where

$$t^2+t+1 = (t+\tfrac{1}{2})^2 + 1 - \tfrac{1}{4} = (t+\tfrac{1}{2})^2 + (\tfrac{\sqrt{3}}{2})^2.$$

Setting

$$t+\frac{1}{2} = \frac{\sqrt{3}}{2}u, \qquad dt = \frac{\sqrt{3}}{2}du,$$

we have

$$A = \int \frac{\frac{\sqrt{3}}{2}\,du}{\frac{3}{4}(u^2+1)} = \frac{2}{\sqrt{3}} \int \frac{du}{u^2+1} = \frac{2}{3}\arctan u + C$$

$$= \frac{2}{3}\arctan \frac{2}{3}(t+\tfrac{1}{2}) + C$$

$$= \frac{2}{3}\arctan \frac{1}{3}(1+2\tan\tfrac{\Theta}{2}) + C$$

Example 2. Evaluate $B = \int \frac{1}{2Sh\theta - 3Ch\theta + 1}\, d\theta$

Solution. $Th \frac{\theta}{2} = t \Rightarrow \theta = 2 \, Argth \, t \Rightarrow d\theta = \frac{2\,dt}{1-t^2}$, and

$$Sh\,\theta = \frac{2t}{1-t^2}, \quad Ch\,\theta = \frac{1+t^2}{1-t^2}$$

$$B = \int \frac{1}{\frac{4t}{1-t^2} - 3\frac{1+t^2}{1-t^2} + 1} \frac{2\,dt}{1-t^2} = -2\int \frac{dt}{4t^2 - 4t + 2}$$

$$= \int \frac{d(2t)}{(2t)^2 - 2(2t) + 2} = -\int \frac{du}{u^2 - 2u + 2} \quad (u = 2t)$$

$$= -\int \frac{d(u-1)}{(u-1)^2 + 1} = -\arctan(u-1) + C$$

$$= -\arctan(2t-1) + C = -\arctan\left(2Th\frac{\theta}{2} - 1\right) + C.$$

The half-angle (half-argument) substitutions works always for the rational integrands in trigonometric (or hyperbolic) functions.

In some cases as the following ones the use of these substitutions not necessary:

Integrals reducible to $\int du/u$:

1. a) $\int \tan\theta \, d\theta$

$\int \tan\theta \, d\theta = \int \frac{\sin\theta}{\cos\theta} d\theta$

$= \int \frac{-d\cos\theta}{\cos\theta} = -\ln|\cos\theta| + c$

a') $\int Th\theta \, d\theta$

$\int Th\theta \, d\theta = \int \frac{Sh\theta}{Ch\theta} d\theta$

$= \int \frac{d\,ch\theta}{ch\theta} = \ln ch\theta + c.$

b) $\int \cot\theta \, d\theta = \int \frac{d\sin\theta}{\sin\theta}$

$= \ln|\sin\theta| + c$

b') $\int Coth\theta \, d\theta = \int \frac{d\,Sh\theta}{Sh\theta}$

$= \ln|Sh\theta| + c$

c) $\int \sec\Theta \, d\Theta$

$$= \int \frac{\sec\Theta(\sec\Theta + \tan\Theta)}{\sec\Theta + \tan\Theta.} d\Theta$$

$$= \int \frac{d(\text{Sec}\Theta + \tan\Theta)}{\sec\Theta + \tan\Theta}$$

$$= \ell n|\sec\Theta + \tan\Theta| + C$$

c') $\int \text{Sech}\Theta \, d\Theta$

$$= \int \frac{\text{Sech}\Theta(\text{Sech}\Theta + \text{Tanh}\Theta)}{\text{Sech}\Theta + \text{Tanh}\Theta} d\Theta$$

$$= \int \frac{d(\text{Sech}\Theta + \text{Tanh}\Theta)}{\text{Sech}\Theta + \text{Tanh}\Theta}$$

$$= \ell n|\text{Sech}\Theta + \text{Tanh}\Theta| + C$$

d) $\int \csc\Theta \, d\Theta$

$$= -\ell n|\csc\Theta + \cot\Theta| + C$$

d') $\int \text{Csch}\Theta \, d\Theta$

$$= -\ell n|\text{Csch}\Theta + \text{Coth}\Theta| + C$$

2. $I = \int \frac{a \cos\Theta + b \sin\Theta}{A \cos\Theta + B \sin\Theta} d\Theta$, $\quad I' = \int \frac{a \text{ Ch}\Theta + b \text{ Sh}\Theta}{A \text{ Ch}\Theta + b \text{ Sh}\Theta} d\Theta$

The method consists of determining constants m, n such that

$$a \cos\Theta + b \sin\Theta = mU + nU' \qquad (1)$$

where $U = A \cos\Theta + B \sin\Theta$. Then

$$I = \int \frac{mU + nU'}{U} d\Theta = m\Theta + n \ \ell n|U| + C$$

(1) $\Rightarrow$

$$a \cos\Theta + b \sin\Theta = m(A \cos\Theta + B \sin\Theta) + n(-A \sin\Theta + B \cos\Theta)$$

$$a \cos\Theta + b \text{ Sin}\Theta = (Am + Bn)\cos\Theta + (Bm - An)\text{Sin}\Theta$$

$$Am + Bn = a \qquad \Delta = \begin{vmatrix} A & B \\ B & A \end{vmatrix} = A^2 + B^2 \neq 0$$

$$Bm - An = b$$

Since $\Delta \neq 0$ there is a unique solution in m, n. Since a, b are not both zero, m, n are not both zero.

I' is solved similarly.

Example. Evaluate $I = \int \frac{11 \cos\Theta - 2 \sin\Theta}{3 \cos\Theta + 4 \sin\Theta} d\Theta$

Solution. We determine the constants m, n such that

$$11 \cos\Theta - 2 \text{ Sin}\Theta = m(3 \text{ Cos}\Theta + 4 \text{ Sin}\Theta) + n(-3 \text{ Sin}\Theta + 4 \text{ Cos}\Theta)$$

$$11 \cos\Theta - 2 \text{ Sin}\Theta = (3m + 4n)\text{Cos}\Theta + (4m - 3n)\text{Sin}\Theta$$

$$3m + 4n = 11, \quad 4m - 3n = -2 \Rightarrow m = 1, \quad n = 2.$$

$$I = \int \left(1 + 2\frac{U'}{U}\right)d\Theta = \Theta + 2\,\ell n|3\cos\Theta + 4\,\mathrm{Sin}\Theta| + c.$$

Integrals reducible to $\int u^{\alpha}\,du$

1. $A = \int \cos^m\Theta\,\mathrm{Sin}^n\Theta\,d\Theta, \qquad A' = \int \mathrm{Ch}^m\Theta\,\mathrm{Sh}^n\Theta\,d\Theta$

We distinguish two cases:

Case 1. At least one of m and n is odd:

Let m be odd with $m = 2k + 1$ $(k\varepsilon\mathbb{N})$. Then

$$A = \int\cos^{2k}\Theta\,\sin^n\Theta\,\cos\Theta\,d\Theta = \int(1-\sin^2\Theta)^k\,\sin^n\Theta\,d\,\sin\Theta$$

$$= \int(1 - s^{2k})\,s^n\,ds = \int P(s)ds \quad (s = \sin\Theta)$$

Case 2. m and n are both even:

Let $m = 2k$, $n = 2r$ $(k, r\varepsilon\mathbb{N})$

$$A = \int (\cos^2\Theta)^k(\sin^2\Theta)^r\,d\Theta = \int \left(\frac{1+\cos 2\Theta}{2}\right)^k\left(\frac{1-\cos 2\Theta}{2}\right)^r d\Theta =$$

$$= \int \sum a_j \cos^j 2\Theta\,d\Theta,$$

each term being evaluated by the case 1 if j is odd and by case 2 if j is even.

The integral A' is evaluated in the same way by the use of $\mathrm{Ch}^2\Theta = \frac{1}{2}(\mathrm{Ch}\,2\Theta + 1)$, $\mathrm{Sh}^2\Theta = \frac{1}{2}(\mathrm{Ch}\,2\Theta - 1)$.

Examples. Evaluate the following

a) $A = \int \cos^3\Theta\,\sin^2\Theta\,d\Theta$ b) $B = \int \cos^2\Theta\,\sin^2\Theta\,d\Theta$

c) $C = \int \mathrm{Ch}^4\Theta\,\mathrm{Sh}^2\Theta\,d\Theta$

Solution.

a) Having an odd exponent we have

$$A = \int\cos^2\Theta\,\sin^2\Theta\,d\,\sin\Theta = \int(1 - \sin^2\Theta)\sin^2\Theta\,d\,\sin\Theta$$

$$= \frac{1}{3}\sin^3\Theta - \frac{1}{5}\sin^5\Theta + c.$$

b) The exponents being equal to each other we have

$$= \int(\cos\Theta\,\sin\Theta)^2\,d\Theta = \frac{1}{4}\int\sin^2 2\Theta\,d\Theta = \frac{1}{4}\int\frac{1-\cos 4\Theta}{2}\,d\Theta = \frac{1}{8}(\Theta - \frac{\sin 4\Theta}{4}$$

c) The exponents being not equal and being both even we get

$$C = \int \left(\frac{Ch\ 2\theta + 1}{2}\right)^2 \frac{Ch\ 2\theta - 1}{2}\, d\theta$$

$$= \frac{1}{8} \int (Ch^2 2\theta + 2Ch\ 2\theta + 1)(Ch\ 2\theta - 1)d\theta$$

$$= \frac{1}{8} \int (Ch^3 2\theta + Ch^2 2\theta - Ch\ 2\theta - 1)d\theta$$

$$= \frac{1}{8} \int Ch^3 2\theta\, d\theta + \frac{1}{8} \int \frac{Ch\ 4\theta + 1}{2}\, d\theta - \frac{1}{8}\,\frac{Sh\ 2\theta}{2} - \frac{\theta}{8}$$

$$= \frac{1}{16} \int (1 - Sh^2 2\theta)d\ Sh\ 2\theta + \left(\frac{1}{16}\,\frac{Sh\ 4\theta}{4} + \frac{\theta}{16}\right) - \frac{Sh\ 2\theta}{16} - \frac{\theta}{8}$$

$$= \frac{Sh\ 2\theta}{16} - \frac{Sh^3 2\theta}{48} + \frac{Sh\ 4\theta}{64} - \frac{Sh\ 2\theta}{16} - \frac{\theta}{16} + c = -\frac{Sh^3 2\theta}{48} + \frac{Sh\ 4\theta}{64} - \frac{\theta}{16} + c$$

2. $A = \int \sec^m\theta \tan^n\theta\, d\theta$, $A' = \int Sech^m\theta\ Tanh^n\theta\, d\theta$

$B = \int \csc^m\theta\ Cot^n\theta\, d\theta$, $B' = \int csch^m\theta\ Coth^n\theta\, d\theta$

To evaluate the integral

$$A = \int \sec^m\theta \tan^n\theta\, d\theta$$

we distinguish three cases:

Case 1. m is an even integer: $m = 2k$ $(k \in N)$

$$A = \int \sec^{2k-2}\theta \tan^n\theta \sec^2\theta\, d\theta$$

$$= \int (1 + \tan^2\theta)^{k-1} \tan^n\theta\, d\tan\theta$$

$$= \int P(t)dt \quad (t = \tan\theta)$$

Case 2. n is an odd integer: $n = 2r + 1$ $(r \in N)$

$$A = \int \sec^{m-1}\theta \tan^{2r}\theta .\ \sec\theta \tan\theta\, d\theta$$

$$= \int \sec^{m-1}\theta\, (\sec^2\theta - 1)^r\, d\sec\theta$$

$$= \int P(s)ds \quad (s = \sec\theta)$$

If both m and n are even or both odd we are in case 1 or case 2.

Case 3. $m = 2k + 1$, $n = 2r$

$$A = \int \sec^{2k+1}\theta \tan^{2r}\theta \, d\theta$$
$$= \int \sec^{2k+1}\theta \, (\sec^2\theta - 1)^r \, d\theta$$
$$= \int \sec^j\theta \, d\theta \text{ (See 4)}$$

The procudere is the same for the integral B, and for hyperbolic ones with $1 - Th^2\theta = Sech^2\theta$ instead of $1+\tan^2\theta=\sec^2\theta$.

Example. Evaluate

a) $I = \int Sec^4\theta \tan^2\theta \, d\theta$ b) $J = \int Sech^3\theta \, Tanh\theta \, d\theta$

c) $K = \int Csc^4\theta \cot^4\theta \, d\theta$

Solution.

a) Since $m = 4$ is even we have by case 1:

$$I = \int \sec^2\theta \tan^2\theta \sec^2\theta \, d\theta$$
$$= \int (1 + \tan^2\theta)\tan^2\theta \, d \tan\theta$$
$$= \int (1 + t^2)t^2 \, dt = \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} + c$$

b) Since $n = 3$ is odd, by case 2 we have

$$J = \int Sech^2\theta \, Tanh^2\theta \, Sech\theta \, Tanh\theta \, d\theta$$
$$= \int Sech^2\theta \, (1-Sech^2\theta) d \, Sech\theta$$
$$= \int s^2(1-s^2)ds = \frac{Sech^3\theta}{3} - \frac{Sech^5\theta}{5} + c$$

c) $$K = \int \csc^2\theta \cot^4\theta \csc^2\theta \, d\theta$$
$$= - \int (1 + \cot^2\theta) \cot^4\theta \, d \cot\theta$$
$$= \frac{\cot^5\theta}{5} - \frac{\cot^7\theta}{7} + c.$$

In case the integrand involves hyperbolic functions (or powers of e^x) integration may be easier, as seen in the example, if it is transformed to powers of e^x (or to hyperbolic functions:)

Example. Evaluate

$$A = \int Sh^2 t\ dt, \qquad B = \int \frac{e^t - e^{-t}}{e^t + e^t}\ dt$$

Solution.

$$A = \int (e^t - e^{-t})^2\ dt = \frac{1}{4} \int (e^{2t} - 2 + e^{-2t})dt$$

$$= \frac{1}{4}\left[\frac{1}{2} e^{2t} - 2t - \frac{1}{2} e^{-2t}\right] + c = \frac{1}{4} Sh\ 2t - \frac{1}{2} t + c$$

$$B = \int \frac{Sh\ t}{Ch\ t}\ dt = \ell n\ Ch\ t + c$$

Integrals evaluated by recurrence formulas

1. $c_n = \int \cos^n\Theta\ d\Theta$ 1'. $C_n = \int Ch^n\Theta\ d\Theta$

$s_n = \int \sin^n\Theta\ d\Theta$ $S_n = \int Sh^n\Theta\ d\Theta$

We establish the formula for c_n; the others we obtained similarly:

$$c_n = \int \cos^n\Theta\ d\Theta = \int \underbrace{\cos^{n-1}\Theta}_{u} \cdot \underbrace{\cos\Theta\ d\Theta}_{dv} \quad (n \geq 2)$$

$$= \cos^{n-1}\Theta \sin\Theta - \int \sin\Theta \cdot (n-1)\cos^{n-2}\Theta\ (-\sin\Theta)d\Theta$$

$$= \cos^{n-1}\Theta \sin\Theta + (n-1) \cos^{n-2}\Theta\ (1-\cos^2\Theta)d\Theta$$

$$= \cos^{n-1}\Theta \sin\Theta + (n-1)\ c_{n-2} - (n-1)I_n$$

$$n\ c_n = (n-1)c_{n-2} + \cos^{n-1}\Theta \sin\Theta$$

$$c_n = \frac{n-1}{n} c_{n-2} + \frac{1}{n} \cos^{n-1}\Theta \sin\Theta$$

$$s_n = -\frac{n-1}{n} s_{n-2} - \frac{1}{n} \sin^{n-1} \cos\Theta$$

$$C_n = \frac{n-1}{n} C_{n-2} + \frac{1}{n} Cosh^{n-1}\Theta\ Sh\ \Theta$$

$$S_n = -\frac{n-1}{n} S_{n-2} + \frac{1}{n} Sinh^{n-1}\Theta\ Ch\ \Theta$$

2. $t_n = \int \tan^n\Theta\ d\Theta$ 2'. $T_n = \int Tanh^n\Theta\ d\Theta$

$t'_n = \int \cot^n\Theta d\Theta$ $T'_n = \int Coth^n\Theta\ d\Theta$

We establish the formula again for t_n; the others are obtained similarly.

$$I_n = \int \tan^n\Theta \, d\Theta \qquad (n \geqslant 2)$$

$$= \int \tan^{n-2}\Theta \tan^2\Theta \, d\Theta$$

$$= \int \tan^{n-2}\Theta \, (\sec^2\Theta - 1)d\Theta$$

$$= \int \tan^{n-2}\Theta \, d\tan\Theta - \int \tan^{n-2}\Theta \, d\Theta$$

$$= \frac{1}{n-1} \tan^{n-1}\Theta - I_{n-2}$$

$$\left.\begin{aligned} t_n &= -t_{n-2} + \frac{1}{n-1} \tan^{n-1}\Theta \\ t'_n &= -t'_{n-2} - \frac{1}{n-1} \cot^{n-1}\Theta \end{aligned}\right\}$$

$$\left.\begin{aligned} T_n &= T_{n-2} - \frac{1}{n-1} \operatorname{Tanh}^{n-1}\Theta \\ T'_n &= T'_{n-2} - \frac{1}{n-1} \operatorname{Coth}^{n-1}\Theta \end{aligned}\right\}$$

3. $c'_n = \int \sec^n\Theta \, d\Theta$ 3' $C'_n = \int \operatorname{Sech}^n\Theta \, d\Theta$

$s'_n = \int \csc^n\Theta \, d\Theta$ $S'_n = \int \operatorname{Csch}^n\Theta \, d\Theta$

Again we obtain the formula for c'_n, the others being obtained similarly.

$$c'_n = \int \sec^n\Theta \, d\Theta \qquad (n \geqslant 2)$$

$$= \int \underset{u}{\sec^{n-2}\Theta} \, \underset{dv}{\sec^2\Theta \, d\Theta}$$

$$= \sec^{n-2}\Theta \tan\Theta - \int \tan\Theta \, . \, (n-2)\sec^{n-3}\Theta \sec\Theta\tan\Theta d\Theta$$

$$= \sec^{n-2}\Theta \tan\Theta - (n-2)\int \sec^{n-2}\Theta \, (\sec^2\Theta - 1)d\Theta$$

$$= \sec^{n-2}\Theta \tan\Theta - (n-2)c'_n + (n-2)c'_{n-2}$$

$$(n-1)c'_n = (n-2)c'_{n-2} + \sec^{n-2}\Theta \tan\Theta$$

$$c_n = \frac{n-2}{n-1} c'_{n-2} + \frac{1}{n-1} \sec^{n-2}\Theta \tan\Theta$$

$$\left.\begin{aligned} c'_n &= \frac{n-2}{n-1} c'_{n-2} + \frac{1}{n-1} \sec^{n-2}\theta \tan\theta \\ s'_n &= \frac{n-2}{n-1} s'_{n-2} - \frac{1}{n-1} \csc^{n-2}\theta \cot\theta \end{aligned}\right\}$$

$$\left.\begin{aligned} C'_n &= -\frac{n-2}{n-1} C'_{n-2} + \frac{1}{n-1} \mathrm{Sech}^{n-2}\theta\ \mathrm{Tanh}\theta \\ S'_n &= -\frac{n-2}{n-1} S'_{n-2} - \frac{1}{n-1} \mathrm{Csch}^{n-2}\theta\ \mathrm{Coth}\theta \end{aligned}\right\}$$

Example. Evaluate $I = \int \sec^5\theta \, d\theta$

Solution. By the recurrence formula,

$$I_5 = \frac{3}{4} I_3 + \frac{1}{4} \sec^3\theta \tan\theta$$

$$I_3 = \frac{1}{2} I_1 + \frac{1}{2} \sec\theta \tan\theta$$

where

$$I_1 = \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta|.$$

Then

$$I_5 = \frac{3}{4}\left(\frac{1}{2} \ln|\sec\theta + \tan\theta| + \frac{1}{2} \sec\theta \tan\theta\right) + \frac{1}{4} \sec^3\theta \tan\theta + c$$

$$= \frac{3}{8} \ln|\sec\theta + \tan\theta| + \frac{3}{8} \sec\theta \tan\theta + \frac{1}{4} \sec^3\theta \tan\theta + c$$

B. INTEGRATION OF MIXED FUNCTIONS:

The integrals that we consider here are listed below where P and R are polynomial and rational function of their arguments respectively:

1. $\int P(x) \ln x \, dx$, $\int P(x) e^{ax} dx$
2. $\int P(x) \cos ax \, dx$, $\int P(x) \sin ax \, dx$; $\int P(x) \mathrm{Ch}ax \, dx$, $\int P(x) \mathrm{Sh}ax \, dx$
3. $\int e^{ax} \cos bx \, dx$, $\int e^{ax} \sin bx \, dx$; $\int e^{ax} \mathrm{Ch}ax \, dx$, $\int P(x) \mathrm{Sh}ax \, dx$
4. $\int \cos ax \cos bx \, dx$, $\int \cos ax \sin bx \, dx$, $\int \sin ax \sin bx \, dx$
5. $\int P(x, \ln x) dx$, $\int P(x, e^x) dx$

6: $\int R(\ln x)dx$ 7. $\int R(e^x)dx$

Integrals 1-6 can be evaluated directly, or after some process, by parts.

The integral (7) can be transformed into integral of a rational function after the substitution $e^x = t$.

Examples. Evaluate

a) $A = \int_0^{\ln 3} e^{3x} \mathrm{Sh}\, 2x\, dx$ b) $B = \int \cos 3x \cos 4x \sin 5x\, dx$

c) $\int_1^e (\ln x + \ln^2 x)dx$ d) $\int_0^{\ln 4} \frac{e^{3x} - e^x}{e^{2x} + 1} dx$

Solution.

a) $A = \int_0^{\ln 3} e^{3x} \frac{1}{2} (e^{2x} - e^{-2x})dx = \frac{1}{2} \int_0^{\ln 3} (e^{5x} - e^x)dx$

$= \frac{1}{2} \left[\frac{1}{5} e^{5x} - e^x \right]_0^{\ln 3} = \frac{1}{2} (\frac{3^5}{5} - 3 - \frac{1}{5} + 1)$

$= \frac{1}{10} (3^5 - 11)$

b) $2B = \int (2 \cos 3x \cos 4x)\sin 5x\, dx$

$= \int (\cos 7x + \cos x)\sin 5x\, dx$

$4B = \int 2 \cos 7x \sin 5x\, dx + \int 2 \cos x \sin 5x\, dx$

$= \int (\sin 12x - \sin 2x)dx + \int (\sin 6x - \sin 4x)dx$

$= -\int \frac{\cos 12x}{12} + \frac{\cos 2x}{2} - \frac{\cos 6x}{6} + \frac{\cos 4x}{4} + C_1$

$B = -\frac{\cos 12x}{48} + \frac{\cos 6x}{24} - \frac{\cos 4x}{16} + \frac{\cos 2x}{8} + C$

c) $C = \int_1^e \ln x\, dx + \int_1^e \ln^2 x\, dx$

$\int_1^e \ln x\, dx = \Big[x \ln x\Big]_1^e - \Big[x\Big]_1^e = e - (e - 1) = 1$

$$\int_1^e \ln^2 x \, dx \quad \left| \begin{array}{ll} u = \ln^2 x, & dv = dx \\ du = 2\ln x \cdot \frac{1}{x}, & V = x \end{array} \right.$$

$$= \left[x \ln^2 x\right]_1^e - 2 \int_1^e \ln x \, dx$$

$$= e - 2 \cdot 1 = e - 2$$

$$C = 1 + (e - 2) = e - 1.$$

d) $$D = \int_0^{\ln 4} \frac{e^{3x} - e^x}{e^{2x} + 1} dx = \int_1^4 \frac{t^2 - 1}{t^2 + 1} dt \quad \text{with } e^x = t$$

$$= \int_1^4 \left(1 - \frac{2}{t^2 + 1}\right) dt = \left[t - 2 \arctan t\right]_1^4$$

$$= 3 - 2(\arctan 4 - \arctan 1) = 3 + \frac{\pi}{2} - 2 \arctan 4.$$

In a more general case the integral

$$\int f(x, T(x)) dx$$

where $T(x)$ is a transcendental function, and f is an algebraic function of its arguments, may be reduced to previous types of integrals after a suitable substitution.

Example. Evaluate

a) $A = \int \frac{\sqrt{x} + x}{x(1 + \sqrt[3]{x})} dx$ b) $B = \int \frac{1 + \cos \frac{x}{2}}{\sin \frac{x}{3}} dx$

c) $C = \int \frac{1 + \sqrt{\cos x}}{\sin x} dx$

Solution.

a) Since $x^{1/2}$, $x^{1/3}$ can be expressed as power of $x^{1/6}$, the substitution $x = t^6$ gives

$$A = \int \frac{t^3 + t^6}{t^6(1 + t^2)} 6t^5 \, dt = 6 \int \frac{t^2 + t^5}{1 + t^2} dt$$

$$= 6 \int (t^3 - t + 1 + \frac{t - 1}{t^2 + 1}) dt =$$

$$= \frac{3}{2} t^4 - 3t^2 + 6t + 3 \ \ell n(t^2 + 1) - 5 \arctan t + C$$

$$= \frac{3}{2} x^{2/3} - 3 x^{1/3} + 6x^{1/6} + 3 \quad n(x^{1/3} \quad 1) - 6 \arctan$$

$$3 \ \ell n(x^{1/3} + 1) - 6 \arctan x^{1/6} + C$$

b) Setting $x = 6t$, we have

$$B = 6 \int \frac{1 + \cos 3t}{\sin 2t} dt = 6 \int \frac{1 + \cos^3 t - 3 \cos t \sin^2 t}{\sin 2t} dt$$

having integrand as a rational function of $\cos t$, and $\sin t$ it can be interated by half-angle substitution.

c) Setting $\cos x = u^2$ $(u > 0)$, we have

$$C = \int \frac{2u \, du}{(1 - u)(1 + u^2)} = \ell n \frac{\sqrt{1 + \cos x}}{1 - \sqrt{\cos x}} - \arctan \sqrt{\cos x} + C.$$

EXERCISES (7. 2)

21. Evaluate $\int_{-\pi}^{\pi} \frac{dx}{1 - \cos\Theta \cos x}$ $(0 < \Theta \leq \pi/2)$

22. Show that $\int_0^{\pi} \frac{d\Theta}{3\cos^2\Theta + 1} = \frac{\pi}{2}$.

23. Evaluate

a) $\int \frac{2 - \sin x}{2 + \cos x} dx$ b) $\int \frac{1 - r \cos x}{1 - 2r \cos x + r^2} dx$

24. Evaluate

a) $\int \cot^3 x \, dx$, b) $\int \tan^3 x \, dx$, c) $\int \csc^3 x \, dx$, d) $\int \csc^4 x \, dx$

25. Evaluate

a) $\int \cos^3\theta \sin^3\theta \, d\theta$ b) $\int \cos^6\theta \, d\theta$

26. Evaluate

a) $\int \frac{dx}{\cos x - \cos a}$, b) $\int \frac{dx}{\sin x - \sin a}$, c) $\int \frac{dx}{\tan x - \tan a}$

27. Evaluate

a) $\int \sec^3 3x \tan^5 3x \, dx$ b) $\int \sin^2 y \cos^3 y \, dy$

28. Evaluate

a) $\int \cos^5 x \, dx$ b) $\int \tan^5 3x \, dx$

29. Evaluate

a) $\int \frac{\cos^3 x}{1-2 \sin x} dx$, b) $\int \frac{\sin^3 x}{1+\cos x} dx$, c) $\int \frac{\tan x}{1+\tan x} dx$

30. Evaluate

a) $\int \sin x \operatorname{Sh} x \, dx$ b) $\int \frac{\operatorname{Ch} x + 1}{\operatorname{Ch} x - 1} dx$ c) $\int \frac{\operatorname{Sh}^2 x}{\operatorname{Ch}^3 x} dx$

31. Evaluate

a) $\int \exp \arcsin x \, dx$ b) $\int x \arccos x^2 \, dx$

c) $\int \frac{\cos x}{x} dx - \int \frac{\sin x}{x^2} dx$ d) $\int \frac{\sec^2 x}{x} dx - \int \frac{\tan x}{x^2} dx$

32. Evaluate

a) $\int_0^{\pi/2} \cos^5 x \, dx$ b) $\int_0^{\pi/2} \sin^6 x \, dx$

33. Evaluate

a) $\int_0^{\pi/4} \tan^3 x \, dx$ b) $\int_0^{\pi/2} \cos^4 3x \, dx$

34. Evaluate

a) $\int_0^{\pi/2} \sin^3 2x \, dx$ b) $\int_{\pi/4}^{\pi/2} \csc^6 x \, dx$

35. Find the area bounded by one arc of the curve $y = \sin^3 3x$ and the x-axis.

ANSWERS TO EVEN NUMBERED EXERCISES

24. a) $-\frac{1}{16}\cos 2x + \frac{1}{48}\cos^3 2x + c,$

b) $\frac{5}{16}x + \frac{1}{4}\sin 2x + \frac{3}{64}\sin 4x - \frac{1}{48}\sin^3 2x + c.$

26. a) $\csc a \,\ell n \left| \dfrac{\sin \frac{1}{2}(x+a)}{\sin \frac{1}{2}(x-a)} \right| + c,$

b) $\sec a \,\ell n \left| \dfrac{\sin \frac{1}{2}(x-a)}{\cos \frac{1}{2}(x+a)} \right| + c,$

c) $\cos^2 a \;\; n\,|\sin(x-a)| - \frac{1}{2}x \sin 2a + c.$

28. a) $\sin x + \frac{2}{3}\sin^3 x + \frac{\sin^5 x}{5} + c$

b) $\frac{1}{12}\tan^4 3x - \frac{1}{6}\tan^2 3x + \frac{1}{3}\ell n|\sec 3x| + c.$

30. a) $\frac{1}{2}(\sin x \operatorname{Ch} x - \cos x \operatorname{Sh} x) + c,$

b) $2 \operatorname{Coth}\frac{x}{2} - x + c,$ c) $3 \arctan e^x + \frac{1}{2}\operatorname{Tanh} x \operatorname{Sech} x + c$

32. a) 28/15, b) $5\pi/256$.

34. a) 1/3, b) 8/15

7.3 INTEGRATION OF FUNCTIONS INVOLVING $Ax^2 + Bx + C$

We treat here the integrals

$$\int f(x, Ax^2 + Bx + C)dx \quad (A \neq 0)$$

where the integrand involves the quadratic expression Ax^2+Bx+C

The following are of this type:

$$\int \frac{dx}{x\sqrt{x^2 - 2x}}, \quad \int \sqrt[3]{9x^2 + 4}\, dx, \quad \int \frac{x - \sqrt{9 + x - x^2}}{1 + \sqrt{9 + x - x^2}}\, dx$$

We transform $Ax^2 + Bx + C$ as follows:

$$Ax^2 + Bx + C = A\left[x^2 + \frac{B}{A}x + \frac{C}{A}\right]$$

$$= A\left[(x + \frac{B}{2A})^2 - \frac{B^2}{4A^2} + \frac{C}{A}\right]$$

$$= A\left[(x + \frac{B}{2A})^2 - \frac{B^2 - 4AC}{4A^2}\right]$$

Setting

$$x + \frac{B}{2A} = u$$

we have

$$Ax^2 + Bx + C = A\left(u^2 - \frac{\Delta}{4A^2}\right) . \qquad (\Delta = B^2 - 4AC)$$

If $A > 0$, then

$$Ax^2 + Bx + C \quad \text{involves} \quad \begin{cases} u^2 - a^2 & \text{when } \Delta > 0 \\ u^2 + a^2 & \text{when } \Delta < 0 , \end{cases}$$

and if $A < 0$, it involves $a^2 - u^2$.

Then we arrange the

TABLE

Integrand involving	Substitution	New integrand involves	du
$u^2 + a^2$	$u = a \tan t$ or $u = a \operatorname{Sh} t$	$a^2 \sec^2 t$ or $a^2 \operatorname{Ch}^2 t$	$a \sec^2 t\, dt$ or $a \operatorname{Ch} t\, dt$
$u^2 - a^2$	$u = a \sec t$ or $u = a \operatorname{Ch} t$	$a^2 \tan^2 t$ or $a^2 \operatorname{Sh}^2 t$	$a \sec t \tan t\, dt$ or $a \operatorname{Sh} t\, dt$
$a^2 - u^2$	$u = a \sin t$ (*) or $u = a \operatorname{Th} t$	$a^2 \cos^2 t$ or $a^2 \operatorname{Sech}^2 t$	$a \cos t\, dt$ or $a \operatorname{Sech}^2 t\, dt$

(*) The substitution $u = a \cos t$ works also, but $u = a \sin t$ is preferred.

After a substitution, the integrand becomes a function of a trigonometric or a hyperbolic function which is in turn can be transformed into a type given in §7. 2, if necessary.

Example. Evaluate

a) $A = \int \frac{x^2}{\sqrt{9 - x^2}} dx$, b) $B = \int \sqrt{x^2 - 4x}\, dx$, c) $C = \int \frac{1}{x\sqrt{4x^2+x-1}} dx$

Solution

a) Setting $x = 3 \sin t$, we have

$$A = \int \frac{9 \sin^2 t}{3 \cos t} \cdot 3 \cos t\, dt = 9 \int \sin^2 t\, dt$$

$$= \frac{9}{2} \int (1 - \cos 2t) dt = \frac{9}{2} (t - \frac{\sin 2t}{2}) + c$$

$$= \frac{9}{2} (\arcsin \frac{x}{3} - \sin t \cos t) + c$$

$$= \frac{9}{2} (\arcsin \frac{x}{3} - \frac{x}{3} \sqrt{1 - \frac{x^2}{9}}) + c$$

Apply the other substitution!

b) $x^2 - 4x = (x - 2)^2 - 4$ $\qquad u = x - 2$

$B = \int \sqrt{u^2 - 4}\, du$. Setting $u = 2\, \mathrm{Ch}\, t$, we have

$$B = \int \sqrt{4 \mathrm{Ch}^2 t - 4} \cdot 2 \mathrm{Sh} t\, dt$$

$$= 4 \int \mathrm{Sh}^2 t\, dt = 2 \int (\mathrm{Ch}\, 2t - 1) dt = \mathrm{Sh}\, 2t - 2t + C$$

$$= 2\, \mathrm{Sh}t\, \mathrm{Ch}t - 2\, \mathrm{ArgCh} \frac{u}{2} + C$$

$$= 2 \sqrt{\frac{u^2}{4} - 1}\, \frac{u}{2} - 2\, \mathrm{Argch} \frac{u}{2} + C$$

$$= \frac{1}{2} (x - 2) \sqrt{x^2 - 4x} - 2 \mathrm{Argch} \frac{x - 2}{2} + c$$

Apply the other substitution!

c) $C = \int \frac{1}{x\sqrt{4x^2 + x - 1}} dx$

If the integrand had no x as a factor on the denominator, the evaluation could be done by the process used in (b). To remove the factor x, the *reciprocal substitution* $x = 1/y$ is applied as follows:

$$C = \int \frac{1}{\frac{1}{y}\sqrt{\frac{4}{y^2} + \frac{1}{y} - 1}}\left(-\frac{dy}{y^2}\right) = -\int \frac{dy}{\sqrt{4 + y - y^2}}$$

which does not contain a factor in the denominator.

$$4 + y - y^2 = 4-(y^2 - y) + 4-(y - \tfrac{1}{2})^2 + \tfrac{1}{4} = \tfrac{17}{4} - (y - \tfrac{1}{2})^2$$

Then setting

$$u = y - \tfrac{1}{2}, \qquad du = dy$$

we have

$$C = -\int \frac{du}{\sqrt{\frac{17}{4} - u^2}}$$

$$u = \frac{17}{2}\,\text{sint},$$

and by

$$C = -\int \frac{\frac{1}{2}\sqrt{17}\,\text{cost}}{\frac{1}{2}\sqrt{17}\,\text{cost}}\,dt = -t + c = -\arcsin\frac{2u}{\sqrt{17}} + c$$

$$= -\arcsin\frac{2}{\sqrt{17}}(y - \tfrac{1}{2}) + c = -\arcsin\frac{2}{\sqrt{17}}\left(\tfrac{1}{x} - \tfrac{1}{2}\right) + c$$

EXERCISES (7. 3)

36. Evaluate

a) $\int \frac{(4x + 9)dx}{(x^2 + 4x + 13)^2}$ b) $\int \frac{(3x + 7)dx}{(x^2 + 2x + 2)(x^2 + 4)}$

37. Evaluate

e) $\int \frac{dx}{3x^2 + 4x + 2}$ b) $\int \frac{e^t\,dt}{e^{2t} + 3e^t + 2}$

38. Evaluate

a) $\int \frac{x+1}{\sqrt{2x-x^2}}\,dx$ b) $\int \frac{dx}{\sqrt{x^2-2x-8}}$

39. Evaluate

a) $\int \frac{dx}{(1+x^2)^{3/2}}$ b) $\int \sqrt{1-x^2}\,dx$

40. Evaluate $\int \tan^5 x\,dx$ by setting $\tan x = t$.

41. Evaluate

a) $\int_0^{\pi/4} \frac{dx}{3-5\sin 2x}$ b) $\int_0^{\pi/2} \frac{dx}{3+2\sin x+\cos x}$

42. Evaluate

a) $\int_0^{\ln 2} \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$ b) $\int_{\ln 2}^{\ln 3} \frac{dt}{e^t-e^{-t}}$

43. Evaluate

a) $\int_0^1 \frac{dx}{3x^2+4x+2}$ b) $\int_0^{\ln 2} \frac{e^t\,dt}{e^{2t}+3e^t+2}$

44. If $Df(x) = x \arcsin x$ and $Dg(x) = x^2 \operatorname{Ch} x$, find $f(x)$ and $g(x)$.

45. Evaluate

a) $A = \int \sqrt{4x^2-8x+13}\,dx$ b) $B = \int \frac{dx}{\sqrt{-x^2+3x-2}}$

46. Evaluate

$$\int_0^3 \frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\,dx$$

47. Evaluate

a) $\int \frac{dx}{x^2\sqrt{x^2+9}}$ b) $\int \frac{dx}{x^2\sqrt{4-x^2}}$

c) $\int \frac{dx}{(4+x^2)^{3/2}}$ d) $\int \frac{dx}{\sqrt{4x-x^2}}$

48. Evaluate

a) $\int (y^3+1)\ln y\,dy$ b) $\int x^4 \ln^4 x\,dx$

49. Evaluate

a) $\int x \arcsin x\,dx$, b) $\int x \ln x\,dx$

c) $\int x \csc^2 x \, dx$ d) $\int x^{3/2} \ln 3x \, dx$

50. Evaluate by the given substitution:

a) $\int_{5/4}^{5/3} \frac{dx}{\sqrt{x^2 - 1}}$, $x = 1/\sin t$ b) $\int_0^1 \frac{x^2 \, dx}{(2 - x^3)\sqrt{1 - x^3}}$, $x = \sqrt[3]{1 - y^2}$

c) $\int_0^{\pi} \frac{\sin x \, dx}{1 - 2a\cos x + a^2}$, $t = \sqrt{1 - 2a \cos x + a^2}$

51. Evaluate

a) $\int_4^8 \frac{dx}{(x^2 - 4)^{3/2}}$ b) $\int_4^6 \frac{dx}{x\sqrt{x^2 - 4}}$

c) $\int_{\sqrt{2}/2}^{\sqrt{3}/2} x^2\sqrt{1 - x^2} \, dx$ d) $\int_0^8 (4 - x^{2/3})dx$

52. Determine convergence or divergence, and find the value when convergent:

a) $\int_0^{\infty} x \, e^{-x} \, dx$ b) $\int_0^{\infty} e^{-x} \sin x \, dx$

53. Evaluate

a) $\int \frac{dx}{(1 - x^2)^{5/2}}$, b) $\int \frac{\sqrt{x} \, dx}{\sqrt{a^3 - x^3}}$, c) $\int \frac{dx}{1 + x\sqrt{1 + x^2}}$

54. Evaluate $A = \int \frac{\sqrt{x} + x}{x(1 + \sqrt[3]{x})} dx$

55. Evaluate

a) $\int \frac{\sqrt{\sin x}}{\cos x} dx$ b) $\int \frac{dx}{\sqrt{\cos x(1 - \cos x)}}$

ANSWERS TO EVEN NUMBERED EXERCISES

36. a) $\frac{x - 34}{18(x^2 + 4x + 13)} + \frac{1}{54} \arctan \frac{x + 2}{3} + c$

b) $\frac{1}{2} \ln(x^2 + 2x + 2) + \frac{1}{2} \arctan (x+1) - \frac{1}{2} \ln (x^2+1) + \frac{1}{4} \arctan \frac{x}{2} + c$

38. a) $2 \arcsin (x-1) - \sqrt{2x - x^2} + c$, b) $\ln|x-1+\sqrt{x^2-2x+8}| + c$

40. $\frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \frac{1}{2}\ln|\sec x| + c$

42. a) $\ln\frac{5}{4}$, b) $\ln\frac{3}{2}$

44. a) $\frac{1}{4}(2x^2 - 1)\arcsin x + \frac{1}{4}x\sqrt{1 - x^2} + c$

b) $(x^2 + 2)\,\mathrm{Sh}\,x - 2x\,\mathrm{Ch}\,x + c$

46. $4\ln\frac{3}{2} - 1$

48. a) $\frac{x^5}{3125}\left[625\ln^4 x - 500\ln^3 x + 300\ln^2 x - 120\ln x + 24\right] + c$

b) $\frac{1}{4}(y^4 + 4y)\ln y - \frac{1}{16}y^4 - y + c$

50. a) $\ln\frac{116}{111}$ b) $\pi/6$ c) $2/a$

52. a) 1 b) 1/2

54. $\frac{3}{2}x^{2/3} - 3x^{1/3} + 6x^{1/6} + 3\ln(1 + x^{1/3}) - 6\arctan x^{1/6} + c$

A SUMMARY
(CHAPTER 7)

7.1 <u>Decomposition of a proper rational function $P(x)/Q(x)$:</u>

To each factor $(x-a)^r$ of $Q(x)$ corresponds a sum

$$\sum_{k=1}^{r}\frac{A_k}{(x-a)^k}$$

of partial fractions, and to each factor $(x^2 + px + q)^r$ with $\Delta < 0$ corresponds the sum

$$\sum_{k=1}^{r}\frac{B_k x + C_k}{(x^2 + px + q)^k}$$

of partial fractions.

7.2 Half-angle substitutions: $\tan\frac{x}{2} = t$

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad \tan x = \frac{2t}{1-t^2}, \quad dx = \frac{2dt}{1+t^2}$$

is applied to $\int R\,dx$ where R is a rational function of trigonometric functions.

Half-argument substitutions: $\mathrm{Th}\frac{x}{2} = t$

$$\mathrm{Sh}\,x = \frac{2t}{1-t^2}, \quad \mathrm{Ch}\,x = \frac{1+t^2}{1-t^2}, \quad \mathrm{Th}\,x = \frac{2t}{1+t^2}, \quad dx = \frac{2\,dt}{1-t^2}$$

is applied to $\int R\,dx$ where R is a rational function of hyperbolic functions.

7.3 In integrals involving $Ax^2 + Bx + C$, the latter is transformed to $u^2 - a^2$ or $u^2 + a^2$ for $A > 0$, and to $a^2 - u^2$ for $A < 0$, and then the following substitutions are applied:

For $u^2 - a^2$: $u = a\,\mathrm{ch}\,t$ or $u = a \sec t$

For $u^2 + a^2$: $u = a\,\mathrm{Sh}\,t$ or $u = a \tan t$

For $a^2 - u^2$: $u = a\,\mathrm{Sh}\,t$ or $u = a \sin t$

MISCELLANEOUS EXERCISES

56. Decompose teh following rational function into partial fractions:

$$\frac{x^5 + 6x}{(x^2 + 2x + 4)^3}$$

57. Same question for $\frac{1}{(x-2)(x^2 - 6x + 10)^2}$

58. Same question for $\frac{1}{(x+1)(x+2)^2}$

59. Evaluate $\int_{\pi/4}^{\pi/2} \frac{4\mathrm{Sin}x - 7\mathrm{Cos}x}{3\mathrm{Sin}x - 2\mathrm{Cos}x}\,dx$

60. Evaluate $\int_{\ln 2}^{\ln(2-\frac{\pi}{3})} e^x \sec^2(2-e^x)dx$

61. Evaluate $\int_1^e \frac{dx}{x(1+\ln^2 x)}$

62. Evaluate $\int \frac{dx}{a^2 - 2ab\cos x + b^2}$

63. Evaluate $\int \sqrt{\frac{1+x}{1-x}}\, dx$

64. Evaluate

a) $\int \frac{x^3 - 2}{x^2 - x}\, dx$ b) $\int \frac{x+1}{x^3(x^2+1)^2}\, dx$

65. Evaluate $\int \frac{x\ dx}{x^4 + 1}$

66. Find the area of the region enclosed by the curves:

$$y = \frac{x}{(x+1)(x-2)(x-4)}, \quad x = -4, \quad x = -2, \quad y = 0$$

67. Determine the constant a such that

$$\int_0^a x^3\sqrt{a^2 - x^2}\, dx = \int_0^a x\sqrt[3]{a^2 - x^2}\, dx$$

68. Evaluate

a) $\int \frac{\cos x}{1+\sin x}\, dx$ b) $\int \frac{x\ dx}{9 + x^4}$

69. Evaluate

a) $\int \frac{e^x\ dx}{1 + 4e^{2x}}$, b) $\int \frac{dx}{4 + 9x^2}$

70. Evaluate $\int \frac{x^2\ dx}{x^4 + 1}$

71. Evaluate

$$\int_0^{\pi/2} \frac{d\theta}{a^2\cos^2\theta + b^2\sin^2\theta}, \quad (\tan\theta = t)$$

72. Evaluate

$$\int \frac{d\theta}{\cos^4\theta + \sin^4\theta}$$

73. Prove

a) $\int_1^n \ln x\, dx = \ln(n^n e^{-n+1})$

b) $\ln\left[(n-1)!\right] < \int_1^n \ln x\, dx < \ln(n!)$ (by lower and upper sums on the regular partition $(1, 2, \ldots, n)$)

c) (a) and (b)

$$(n-1)! < n^n e^{-n} e < n!$$

74. Prove

a) $\sqrt[n]{e} \cdot \frac{1}{e} < \frac{\sqrt[n]{n!}}{n}$ (from the right inequality in Example 73c)

b) $\frac{\sqrt[n]{n!}}{n} < \sqrt[n]{n} \cdot \sqrt[n]{e} \cdot \frac{1}{e}$ (from the left inequality in Example 73c)

c) (a) and (b)

$$\lim_{n \to \infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e} \quad \text{(using } \sqrt[n]{n} \to 1 \text{ and } \sqrt[n]{e} \to 1\text{)}$$

75. Use the result in Example 74c to evaluate

a) $\lim_{n \to \infty} \frac{\sqrt[kn]{(kn)!}}{n}$

b) $\lim_{n \to \infty} \frac{\sqrt[n]{(3n)!}}{9n^3}$

c) $\lim_{n \to \infty} \frac{\sqrt[3n]{n!}}{27\sqrt[3]{n}}$

d) $\lim_{n \to \infty} \frac{\sqrt[n]{(2n)!\,(3n)!}}{n^2}$

76. Evaluate $\int_0^2 \frac{dx}{2x + \sqrt{4 - x^2}}$

77. Evaluate

a) $\int \frac{1 + x}{1 + \sqrt{x}}\, dx$

b) $\int \frac{dx}{\sqrt[3]{1 + x} - \sqrt{1 + x}}$

78. Evaluate $\int \frac{x^{1/3} + x^{5/6}}{x^{1/3} + 1}\, dx$

79. Evaluate $\int x\sqrt{\frac{x + 3}{x - 2}}\, dx$

80. Evaluate

a) $\int x\sqrt{\frac{1+x}{1-x}}\,dx$ b) $\int_0^2 \frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\,dx$

81. Evaluate $\int_a^b \frac{dx}{\sqrt{(x-a)(b-x)}}$

82. If $R(x, \sqrt{ax^2+bx+c})$ is a rational function of its arguments, show that it becomes a rational function of t upon the substitution:

a) $t = \sqrt{a}\,x + \sqrt{ax^2+bx+c}$ when $a > 0$,

b) $t = \sqrt{(-a)\cdot\frac{x-x_1}{x_2-x}}$ when $a < 0$, where x_1, x_2 are the (real) roots of $ax^2+bx+c=0$ $(x_1 < x_2)$

83. Apply the substitution given in Exercise 82 to transform

a) $\int \frac{3-\sqrt{4x^2+x-1}}{x+\sqrt{4x^2+x-1}}\,dx$ b) $\int \frac{\sqrt{x-x^2}}{1+\sqrt{x-x^2}}\,dx$

into ones with integrand as rational function of t.

84. Evaluate $\int \frac{1-\sqrt[3]{x}}{\sqrt{x}}\,dx$

85. Show that $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx = \frac{\pi^2}{4}$

86. Find the area of the region bounded by the x-axis, the curve $y = x\,e^{-x}$ and the vertical line through the maximum point.

87. Evaluate $\int x\,e^x \cos x\,dx$

88. Find the area between the two curves:

a) $y = \ln x$, $y = \ln\frac{1}{x}$, $1 < x < e$

b) $y = \sin^2 x$, $y = \sec x$, $\frac{2\pi}{4} < x < \frac{5\pi}{4}$

89. Given $I_n = \int \cos(n\arctan x)dx$, show that

$$I_{n+2} + 2I_n + I_{n-2} = \frac{4}{n}\sin(n\arctan x) + c$$

90. Determine the convergence or divergence, and find the value

if convergent:

a) $\int_0^{\infty} x^2 e^{-3x} dx$ b) $\int_{-\infty}^{0} e^{2x} \cos x \, dx$

91. Find a relation between the constants a, b for which a primitive of

$$\frac{x^2 - 1}{(x-a)^2(x-b)^2}$$

be a rational function.

92. Find the area of the region bounded by the curve $y = \frac{1}{x^2(x^2+4)}$ and the lines $x = 2$, $x = 2\sqrt{3}$ and $y = 0$.

93. Find the area of the region enclosed by one loop of the curve

$$y^2 = x^4(1 - x^2)^3$$

94. Same question for $x^2 = y^4(1 - y^2)$

95. Find the area enclosed by the curve of

$$x^{2/3} + y^{2/3} = a^{2/3} \quad (a > 0).$$

ANSWERS TO EVEN NUMBERED EXERCISES

56. $\frac{x - 4}{x^2 + 2x + 4} + \frac{4x + 24}{(x^2 + 2x + 4)^2} - \frac{16x - 32}{(x^2 + 2x + 4)^3}$

58. $\frac{1}{x + 1} - \frac{1}{x + 2} - \frac{1}{(x + 2)^2}$

60. $-\sqrt{3}$

62. $\frac{2}{a^2 - b^2} \arctan \left(\frac{a + b}{a - b} \arctan \frac{x}{2}\right) + c.$

64. a) $\ln \left| \frac{x^2}{x - 1} \right| + \frac{1}{2} x^2 + x + c$

b) $-2 \ln|x| - \frac{1}{x} - \frac{1}{2x^2} + \ln(x^2+1) - \frac{3}{2} \arctan x - \frac{1}{2} \frac{x+1}{x^2+1} + c$

66. $\frac{4}{5}\ln 3 - \frac{17}{15}\ln 2.$

68. a) $\arctan \sin x + c,$ b) $\frac{1}{6}\arctan \frac{x^2}{3} + c$

70. $\frac{\sqrt{2}}{8}\left[\ln \frac{x^2 - x\sqrt{2} + 1}{x^2 + x\sqrt{2} + 1} - \arctan \frac{1}{x^2}\right] + c$

72. $\frac{\sqrt{2}}{2}\arctan\left(\frac{\sqrt{2}}{2}\tan 2x\right) + c$

76. $\frac{\pi}{10} + \frac{2}{5}\ln 2.$

78. $\frac{2}{3}x^{3/2} - \frac{6}{7}x^{7/6} + x + \frac{6}{5}x^{5/6} - \frac{3}{2}x^{2/3} + 3x^{1/3} + 6x^{1/6}$

$- 6\ln(1 + x^{1/3}) - 6\arctan x^{1/6} + c.$

80. a) $14\arctan\sqrt{\frac{1+x}{1-x}} - 5\sqrt{1-x^2} - 20\sqrt{\frac{1+x}{1-x}} + 2(1-x)\sqrt{1-x^2} + c$

b) $4\ln\frac{3}{2} - 1.$

84. $2\sqrt{x} - \frac{6}{5}\sqrt[6]{x^5} + c$

86. $1 - 2/e$

88. a) 2, b) $\frac{\pi}{4} + \ln(3 + 2\sqrt{2})$

90. a) 2/27, b) 2/5

92. $(12 - \pi - 4\sqrt{3})/96$

94. $\pi/8.$

CHAPTER 8

APPLICATIONS OF DEFINITE INTEGRAL

8. I GEOMETRIC APPLICATIONS

A. AREA:

We recall the area formulas

$$|R_{xy}| = \int_a^b \left[y_2(x) - y_1(x)\right]dx, \quad R_{yx} = \int_c^d \left[x_2(y) - x_1(y)\right]dy$$

for normal regions

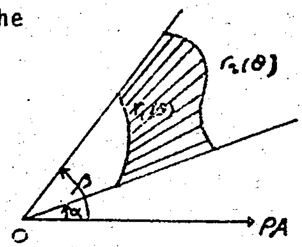
$$R_{xy} = \left[a,\ b;\ y_1(x),\ y_2(x)\right], \quad R_{yx} = \left[c,\ d;\ x_1(y),\ x_2(y)\right].$$

In polar coordinates normal regions are similarly defined; one type is denoted by $R_{\theta r}$ the other being $R_{r\theta}$:

The region

$$\{(\theta,\ r):\ \alpha \leqslant \theta \leqslant \beta;\quad r_1(\theta) \leqslant r \leqslant r_2(\theta)\}$$

bounded by the rays $\theta = \alpha$, $\theta = \beta$ and by the curves $r = r_1(\theta)$; $r = r_2(\theta)$ is called a normal region denoted by $R_{\theta r}$, written shortly



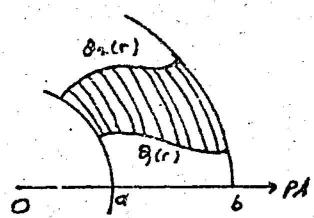
$$R_{\theta r} = \left[\alpha,\ \beta;\ r_1(\theta),\ r_2(\theta)\right]$$

where $r = r_1(\theta)$, $r = r_2(\theta)$ are called the *inner (first)* and *outer (second)* curves respectively.

The region

$$\{(\theta,\ r):\ a \leqslant r \leqslant b;\quad \theta_1(r) \leqslant \theta \leqslant \theta_2(r)\}$$

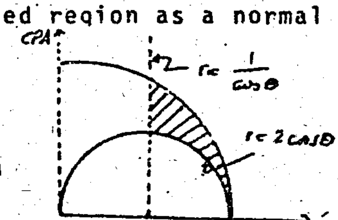
bounded by the circles $r = a$, $r = b$ and by the curves $\theta = \theta_1(r)$, $\theta = \theta_1(r)$ is called a normal region denoted by $R_{r\theta}$ and written



$$R_{r\theta} = \left[a,\ b;\ \theta_1(r),\ \theta_2(r)\right]$$

where $\theta = \theta_1(r)$, $\theta = \theta_2(r)$ are called the *initial (first)* and *terminal (second)* curves respectively.

<u>Example 1</u>. Express the given shaded region as a normal region $R_{\theta r}$ or as union of such regions when necessary.



<u>Solution</u>. The region lies between

the rays $\Theta = 0$, $\Theta = \frac{\pi}{3}$. The outer curve is the circle $r = 2$, but the inner one consists of two curves, namely the circle $r = 2\cos\Theta$ and the line $r = 1/\cos\Theta$. Then the region is to be written as union

$$R = R'_{\Theta r} \cup R''_{\Theta r}$$

$$= \left[0, \frac{\pi}{4} ; 2\cos\Theta, 2\right] \cup \left[\frac{\pi}{4}, \frac{\pi}{3} ; \frac{1}{\cos\Theta}, 2\right]$$

Example 2. Express the above region as normal region $R_{r\Theta}$ or as union of such regions.

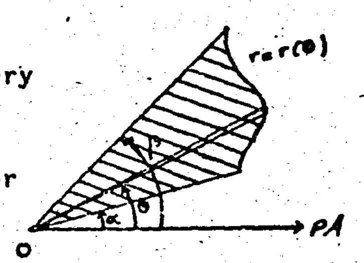
Solutions. The region is bounded by the circles $r = \sqrt{2}$ and $r = 2$ and the circle $r = 2\cos\Theta$ (the initial curve), the line $r = 1/\cos\Theta$ (the terimal curve). Hence the region is a normal region of type $R_{r\Theta}$ and

$$R_{r\Theta} = \left[\sqrt{2}, 2; \arccos\frac{r}{2}, \operatorname{arcsec} r\right].$$

The area of $R_{\Theta r}$:

We first treat the area of a normal region $R_{\Theta r}$ for $r_1(\Theta) = 0$, that is, the area of a region bounded by two rays and a curve.

Consider an element of area in the shape of wedge with sides r, $r + \Delta r$ having an angle $\Delta\Theta$ between them. For very small $\Delta\Theta$ this wedge is in the shape of triangle (in a partition of (α, β) for large n). Then



$$\Delta A = \frac{1}{2} r(r + \Delta r) \sin \Delta\Theta \quad \text{(From } S = \frac{1}{2} bc \sin A)$$

$$\frac{\Delta A}{\Delta\theta} = \frac{1}{2} r\ (r + \Delta r)\ \frac{\sin\ \Delta\theta}{\Delta\theta} \Rightarrow \frac{dA}{d\theta} = \frac{1}{2} r^2$$

$$\Rightarrow \quad dA = \frac{1}{2} r^2\ d\theta, \qquad A = \frac{1}{2} \int_{\alpha}^{\beta} r^2\ d\theta$$

Corollary. The area of the normal region

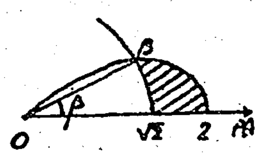
$$R_{\theta r} = \left[\alpha, \beta;\ r_1(\theta)\ r_2(\theta)\right]$$

is

$$|R_{\theta r}| = \frac{1}{2} \int_{\alpha}^{\beta} \left[r_2^2(\theta) - r_1^2(\theta)\right] d\theta.$$

Example. Compute the area of the region inside the lemiscate $r^2 = 4 \cos 2\theta$ and outside the circle $r = \sqrt{2}$.

Solution. From symmetry of the figure with respect to polar and copolar axes, the required area is four times that of the shaded region.



From simultaneous solution we get $B(\sqrt{2},\ \pi/6)$. Then

$$A = 4 \cdot \frac{1}{2} \int_{0}^{\pi/6} (4\ \cos 2\theta\ - 2) d\theta = 2\sqrt{3} - 2\pi/3.$$

The area of $R_{r\theta}$: Let

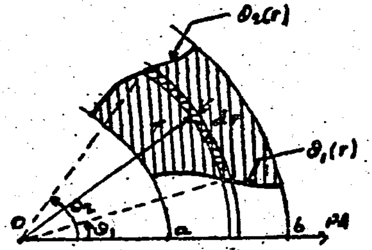
$$R_{r\theta} = \left[a,\ b;\ \theta_1(r),\ \theta_2(r)\right].$$

Then

$$|R_{r\theta}| = \int_{a}^{b} \left[\theta_2(r) - \theta_1(r)\right] r\ dr.$$

Indeed, consider $R_{r\theta}$ as the shaded region in the given figure. Here an element of area is a part of a circular ring of radius r, central angle $\theta_2(r) - \theta_1(r)$ and width dr. Then

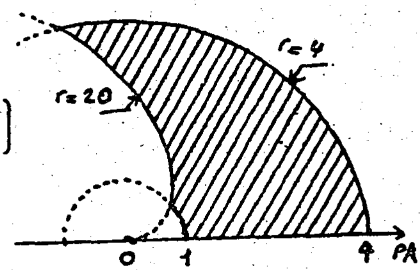
$$dA = r.(\theta_2 - \theta_1) dr$$



implying

$$A = \int_a^b (\theta_2 - \theta_1) r\, dr.$$

<u>Example</u>. Find the area of the region bounded by the circles $r = 1$, $r = 4$, the line $\theta = 0$ and the ARCHEMEDEAN spiral $r = 2\theta$.



<u>Solution</u>. The region is

$$R_{r\theta} = \{1, 4; \ \theta_1 = 0, \ \theta_2 = r/2\}$$

and

$$A = \int_1^4 \left(\frac{r}{2} - 0\right) r\, dr = 21/2.$$

Verify this result by interpreting the region as $R_{\theta r}$!

When the two curves enclosing the region are given in parametric form, substitute them in the integral.

<u>Example</u>. Compute the area of the region under the arc of the curve $x = t + 1/t$, $y = t - 1/t$ from $t = 1$ to $t = 2$.

<u>Solution</u>. The area is above the x-axis for $t \in (1, 2)$. Then

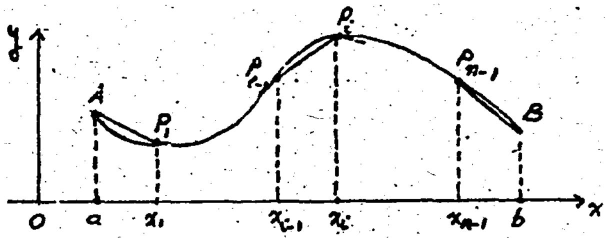
$$A = \int_a^b y\, dx = \int_1^2 \left(t - \frac{1}{t}\right)\left(1 - \frac{1}{t^2}\right) dt = \frac{15}{8} - 2 \ln 2.$$

B. Arc Length.

Consider a function $y = f(x)$ differentiable (and hence continuous) on a closed interval $[a, b]$ with graph as shown in the figure.

We wish to express the arc length of this curve between its end points $A(x = a)$ and $B(x = b)$. The method is similar to one given for evaluating the area under a curve. It consists of partioning $[a, b]$ by points

$$a(=x_0),\ \dots,\ x_{i-1},\ x_i,\ \dots,\ b(=x_n)$$



and considering the corresponding points $P_i(x_i)$ on the arc, which when joined successively giving the polygonal line $AP_1 \dots P_{i-1}P_i \dots P_{n-1}B$ whose length

$$\sum_{i=1}^{n} |P_{i-1}P_i|$$

is a rough approximation of the required arc length s of the curve. Approximation gets better and better as $n \to \infty$ and

$= \max\{|P_0P_1|, \dots, |P_{n-1}P_n|\} \to 0$, and the

$$\lim_{\substack{n \to \infty \\ \to 0}} \sum_{i=1}^{n} |P_{i-1}P_i|$$

gives s by definition, where

$$|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}.$$

By MVT

$$f(x_i) - f(x_{i-1}) = (x_i - x_{i-1})f'(\xi_i),\quad x_{i-1} < \xi_i < x_i$$

implying

$$|P_{i-1}P_i| = \sqrt{1 + \left[f'(\xi_i)\right]^2}\,(x_i - x_{i-1}).$$

Then

$$s = \lim_{\substack{n \to \infty \\ \to 0}} \sum_{i=1}^{n} \sqrt{1 + \left[f'(\xi_i)\right]^2}\ \Delta x_i$$

which is the RIEMANN integral

$$s = \int_a^b \sqrt{1 + f'^2(x)}\,dx \tag{1}$$

when $f'(x)$ is continuous in the interval.

Corollary 1. If $y = f(x)$ has continuous derivative in $[a, b]$, then for the element of arc ds we have

a) $ds = \sqrt{1 + f'^2(x)}\, dx$ (from the Fund. Theorem of Calculus)

b) $ds = \sqrt{dx^2 + dy^2}$ (since $f'(x) = dy/dx$)

c) $ds = \sqrt{(\frac{dx}{dy})^2 + 1}\, dy$

When the curve is given by the equation

$$x = g(y)$$

where $g'(y)$ is continuous in $[c, d]$, we have

$$s = \int_c^d \sqrt{1 + g'^2(y)}\,dy \qquad (1')$$

from Corollary 1c.

Example. Find the arc length of $y = x^2$ from $x = 0$ to $x = 2$.

Solution.

$$s = \int_0^2 \sqrt{1 + 4x^2}\, dx = \frac{1}{2}\left[\ln(4 + \sqrt{17}) + 4\sqrt{17} - 1\right].$$

Compute the same arc length by (1') above!

Corollary 2. If $x = x(t)$, $y = y(t)$ are the parametric equations of a curve with continuous derivatives in $[t_1, t_2]$, then

a) $ds^2 = dx^2 + dy^2$ $(dx = \dot{x}(t)dt,\ dy = \dot{y}(t)dt)$

b) $s = \int_{t_1}^{t_2} \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}\, dt.$

Corollary 3. If $r = f(\theta)$ is the polar equation of a curve with continuous derivative in $[\theta_1, \theta_2]$, then

a) $ds^2 = dr^2 + r^2\, d\theta^2$ b) $s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + r'^2}\, d\theta$

Proof. Setting in Corollary 2 b,

$$x = r(\theta)\cos\theta, \quad y = r(\theta)\sin\theta$$

we have

a) $ds^2 = dx^2 + dy^2$

$$= (\cos\theta\, dr - r\sin\theta\, d\theta)^2 + (\sin\theta\, dr + r\cos\theta\, d\theta)^2$$

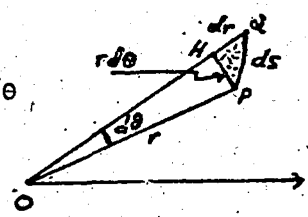
$$= dr^2 + r^2 d\theta^2.$$

Observe this relation in the infinitely small triangle HPQ.

b) $ds = \sqrt{dr^2 + r^2 d\theta^2}$

$$= \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta = \sqrt{r^2 + r'^2}\, d\theta$$

$$s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + r'^2}\, d\theta$$



Element of arc in polar coordinates

Show that for the curve

$$\theta = g(r),$$

s is given by

$$s = \int_{r_1}^{r_2} \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\, dr$$

Example. Find the length of curve

$$x = e^t, \quad y = \frac{1}{2}e^{2t} - \frac{t}{4} \quad \text{for} \quad t\varepsilon[0, 1].$$

Solution.

$$S = \int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2}\,dt = \int_0^1 \sqrt{e^{2t} + e^{4t} - \frac{1}{2}e^{2t} + \frac{1}{16}}\, dt$$

$$= \int_0^1 \sqrt{e^{2t} + \frac{1}{2}e^{2t} + \frac{1}{16}}\, dt = \int_0^1 (e^{2t} + \tfrac{1}{4})dt = e^2/2 - 1/4.$$

Example. Find the perimeter of the cardioid $r = a(1+\cos\theta)$.

Solution. The graph being symmetric with respect to the polar axis, we have

$$s = 2\int_0^{\pi} \sqrt{r^2 + r'^2}\,d\theta = 2a\int_0^{\pi} \sqrt{2 + 2\cos\theta}\, d\theta = 4a\int_0^{\pi} \cos\frac{\theta}{2}\, d\theta = 8a.$$

Polar slope:

If we call the slope

$$m = \tan\alpha = \frac{dy}{dx}$$

of the tangent line to a curve $y = y(x)$ in cartesian coordinates, the *cartesian slope* of the curve at (x, y), where α is the angle that the tangent line makes with positive x-axis, we define the *polar slope* of the curve $r = f(\theta)$, at $P(\theta, r)$ as

$$\mu = \tan\psi$$

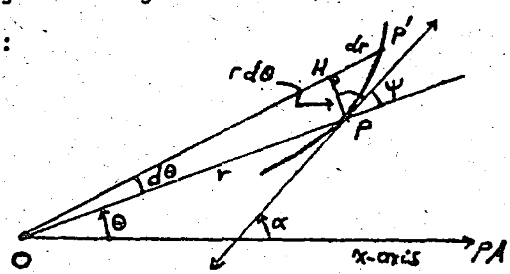
where ψ is the angle that the tangent line makes with the ray (OP).

The expression for polar slope μ is easily obtained by considering the elementary right triangle PHP' with sides $rd\theta$, dr and ds:

$$\mu = \tan\psi = \cot(\frac{\pi}{2} - \psi)$$

$$= \frac{r\,d\theta}{dr} = \frac{r}{dr/d\theta}$$

$$\mu = \tan\psi = \frac{r}{r'}$$



As the relation

$$m_1 m_2 = -1$$

indicates the orthogonality (perpendicularity) of two interesting curves in cartesian coordinates, the relation

$$\mu_1 \mu_2 = -1$$

gives orthogonality of two intersecting curves in polar coordinates.

Example 1. Find the points on the lemniscate $r^2 = 9\cos 2\theta$ at which the tangent lines are parallel to polar axis and find ψ.

Solution. For a horizontal tangent line we have $\psi = \pi - \theta$, (or $\psi = \theta$). Then

$$\frac{r}{r'} = \tan\psi = -\tan\theta$$

$$\Rightarrow \quad -\tan\theta = \frac{r}{r'} = \frac{r^2}{rr'} = \frac{9\cos 2\theta}{-9\sin 2\theta} = -\cot 2\theta$$

$$\Rightarrow \quad \tan\theta \tan 2\theta = 1 \Rightarrow \tan\theta = \pm 1/\sqrt{3} \Rightarrow \theta = \pm \pi/6 + \pi$$

$$\Rightarrow \quad \theta_{1,2} = \pm \pi/6, \quad \theta_{3,4} = \pm \pi/6 + \pi$$

and

$$\psi_{1,2} = \pi - \theta_{1,2} = \pi \pm \frac{\pi}{6}, \quad \psi_{3,4} = \pm \frac{\pi}{6}.$$

Example 2. Show that the circles $r = 2\sin\theta$ and $r = 4\cos\theta$ intersect orthogonally.

Solution. $2\sin\theta = 4\cos\theta \Rightarrow \tan\theta = 2 \Rightarrow \theta_0 = \arctan 2$.

$$\tan\psi_1 = \frac{2\sin\theta}{2\cos\theta} = \tan\theta = 2, \quad \tan\psi_2 = \frac{4\cos\theta}{-4\sin\theta} = -\cot\theta = -1/2$$

Then

$$\mu_1 \cdot \mu_2 = -1$$

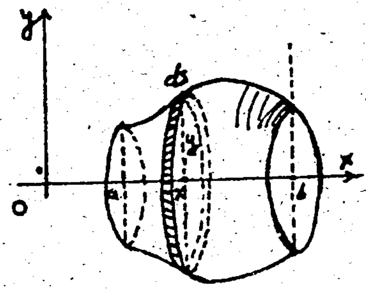
C. Surface Area of a Surface of Revolution

A surface of revolution is a surface generated when a curve is revolved about a (straight) line. This line is called the symetry axis of the surface.

Sphere is a familiar example of a surface of revolution.

Let $y = f(x)$ be a function with continuous derivative on $[a, b]$. When the curve is rotated about the x-axis (or y-axis) it generates a surface of revolution of area S_{ox} (or S_{oy}).

Consider an element of arc ds. After revolution it generates an element of surface in the shape of a slice of a cone whose lateral area is



$$dS_{ox} = 2\pi y\, ds, \quad dS_{oy} = 2\pi x\, ds$$

which by integration gives

$$S_{ox} = 2\pi \int_a^b y(x)ds, \quad S_{oy} = 2\pi \int_c^d x(y)$$

If the curve is given by parametric equation we have

$$S_{ox} = 2\pi \int_{t_1}^{t_2} y\sqrt{\dot{x}^2 + \dot{y}^2}\, dt, \quad S_{oy} = 2\pi \int_{t_1}^{t_2} x\sqrt{\dot{x}^2 + \dot{y}^2}\, dt$$

Example 1. Compute the surface area of the surface generated when the curve

$$y = \sqrt{x}, \quad x\varepsilon[1, 4]$$

is rotated about x-axis.

Solution.

$$S_{ox} = 2\pi \int_a^b y\, ds = 2\pi \int_1^4 y\sqrt{1 + y'^2}\, dx$$

$$= 2\pi \int_1^4 \sqrt{x}\sqrt{1 + \frac{1}{4x}}\, dx = \pi \int_1^4 \sqrt{1 + 4x}\, dx$$

$$= (\pi/6)\left[17\sqrt{17} - 5\sqrt{5}\right]$$

Example 2. Compute the area of the surface obtained when the curve

$$x = 3t(t - 2), \quad y = 8t^{3/2} \quad t\varepsilon[0, 1]$$

is rotated about x-axis.

Solution.

$$S_{ox} = 2\pi \int_0^1 y(t)\sqrt{\dot{x}^2 + \dot{y}^2}\, dt$$

$$= 2\pi \int_0^1 8t\sqrt{t}\ \sqrt{(6t - 6)^2 + 12^2 t}\, dt$$

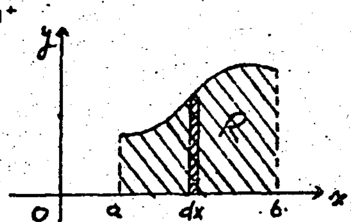
$$= 96\pi \int_0^1 t\sqrt{t}\ (t + 1)dt = \frac{2304}{35}\ \pi.$$

D. VOLUME OF A SOLID OF REVOLUTION:

A *solid of revolution* is the solid generated by revolving a plane region about a (straight) line. This line is called the symmetry axis of the solid. The boundary of a surface of revolution is certainly a surface of revolution.

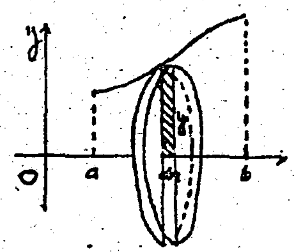
Consider first a region under the curve of a continuous positive function $y = f(x)$ bounded by the lines $x = a$, $x = b$.

When this region is revolved about the x-axis (or y-axis), it generates a solid of revolution whose volume is denoted by V_{ox} (or V_{oy}).

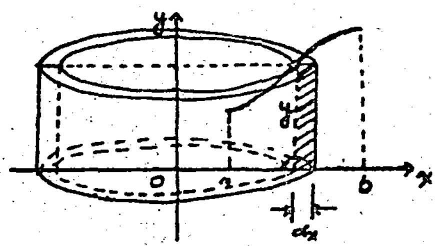


For this region, for convenience V_{ox} will be evaluated by what we call *disc method*, while V_{oy} by *shell method* as explained below.

Consider an element of area as a vertical strip in the region R. When R is rotated about x-axis (y-axis) the strip generates an element of volume in the form of a *disc* (a *shell*)



Disc of radius y and thichness dx



Shell of inner radius x, thickness dx and height y.

The volume of this disc:

$$dV_{ox} = \pi y^2\ dx$$

$$V_{ox} = \pi \int_a^b y^2\ dx.$$

The volume of this shell is the differential of the volume of inner cylinder (with constant height):

$$dV_y = d(\pi x^2).y = 2\pi xy\ dy$$

$$V_y = 2\pi \int_a^b xy\ dx$$

These formulas are for special normal region $(a,\ b;\ 0,\ f(x))$. More generally, if the normal region

$$R_{xy} = \left[a,\ b;\ y_1(x),\ y_2(x)\right]$$

is revolved about x-axis (y-axis) they become

$$V_{ox} = \pi \int_a^b (y_2^2 - y_1^2)dx$$

It is obtained by taking the difference between the volumes of disc relative to upper and lower curves

$$V_{oy} = 2\pi \int_a^b x\ .\ (y_2 - y_1)dx$$

It is obtained by taking the difference between the volumes of shells relative to upper and lower curves.

If the normal region is

$$R_{yx} = \left[c,\ d;\ x_1(y),\ x_2(y)\right]$$

then V_{oy} is obtained more easily by disc, and V_{ox} by shell method. The formulas for this region is obtained from the formulas for the region R_{xy} by interchanging the roles of x and y:

$$V_{ox} = 2\pi \int_c^d y(x_2 - x_1)dy, \qquad V_y = \pi \int_c^d (x_2^2 - x_1^2)dy.$$

For a given region, one of these methods is applicable more easier than the other in general.

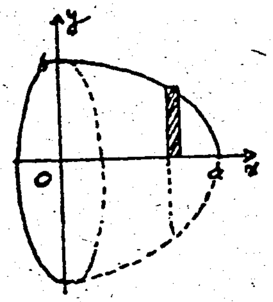
If the given region is not normal one, decomposition of it into normal ones is necessary, Example 1. Compute the volume of the solid generated by revolving an ellipse about its major axis, by two methods.

Solution. Let $b^2x^2 + a^2y^2 = a^2b^2$ be the equation of the ellipse.

a) By disc method:

$$\frac{1}{2} V_{ox} = \pi \int_0^a y^2 dx = \pi \frac{b^2}{a^2} \int_0^a (a^2-x^2)dx$$

$$V_{ox} = \frac{4}{3} \pi ab^2$$



b) By shell method:

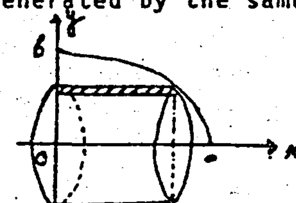
$$\frac{1}{2} V_{ox} = 2\pi \int_0^a yx\, dy$$

$$= 2\pi \frac{a}{b} \int_0^b y\sqrt{b^2 - y^2}\, dy$$

$$V_{ox} = \frac{4}{3} \pi ab^2.$$

Show that the volume of the solid generated by the same ellipse when revolved about y-axis is

$$V_{oy} = \frac{4}{3}\pi a^2 b.$$



(by two methods)

Example 2. Find the volume of the solid generated by revolving about y-axis the region bounded by

$$y = e^{x^2}, \quad x = 0, \quad x = 2, \quad y = 0$$

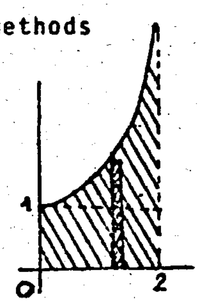
in two ways, and compare the convenience of the methods

Solution.

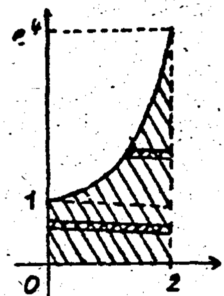
a) By shell method:

$$V_{oy} = 2\pi \int_0^2 xy\, dx = \pi \int_0^2 2x\, e^{x^2}\, dx$$

$$= \pi\left(e^{x^2}\right)_0^2 = \pi(e^4 - 1)$$



b) By disc method:



$$V_{oy} = \pi \int_0^1 (2^2 - 0)dy + \int_1^{e^4} (2^2 - \ell ny)dy$$

$$= 4\pi + 4\pi\ (e^4 - 1) - \int_1^{e^4} \ell ny\ dy$$

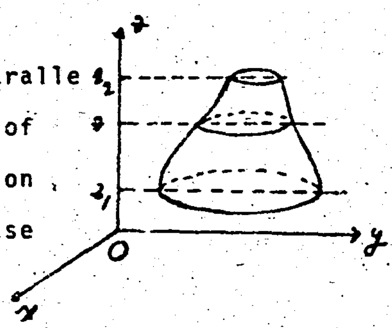
$$= 4\pi\ e^4 - \pi\Big[y\ \ell ny - y\Big]_1^{e^4}$$

$$= 4\pi\ e^4 - \pi\Big[4e^4 - e^4 - 0 + 1\Big] = \pi(e^4 - 1).$$

It is seen that the shell method is easier than the other.

E. Volume of a Solid with given Cross Sections Perpendicular to a given Line

We place the cartesian xyz-system in such way that cross sections be perpendicular to z-axis. The cross sections of the solid are all parallel to xy-plane.

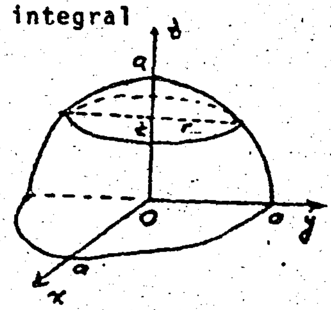


Let the solid lie between the paralle planes $z = z_1$ and $z = z_2$. If the area of any cross section is $A(z)$ as a function of z, then the volume of a slice of base $A(z)$ and hight dz will be

$$dV = A(z)dz$$

which gives the volume of the solid as the integral

$$V = \int_{z_1}^{z_2} A(z)dz$$



Example. Find the volume of the hemisphere of radius a.

Solution. Placing the hemisphere

as shown in the Fig., the area $A(z)$ of the cross section being

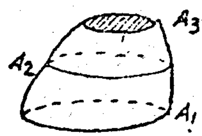
$$A(z) = \pi r^2 = \pi(a^2 - z^2),$$

we have

$$V = \pi \int_0^a (a^2 - z^2)dz = \pi a^3 - \pi \frac{a^3}{3} = \frac{2}{3} \cdot \pi a^3.$$

Now we estabhish an interesting formula which is called the *three level formula.*

Corollary. If the parallel cross sections of a solid are a quadratic function of z and if A_1, A_2 and A_3 are the areas of bottom, mid and top cross sections and h is the height, then the volume



$$V = \frac{h}{6} \left[A_1 + 4A_2 + A_3\right]$$

Proof. The proof is the same as that of the SARRUS formula.■

An immediate application is the volume of sphere. The bottom and top areas are zero and the mid one is πR^2. Hence

$$V = \frac{2R}{6} \left[0 + 4\pi R^2 + 0\right] = \frac{4}{3} \pi R^3.$$

A slightly modified problem is:

PROBLEM. Find the volume of a solid of given base, and of given parallel cross sections perpendicular to the base.

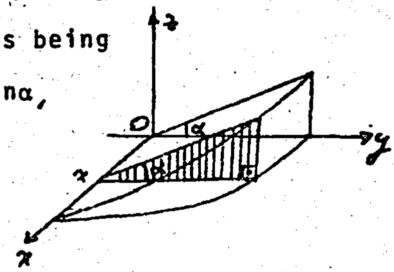
Let the base be taken as a region R on xy-plane. Let the cross sections be parallel to yz-plane. Since the area of a variable cross section A is a function of x the volume of a slice parallel to y-axis will be

$$dV = A(x)dx \qquad V = \int_{x_1}^{x_2} A(x)dx$$

Example. Find the volume of the solid whose base is a quarter of circle with radius a, and cross sections perpendicular to one of the bounding radius are right triangles with acute angle α vertex at that radius. (See Fig.)

Solution. For the selected coordinate system in the figure the base is

$$\{(x, y): x^2 + y^2 \leq a^2, \ x \geq 0, \ y \geq 0\}$$



The right sides of the triangles being

$$\sqrt{a^2 - x^2} \quad \text{and} \quad \sqrt{a^2 - x^2} \cdot \tan\alpha,$$

the volume of the slice is

$$dV = \frac{1}{2}(a^2 - x^2)\tan\alpha \, dx$$

implying

$$V = \int_0^a (a^2 - x^2)\tan\alpha \, dx = \frac{1}{3} a^3 \tan\alpha$$

EXERCISES (8. I)

1. Express the given normal regions in $R_{r\theta}$ type

   a) $R_{\theta r} = \left[0, \frac{\pi}{2}; 2\cos\theta, 2\right]$, b) $R_{\theta r} = \left[0, \pi; 1 + \cos\theta, 2\right]$.

2. Find the area of the region enclosed by the curves:

   a) $r = 4\cos\theta$, $r = 2$ $(r \geq 2)$ b) $r = 3\sin 2\theta$

   c) $r = 3\cos 3\theta$ d) $r = 2 + \sin\theta$, $0 \leq \theta \leq 2\pi$

3. Find the area of the region bounded by the curves

   $r = \sqrt{3}\sin\theta$, $r = \cos\theta$, and $\theta = 0$.

4. Compute the area of the region bounded by

   $r = \sqrt{\cos 2\theta}$ from $\theta = 0$ to $\theta = \pi/4$.

5. Compute the area of the polar curve $r = 4\cos\theta + 2\sin\theta$ from $\theta = 0$ to $\theta = \pi$.

6. Compute the area bounded by the curve $r = 1/(1-\cos\theta)$ and the

rays $\theta = \pi/4$, $\theta = \pi/2$

7. Find the common area enclosed by the following pairs of curves:

   a) $r = 3\cos\theta$, $r = 1 + \cos\theta$ b) $r = 3 - 2\cos 2\theta$, $r = 2$.

8. Find the area of the region enclosed by each of the following curves:

   a) $r = 2\cos 3\theta$, b) $r = 8\sin^2 \frac{\theta}{2}$, c) $r = 2 - \cos\theta$, d) $r^2 = 4\cos 2\theta$.

9. Find the area of the region bounded by the given curves:

   a) $r = \sin 2\theta$, $0 \leq \theta \leq \pi/2$ b) $r = 1 - \sin\theta$, $\pi/2 \leq \theta \leq 5\pi/2$

10. Show that the lemniscates $r^2 = a^2 \cos 2\theta$ and $r^2 = b^2 \sin 2\theta$ intersect orthogonally.

11. Find the length of the curve of

    a) $f(x) = x^{3/2}$ between $x = 1$ and $x = 2$

    b) $f(x) = (x + 1)^{3/2}$ between $x = 3$ and $x = 8$

12. Find the length of the arc of the following curves in the given interval

    a) $y = x^3/6 + 1/2x$, $1 \leq x \leq 2$ b) $y = \sqrt{1 - x^2}$, $0 \leq x \leq 1$

    c) $y = \sqrt{x/3}\,.\,(3 - x/3)$, $0 \leq x \leq 3$, d) $y = \text{Cosh}\ x$, $0 \leq x \leq 2$

13. Find the arc length of the curve:

$$x = \frac{3}{16} y^{4/3} - \frac{3}{2} y^{2/3} \text{ for } y \in [0, 8]$$

14. Find the length of the curve on the given interval:

    a) $x = 4(2t + 3)^{3/2}$, $y = 3(t + 1)^2$, $t \in [-3, 1]$

    b) $x = 3(t^2 - 2t)$, $y = 5t^{3/2}$, $t \in [-1, 1]$

16. Compute the perimeter of the following regions bounded by the given curves:

a) $y = x^2$, $y = 4x$ b) $y = \ln \sec x$, $x = 0$, $y = \ln 2$

16. Find the length of curve for the following function in the given interval:

a) $y = \ln \cos x$, $0 < x < \pi/3$ b) $y = 4 - x^2$, $-2 < x < 2$

17. Find the specified arc length for the following curves:

a) $r = e^{\theta}$, $0 \leq \theta \leq \ln 4$ b) $r = \sin^2 \frac{\theta}{2}$, $0 \leq \theta \leq \pi$.

18. Find the length of the curves in the given intervals:

a) $r = a\theta^2$, $(0, 2\pi)$ b) $r = a \sin^2 \frac{\theta}{2}$, $(0, \pi)$

19. Find the length of the following curves in the given interval:

a) $x = 8t^3$ $y = 6t^4 - 3t^2$, $-1 \leq t \leq 1$

b) $x = t^2$, $y = t^3$, $0 \leq t \leq 1$

c) $x = \cos^3 t$, $y = \sin^3 t$, $0 \leq t \leq 2$

d) $x = 3t^4$, $y = 3t^2 - t^6$, $0 \leq t \leq 1$

20. Find the length of each of the following curves:

a) $r = a \cos^2 \frac{\theta}{2}$ b) $r = a \sin^3 \frac{\theta}{3}$

21. Find V_{ox} for the region

a) $R_{xy} = \left[1, 2;\ y_1 = 0,\ y_2 = \ln x\right]$

b) $R_{xy} = \left[1, 2;\ y_1 = 0,\ y_2 = \sqrt{x}\, e^{x/2}\right]$

22. Find the area of the surface of revolution obtained by revolving the given arc about y-axis:

a) $2y = x^2$, $0 \leq x \leq 3$ b) $y = 3\sqrt[3]{x}$, $2 \leq x \leq 8$.

23. Find the area of the surface of revolution obtained by revolving about the polar axis the given arc:

a) $\theta = \pi/n$, $0 \leq r \leq 1$ b) $r = e^{\theta}$, $0 \leq \theta \leq \pi/2$.

24. Find the area of the surface of revolution obtained by revolving about the x-axis the given arc:

a) $y = x^3/3, \quad 0 \leqslant x \leqslant 2$ b) $y = \cos x, \quad -\pi/2 \leqslant x \leqslant \pi/2$.

25. Find the volume of the solid of revolution when the given region is revolved about the x-axis:

a) $\left[0, \pi/2; \quad y_1 = 0, \quad y_2 = 2 \cos x\right]$

b) $\left[0, \pi; \quad y_1 = 0, \quad y_2 = \sin \frac{x}{2}\right]$

26. Find the volume of the solid of revolution generated when the given region is revolved about the x-axis:

a) $y - \sqrt{x} \leqslant 0, \quad y - 1 \leqslant 0, \quad 0 \leqslant x \leqslant 1$

b) $y - \sqrt{a^2 - x^2} < 0, \quad y - c \geqslant 0,$ where $0 \leqslant c \leqslant a, \quad |x| \leqslant a$

27. Find the volume of the solid obtained by rotating each region about y-axis:

a) $y = \sqrt{x}, \quad x = 4, \quad y = 0$ b) $y^3 = x, \quad x = 8, \quad y = 0$

c) $y = 1/x, \ x=1, \ x=4, \ y=0$ d) $x^2 - y^2 = 1, \quad x = 3$

28. Find the volume of the solid obtained by rotating each region about x-axis:

a) $y = \sqrt{4+x}, \quad x=0, \quad y=0$ b) $y = 1/x, \quad x = 1, \ x = 3, \quad y = 0$

c) $y = x^2 - x, \quad y = 0$ d) $y = 1/(x-1)^3, \ x = -1, \ x = 0, \ y = 0$.

29. A sector of a circle having centeral angle θ is rotated about one of its sides. Find the volume of the solid generated.

30. Compute the volume of the solid generated by revolving a unit circle about one of its tangent lines.

31. Find the volume by the method of cylindrical shells of the solid generated by revolving about y-axis the region bounded by

a) $x = y^2$, $x = y^4$, $y > 0$ b) $y = x^2$, $y = 4x$

c) $y = x$, $y = \sqrt{x}$ d) $y = 9 - x^2$, $y = 9 - 3x$.

32. A solid has in xy-plane, a base of circular disc of radius a, and any cross section of the solid by a plane perpendicular to x-axis is a square. Find the volume of the solid.

33. The base of a certain solid in the portion of the xy-plane for which $x^2 + y^2 \leqslant 1$. If every section of the solid perpendicular to y-axis is a square, find the volume of the solid.

34. A hole of radius r is drilled through the center of a sphere of radius R. Find the volume of the remaining solid

35. Applying the three-level-formula, evaluate the volume of a

a) cone of height h and with circular base of radius R,

b) frustum of cone with lower and upper base radius R, r.

ANSWERS TO EVEN NUMBERED EXERCISES

2. a) $2(\frac{4}{3}\pi - 1)$, b) $\frac{9}{2}\pi$, c) $\frac{9}{4}\pi$, d) $\frac{9}{2}\pi$

4. $\frac{1}{4}$

6. $\frac{1}{2} + \frac{2}{3}\sqrt{2}$

8. a) π, b) 24π, c) $\frac{9}{2}\pi$, d) 4

12. a) $\frac{17}{12}$, b) π, c) 4, d) Sh 2

14. a) 30, b) 12

16. a) $\ln(2 + \sqrt{3})$, b) $2\sqrt{17} + \frac{1}{2}\ln(4 + \sqrt{17})$

18. a) $\frac{8}{3}|a|\left[(\pi^2 + 1)^{3/2} - 1\right]$, b) $2|a|$

20. a) $4|a|$, b) $\frac{3}{2}\pi|a|$

22. a) $\frac{2}{3}\pi(10\sqrt{10} - 1)$, b) $\pi(17\sqrt{17} - 2\sqrt{2})$

24. a) $\frac{\pi}{9}\,(17\sqrt{17} - 1)$, b) $2\pi\left[\ln(1 + \sqrt{2}) + \sqrt{2}\right]$

26. a) $\frac{\pi}{2}$, b) $\frac{4}{3}\,\pi(a^2 - c^2)^{3/2}$

28. a) 8π, b) $\frac{2\pi}{3}$, c) $\frac{\pi}{30}$, d) $\frac{31\pi}{160}$

30. $2\pi^2$

32. $\frac{16}{3}\,a^3$

34. $\frac{4}{3}\,\pi(Q^2 - r^2)^{3/2}$.

8. 2. Physical Applications

A. Work.

A definite integral $\int_a^x f(x)dx$ may be interpreted* as work done against (or by) the force $f(x)$ corresponding to a displacement from a to x, and denoted by $W(x)$:

$$W(x) = \int_a^x f(x)dx$$

Then the differential of $W(x)$ is

$$dW = f(x)dx$$

When one is solving a problem on "work", as a first step it is usual to write the differential of work as the product of "force times differential or displacement" or "displacement times differential of force", as illustrated in the following examples:

Example 1. Find the work done to stretch a spring from its equilibrum position to a distance 20 cm if the spring constant is $k(> 0)$.

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* This definite integral can also be interpreted physically giving different meanings to $f(x)$ and dx, such as fluid pressure.

Solution. From Hooke's law, the force against which the work done being $F = -k|x|$ gr, the force doing the work is $F = k|x|$ and we have

$$dW = k|x|\,dx$$

$$W = \int_0^{20} k|x|\,dx = 200\ k\ (\text{gr-cm})$$

(In the problem the weight of the spring has been neglected.)

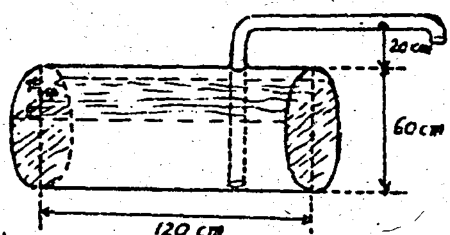
Example 2. A cylindrical tank with size given in the figure is full of water. Find the work done to pump out the content.

Solution. The elementary work done by the pump against the weight of elementary rectangular prizm with size $12 \times 2r \times dx$ in decimeters, is

$$dW = dV = 24\ r\ dx\ \text{kg},$$

where

$$r = \sqrt{x(6 - x)}.$$



Then

$$dW = (x + 2)\ .\ 24\ .\ x(6 - x)dx$$

$$W = 24 \int_0^6 (x + 2)\sqrt{6x - x^2}\ dx = 540\ \text{kg-dm}$$

B. Mass, Moments, Center of Mass, Centroid, Moments of Inertia

MASS: If a mass m is concentrated at a point P we have that we call a *particle*, written $P(m)$.

As a continuous case a mass m may be distributed along a curve with *density* (mass per unit length) so that we have

$$dm = \delta\ ds$$

Another continuous case is the distribution of a mass m over a plane region R with density δ (mass per unit area) so that

$$dm = \delta \, dA$$

If a curve (or a region) is given with corresponding densities, the total mass is obtained by integrating δds or δdA, but in the second case the evaluation is possible when δ is function of x (or y) alone, which in the first case may be given as $\delta(x, y)$ and reducible to $\delta(x)$ or to $\delta(y)$ since x, y are related by the equation of the curve.

Mass of a wire: Let the wire be in the shape of the curve

$$y = f(x) \varepsilon D[a, b] \quad \text{or} \quad x = g(y) \varepsilon D[c, d]$$

Then

$$m = \begin{cases} \int_a^b \delta(x) \sqrt{1 + f'^2(x)}\,dx \\ \int_c^d \delta(y) \sqrt{1 + g'^2(y)}\,dy \end{cases}$$

Mass of a plate: Let the plate be in the shape of the region

$$R_{xy} = [a, b; y_1(x), y_2(x)] \quad \text{or} \quad R_{yx} = [c, d; x_1(y), x_2(y)]$$

Then

$$m = \begin{cases} \int_a^b \delta(x) [y_2(x) - y_1(x)]\,dx \\ \int_c^d \delta(y) [x_2(y) - x_1(y)]\,dy \end{cases}$$

Example. Find the total mass of a wire bent to form the semicircle $x^2 + y^2 = a^2$, $y \geq 0$ with $\delta(y) = 2y$.

<u>Solution</u>.

$$m = \int_0^a \delta \, ds = \int_0^a 2y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

where, from

$$x = \sqrt{a^2 - y^2}, \quad \frac{dx}{dy} = -\frac{y}{\sqrt{a^2 - y^2}},$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + \frac{y^2}{a^2 - y^2}} = \frac{a}{\sqrt{a^2 - y^2}} . \text{ Then}$$

$$m = 2a \int_0^a \frac{y\, dy}{\sqrt{a^2 - y^2}} = -2a\sqrt{a^2 - y^2}\Big|_0^a = 2a^2 \text{ units.}$$

<u>Example</u>. Find the total mass of a plate bounded by $y=x^2$, $x > 0$, $y = 4$ with $\delta = x$.

<u>Solution</u>. Since $= (x) = x$, consider vertical strips:

$$m = \int_0^2 \delta(x)(y_2 - y_1)dx = \int_0^2 x(4 - x^2)dx =$$

$$= \left(2x^2 - \frac{x^4}{4}\right)\Big|_0^2 = 8 - 4 = 4 \text{ units.}$$

From this result we have as the average density $\bar{\delta}$ of the plate

$$\bar{\delta} = \frac{m}{A} \quad \frac{4}{8 - \int_0^2 x^2 dx} = \frac{3}{4} \text{ (from } \delta = x,\ \delta_{max} = 0,\ \delta_{min} = 2)$$

<u>Note</u>. Mass of a shell and mass of a solid can be computed in a similar manner, but when δ is a function of more than one variable, one needs multiple integrals for evaluation (a subject of Calculus II). Therefore the given quantities involved in a physical problem are to be functions of the same variable for the problem to be solved by definite integrals.

Same conditions are valid in polar coordinates, i.e., the density δ must be a function of a single variable θ or r, in other words δ must be a constant on a ray or on a circle.

Under all these restrictions on distance and density, the solution of a physical problem by a definite integral gives certain difficulties which can be eleminated by carefullness.

These are the reasons why the subject is treated usually in the simpler way by multiple integrals.

MOMENTS:

The moment of a particle $P(m)$ of mass m located at P with respect to a point O (or a line ℓ, or a plane π) is the product of m and its distance from O (or ℓ, or π).

$$M_O = m|PO|, \quad M_\ell = m|P\ell|, \quad M_\pi = m|P\pi|$$

where $|PO|$, $|P\ell|$, $|P\pi|$ denote distances of P from O, ℓ, π respectively.

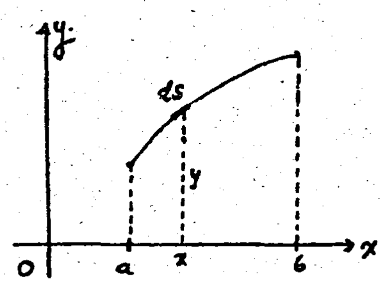
a) Moment and center of mass of an arc with mass:

Let

$$y = f(x) \varepsilon D[a, b] \quad \text{or} \quad x = g(y) \varepsilon D[c, d]$$

be an arc charged with density $\delta (= dm/ds)$. Then the moments of the element of arc ds with density δ with respect to x- and y-axis being

$$d\,M_{ox} = y\,dm = y\,\delta\,ds, \qquad d\,M_{oy} = x\,dm = x\,\delta\,ds,$$



we have, as moments of arc

$$M_{ox} = \begin{cases} \int_a^b f(x)\ \delta(x)\ \sqrt{1 + f'^2(x)}dx \\ \int_c^d y\ \delta(y)\ \sqrt{1 + g'^2(y)}dy, \end{cases}$$

$$M_{oy} = \begin{cases} \int_a^b x\ \delta(x)\ \sqrt{1 + f'^2(x)}dx \\ \int_c^d g(y)\ \delta(y)\ \sqrt{1 + g'^2(y)}dy. \end{cases}$$

We define the *center of mass (center of gravity)* of the arc with mass as the point $G(\overline{x}, \overline{y})$ such that the moments $m\,\overline{y}$, $m\,\overline{x}$ of the particle $G(m)$ are the same as the moments M_{ox}, M_{oy} of the arc where m is the total mass of the arc:

$$m\,\overline{x} = M_{oy}, \qquad m\,\overline{y} = M_{ox}$$

These defined equalities give

$$\overline{x} = \frac{M_{oy}}{m}, \qquad \overline{y} = \frac{M_{ox}}{m}$$

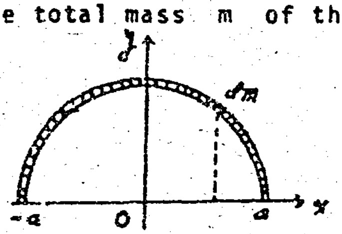
as coordinates of the center of mass G.

Example. Find the center of mass of a wire bent in the shape of semi circle $x^2 + y^2 = a^2$ if the density is $\delta = 2y$.

Solution. Since the arc and the density function are symmetric with respect to y-axis, it follows that G lies on y, axis, and $\overline{x} = 0$.

To find $\overline{y}$, we evaluate first the total mass m of the wire:

$$m = 2\int_0^a 2y\sqrt{1 + \frac{y^2}{x^2}}\,dy = 4a^2$$



Then

$$\overline{y} = \frac{2}{m} \int_0^a y \quad 2y\sqrt{1 + \frac{y^2}{x^2}}\, dy = \frac{\pi}{4} a \quad \text{and} \quad G(0, \frac{\pi}{4} a)$$

b) <u>Moment and center of mass of a region with mass (plate):</u>

Consider a region with mass, i.e., a plate. We define the moments M_{ox}, M_{oy} of this plate with respect to x-and y-axis as the definite integrals, in related normal regions,

$$M_{ox} = \int y \, dm = \int y \, \delta \, dA, \quad M_{oy} = \int x \, \delta \, dA$$

where dA is an element of area. (Limits of integration cannot be written at this instant since the variable of integration is uncertain)

The center of mass $G(\overline{x}, \overline{y})$ of the plate is defined in exactly the same manner as that of an arc:

$$m \, \overline{x} = M_{oy} , \qquad m \, \overline{y} = M_{ox}$$

where m is the total mass of the plate. We have then

$$\overline{x} = \frac{M_{oy}}{m} , \qquad \overline{y} = \frac{M_{ox}}{m}$$

Now let the plate be in the shape of the normal region

$$R_{xy} = \left[a, b; y_1(x), y_2(x)\right] \quad \text{or} \quad R_{yx} = \left[c, d; x_1(y), x_2(y)\right]$$

with density δ.

The moments M_{ox}, M_{oy} can be evaluated by definite integral when the density is a function of x (or y) alone.

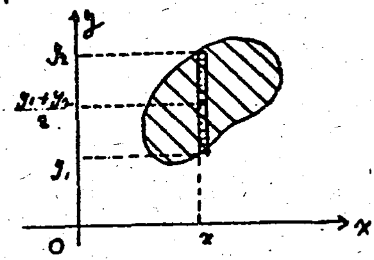
<u>Let $\delta = \delta(x)$:</u>

The density being constant along each vertical strip, the center of mass of such a strip is at its center

$$(x, \frac{y_1 + y_2}{2})$$

with mass

$$dm = \delta(x)(y_2 - y_1)dx$$



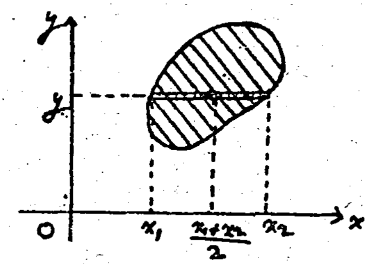
and we have

$$M_{0x} = \int_a^b \frac{y_1 + y_2}{2}\,\delta(x)(y_2 - y_1)dx$$

$$M_{0y} = \int_a^b x\,\delta(x)(y_2 - y_1)dy$$

<u>Let $\delta = \delta(y)$</u>:

By similar reasoning, we have



$$M_{0x} = \int_c^d y\,\delta(y)\,(x_2 - x_1)dy$$

$$M_{0y} = \int_c^d \frac{x_1 + x_2}{2}\,\delta(y)\,(x_2 - x_1)dy$$

<u>Example</u>. Find the center of gravity of the plate in the shape of region bounded by $y = x^2$, $x > 0$, $y = 4$ with density $= x$.

<u>Solution</u>.

$$m = \int_0^2 \delta(x)(y_2 - y_1)dx = \int_0^2 x(4 - x^2)dx = 4 \text{ units.}$$

$$\overline{x} = \frac{1}{m}\int_0^2 x\,\delta(x)(y_2 - y_1)dx = \frac{1}{4}\int_0^2 x\,.\,x(4 - x^2)dx = 16/15,$$

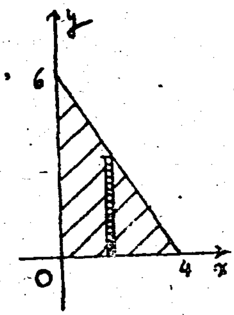
$$\overline{y} = \frac{1}{m}\int_0^2 \frac{y_1 + y_2}{2}\,\delta(x)(y_2 - y_1)dx = \frac{1}{4}\int_0^2 \frac{x^2+4}{2}\,x(4-x^2)dx = 8/3$$

The center of gravity $G(\overline{x}, \overline{y})$ is the *centroid* of the same figure in case the density is constant throughout the figure. Note that the constant can be taken out of the integrals for moments and mass.

<u>Example</u>. Find the centroid (constant density) of the triangular region $R = \{0 \ 4 \ \ldots - 0 \ \ldots \ 12 - 3x\}$

Solution. The region is the triangular region bounded by the line $x/4 + y/6 = 1$ and coordinate axes.

$$A = 12 \Rightarrow m = \delta A = 12.$$



$$M_{Ox} = \delta \int_0^4 \frac{1}{2} \left(\frac{12 - 3x}{2}\right)^2 dx = 24\delta,$$

$$M_{Oy} = \delta \int_0^4 x \cdot \frac{12 - 3x}{2}\, dx = 16\delta$$

$$\Rightarrow \quad \bar{x} = \frac{16}{12} = \frac{4}{3}, \quad \bar{y} = \frac{24}{12} = 2$$

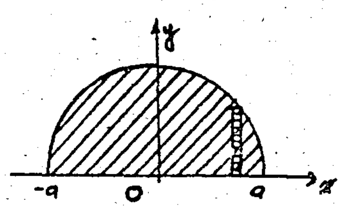
$$\Rightarrow \quad G(4/3, 2),$$

a well known result.

c) Moments of inertia

In the integral formulas for moments with respect to a point O (or a line ℓ, or a plane π), when distance is replaced by its square, one obtains what we call the *moment of inertia (second moment)* of the same figure with mass, with respect to a point (or line, or plane): I_O, I_ℓ, I_π.

Example. Find the moment of inertia of a plate in the shape of semicircle with radius a and constant density, with respect to its diameter.



Solution.

$$I_{Ox} = \int_{-a}^{a} \left(\frac{1}{2} y\right)^2 \delta y \, dx$$

$$= \frac{\delta}{4} \int_{-a}^{a} (a^2 - x^2)^{3/2} dx = \frac{3}{12} \pi \delta a^4$$

C. PAPPUS THEOREMS

Below we state two theorems expressing a relation between area (volume) of a surface (solid) of revolution and the centroid of the generating arc (region). They are extremely usefull for finding the centroid when surface area (volume) is known, and for finding the latter when the centroid is known.

Theorem 1. The area of a surface of revolution generated by revolving an arc about a line in its plane not cutting the arc, is equal to the product of the length of arc and the circumference of the circle described by the centroid of the arc:

$S_{ox} = s \cdot 2\pi\bar{y}$.

Proof. Let the arc of the curve

$$y = f(x) \varepsilon D[a, b], \quad y \geqslant 0$$

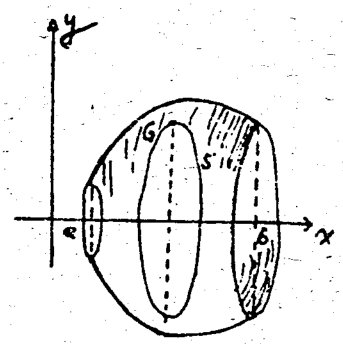
be revolved about the x-axis. We have

$$S_{ox} = 2\pi \int_a^b y\, ds$$

as the area of the surface, and

$$s\,\bar{y} = M_{0x} = \int_a^b y\, ds \quad (\delta = 1)$$

$$S_{0x} = 2\pi \cdot s\bar{y} = s \cdot 2\pi\bar{y}.$$



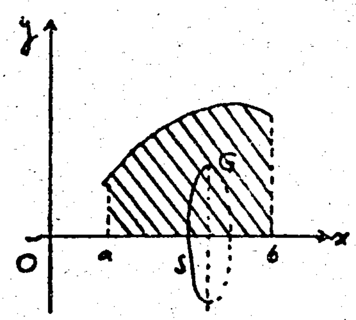
Theorem 2. The volume of a solid of revolution generated by revolving a region about a line in its plane not cutting the region, is equal to the product of the area of the region and the circumference of the circle described by the centroid of the region.

Proof. Let the region be

$$R_{xy} = \left[a, b; \quad y_1(x), \quad y_2(x)\right]$$

be revolved about the x-axis. We have

$$V_{ox} = \pi \int_a^b (y_2^2 - y_1^2)dx$$

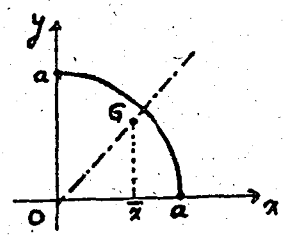


as the volume of the solid, and

$$A\bar{y} = M_{ox} = \int_a^b \frac{y_1 + y_2}{2}(y_2 - y_1)dx = \frac{1}{2}\int_a^b (y_2^2 - y_1^2)dx \quad (\delta=1)$$

$$V_{ox} = 2\pi \, . \, A\bar{y} = A \, . \, 2\pi\bar{y}.$$

Example 1. Find the centroid of the quarter of circle (arc) of radius a.



Solution. The centroid certainly lies on the radius bisecting the arc. Referring to coordinate system of the figure we have $\bar{x} = \bar{y}$

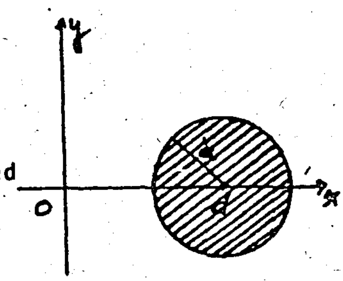
When the arc is revolved about x-axis a hemisphere is obtained with known area $S_{ox} = 2\pi a^2$. Then by PAPPUS Theorem 1 we have

$$2\pi a^2 = s \, . \, 2\pi\bar{y} \qquad (s = \frac{1}{2}\pi a)$$

from which we get

$$\bar{x} = \bar{y} = \frac{a^2}{\frac{1}{2}\pi a} = \frac{2}{\pi} a \Rightarrow G(\frac{2}{\pi} a, \frac{2}{\pi} a).$$

Example 2. Find the volume of the solid generated by revolving a circle of radius "a" about a line where distance from the center is d $(> a)$.



Solution. Referring to coordinate system of the figure, $(d, 0)$ is the center, also, the centroid of the circle.

When the circular region is revolved about y-axis the circumference generates a surface called *torus* which bounds a solid of which the volume is

$$V_{oy} = A \, . \, 2\pi\bar{x} = \pi a^2 \, . \, 2\pi d = 2\pi^2 a^2 d$$

EXERCISES (8. 2)

36. A conical cistern is 20 m heigh and 20 m accross the top, and water is now 7 m deep in the cisterm. Find the work in filling the cistern from a source 5 m below the bottom of the cistern.

37. How much work is done in streching the end of an elastic spring 3 cm from its released position if the spring constant is 15 gr-cm/sec^2. ($F = kx$ where k is the spring constant).

38. How much work is done in lifting a body whose mass is 10 gr, from a height of 100 cm to a height of 200 cm?

39. How much work is done in pumping water from a conical reservoir of vertex down with radius 2 m and height 8 m if it is full of water? (water is pumped from the upper level of reservoir).

40. Find the amount of work done in stretching a spring from its natural length of 6 cm to double that length if a force of 20 kg is needed to double its natural length.

41. Water is to be pumped out of a conical tank vertex down, $r = 2m$, $h = 8m$, to a point 10 m above the top of the tank. Find the amount of work required to pump out to a level of 4 m deep.

42. A conical container (vertex down) of radius r ft and height h ft is full of liquid weighing δ lb/ft^3. Find the work done in pumping out to a level of $h/2$ feet deep.

a) to the top of the tank,

b) to a level k ft above the top of the tank.

43. A trough 20 ft long has a cross section in the shape of an isoceles trapezoid with a lower base 4 ft long, an upper base 10 ft long and an altitude of 4 ft. How much work is done in filling the trough with water if the bottom of the trough is located 20 ft above the pump and the water is pumped in through a valve in the bottom of the trough.

44. A force of 20 kg is required to compress a spring 20 cm long to 19 cm. What is the work done in skreching the spring from a length of 24 cm to 30 cm?

45. A vertical cylindrical tank 6 dm in diameter and 10 dm heigh is half full of water. Find the amount of work done in pumping all the water to the top of the tank.

46. According to NEWTON's law of universal gravitation two objects of weights w_1, w_2 kg are attracted to each other by a force of

$$k \frac{w_1 w_2}{x^2} \text{ kg},$$

where x is the distance between the objects and k is a constant. Find the work done in separating the objects from a distance of "a meters" to a distance of "b meters" apart.

47. Find the amount of work done in skretching a spring from its natural length of 8 cm to triple that length if a force of 15 kg. is needed to triple it natural length.

48. Find moments of the following arc with respect to x- and y-axes, and find also the centroid of the arc if $\delta = 1$.

a) $y = x - 4$, $x = 0$, $x = 4$ b) $(x-2)^2 + (y-3)^2 = 1$

49. Find the centroid of the regions bounded by the given curves:

a) $y^2 = x^3$, $y = 2x$ b) $y = x^2$, $y = x^3$

50. Same question for

a) $y = x^2 - 4$, $y = 2x - x^2$

b) $y + x^2 = 0$, $y + 2 = x$, $y + 2 = -x$, $y = 2$.

51. Find the centroid of each of the regions bounded by the following curves:

a) $2x + y = 6$, $x = 0$, $y = 0$ b) $y = 2x + 1$, $x + y = 7$, $x = 8$

c) $y^2 = x$, $y = x - 2$ d) $y = x^3$, $y = 4x$, $x \geqslant 0$, $y \geqslant 0$

52. Find M_{0x}, M_{0y} and G of the regions given by

a) $y = x^2 + x$, $y = 4x$, $0 \leqslant x \leqslant 3$ b) $y = 9 - x^2$, $y = 9 - 3x$, $0 \leqslant x \leqslant 3$

53. Find M_{0x}, M_{0y} and G for each region:

a) $R_{xy} = \left[0,\ 1;\ r^2 - x^2\right]$. $(r \geqslant 1)$ b) $R_{xy} = \left[0,\ 1;\ 0,\ e^x\right]$

54. Same question for

a) $R_{xy} = \left[0,\ 1;\ x^2,\ \sqrt{x}\right]$ b) $R_{xy} = \left[0,\ \pi/4;\ \sin x,\ \cos x\right]$

55. Find the centroid of the region bounded by the parabola $y = x^2$ and the line $y = 4$.

56. Find the centroid of the region bounded by the curves:

a) $y = \sqrt{x}$, $y = 0$, $x = 4$ b) $y = x^2$, $x = 0$, $x = 2$, $y = 0$

57. Same question for

a) $y = \cos x$, $x = 0$, $x = \frac{\pi}{2}$, $y = 0$

b) $y = \ln x$, $x = 1$, $x = e$, $y = 0$

58. Find the center of gravity of each of the following system of particles. ($m(x, y)$ denotes particle with mass m at (x, y))

a) 3(2, 2), 4(2, -2), 5(-2, 2), 2(-2, -2)

b) 6(0, 0), 6(8, 0), 6(8, 8), 3(4, 4)

c) 2(1, 3), 7(4, 2), 6(3, -3), 8(-4, 2), 5(-3, -4).

59. Use PAPPUS' Theorem to find the centroid of the region of a semicircle of radius a.

60. Use PAPPUS' Theorem to find the volume of the torus generated by revolving the area of a circle of radius a, about an axis b(> a) units from the center of the circle.

ANSWERS TO EVEN NUMBERED EXERCISES

36. $\frac{500}{3}\pi(12{,}25 + \sqrt{35} + 100)$

38. 1000 g gr-cm.

40. 60 kg-cm.

42. a) $11\pi\delta r^2h^2/492$, b) $\pi\delta r^2h(11\,h/492 + 7k/24)$

44. 840 kg-cm.

46. $kw_1\, w_2\, (1/a - 1/b)$ kg-m

48. a) $M_{ox} = -8\sqrt{2}$, $M_{oy} = 8\sqrt{2}$, G(2, -2)

b) $M_{0x} = 12\pi$, $M_{0y} = 8\pi$, G(2, 3).

50. a) (1/2, -3/2), b) (10, 192/205)

52. a) 459/20, 27/4, (3/2, 51/10) b) 243/10, 27/4, (3/2, 27/6)

54. a) $M_{ox} = M_{oy} = 3/20$, G(9/20, 9/20)

b) $M_{ox} = 1/4$, $M_{oy} = (\pi/2\sqrt{2})-1$, $G(1/4\ (\sqrt{2}-1),\ (\pi-2\sqrt{2})/2\sqrt{2}(\sqrt{2}-1)$

56. a) (12/5, 3/4), b) (8/5, 16/7)

58. a) (0, 2/7), b) (4, 4), c) (1/28, -1/14)

60. $2\pi^2\, a^2b$.

A SUMMARY
(CHAPTER 8)

8. 1 $R_{xy} = \left[a,\ b;\ y_1(x),\ y_2(x)\right]$, $\quad R_{yx} = \left[c,\ d;\ x_1(y),\ x_2(y)\right]$

$$|R_{xy}| = \int_a^b \left[y_2(x) - y_1(x)\right]dx, \qquad |R_{yx}| = \int_c^d \left[x_2(y) - x_1(y)\right]dy$$

$$R_{\Theta r} = \left[\alpha,\ \beta;\ r_1\Theta,\ \ r_2(\Theta)\right], \qquad R_{r\Theta} = \left[a,\ b;\ \Theta_1(r),\ \Theta_2(r)\right]$$

$$R_{\Theta r} = \frac{1}{2}\int \left[r_2^2(\Theta) - r_1^2(\Theta)\right]d\Theta \qquad |R_{r\Theta}| = \int_a^b \left[\Theta_2(r) - \Theta_1(r)\right]r\,dr$$

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx, \qquad s = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$$

$$s = \int_{t_1}^{t_2} \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}\,dt$$

$$s = \int_{\Theta_1}^{\Theta_2} \sqrt{r^2 + \left(\frac{dr}{d\Theta}\right)^2}\ d\Theta, \qquad s = \int_{r_1}^{r_2} \sqrt{1 + r^2\left(\frac{d\Theta}{dr}\right)^2}\ dr$$

$$\mu = \tan\psi = \frac{r}{r'} \quad \text{(polar slope)}$$

$$S_{ox} = 2\pi \int_h^k y\ ds = \begin{cases} 2\pi \displaystyle\int_a^b y(x)\ \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx \\[2ex] 2\pi \displaystyle\int_c^d y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy \end{cases}$$

$$S_{oy} = 2\pi \int_h^k x\ ds = \begin{cases} 2\pi \displaystyle\int_a^b x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx \\[2ex] 2\pi \displaystyle\int_c^d x(y)\ \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy \end{cases}$$

$$V_{ox} = \pi \int_a^b \left[y_2^2(x) - y_1^2(x)\right]dx, \qquad V_{oy} = \pi\int_c^d \left[x_2^2(y) - x_1^2(y)\right]dy$$

(disc method)

$$V_{ox} = 2\pi \int_c^d \left[x_2(y) - x_1(y)\right] y \, dy, \quad V_{oy} = 2\pi \int_c^d \left[y_2(x) - y_1(x)\right] x \, dx$$

(shell method)

$$V = \frac{h}{6} \left[A_1 + 4A_2 + A_3\right],$$ A_1, A_2, A_3 are lower, mid and upper bases and h is the altitude.

8. 2

$$\bar{x} = \frac{M_{oy}}{m} = \begin{cases} \dfrac{\int \delta x \, ds}{\int \delta \, ds} & \text{for arcs (wire)} \\[2ex] \dfrac{\int \delta y \, dA}{\int \delta \, dA} & \text{for regions (plate)} \end{cases}$$

$$\bar{y} = \frac{M_{ox}}{m} = \begin{cases} \dfrac{\int \delta y \, ds}{\int \delta \, ds} & \text{for arcs (wire)} \\[2ex] \dfrac{\int \delta y \, dA}{\int \delta \, dA} & \text{for regions (plate)} \end{cases}$$

MISCELLANEOUS EXERCISES

61. Find the area which is inside the first and outside the second curve:

    a) $r = 5 \sin\theta$, $r = 2 + \sin\theta$, b) $r^2 = 2 \cos 2\theta$, $r = 1$

62. Changing to polar coordinates find the area of the region enclosed by the curve $(x^2 + y^2)^3 = 4a^2x^2y^2$.

63. Find the area of the region bounded by:

    a) $r = \dfrac{2}{1 + \cos\theta}$, $r = \dfrac{2}{1 - \cos\theta}$, b) $r = \tan\theta$, $r = \cot\theta$

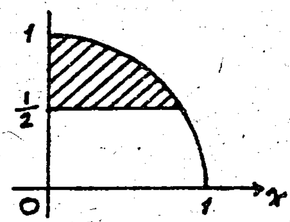
64. Find the area of the region

a) bounded by the circles $r = 2a\cos\theta$, $r = 2a\sin\theta$

b) inside $r^2 = 2a^2\cos 2\theta$, outside $r = a$

c) inside $r = 3a\cos\theta$, outside $r = a(1 + \cos\theta)$

65. Find the area of the region enclosed by one loop of the curve $r = \sin^2\theta$ and find also the length of one loop of the curve.



56. Write the given shaded region

a) as $R_{\theta r}$ or as union of them,

b) as $R_{r\theta}$ or as union

67. Find the area of the smaller loop of the limaçon $r = a(1-2\cos\theta)$

68. Find the area of a loop of the curve $r^2 = a^2\cos 2\theta$.

69. Find the length of the curve:

a) $x = 4(2t+3)^{3/2}$, $y = 3(t+1)^2$ for $t \in (-3, 1)$

b) $x = 3(t^2 - 2t)$, $y = 8t^{3/2}$ for $t \in (-1, 1)$

70. Find the arc length of

a) $r = 2(1-\cos\theta)$ from $\theta = 0$ to $\theta = \pi$.

b) $r = \sin\theta + \cos\theta$ from $\theta = 0$ to $\theta = \pi/2$.

71. Find the arc length of

a) $x = \frac{1}{3}(2+y^2)^{3/2}$ from $y = 0$ to $y = 3$.

b) $y = \frac{x^4}{4} + \frac{1}{8x^2}$ from $x = 1$ to $x = 2$.

c) $4y^3 = (x+1)^2$ from $y = 0$ to $y = 1$.

d) $x = \frac{y^3}{3} + \frac{1}{4y}$ from $y = 0$ to $y = 3$.

72. Find the length of the curve

$$y = \frac{1}{2} \frac{a}{(1+p)^b} x^{1+p} - \frac{1}{2} \frac{b}{(1-p)^a} x^{1-p} \qquad x \varepsilon (0, \lambda)$$

where $a, b, \lambda > 0$, $p \varepsilon R$.

73. Determine $y = f(x)$ passing through the origin such that arc length from $(0, 0)$ to $(x, f(x))$ is equal to $e^x + y - 1$.

74. Find the arc length of the curve $f(x)$ which is any solution of the differential equation

$$\frac{dy}{dx} = \sqrt{x^4 - 1}$$

from $x = 1$ to $x = a$ (> 1).

75. Find the arc length of the following parametric curves in the given interval:

a) $x = \cos t$, $y = \cos^2 t$, $0 < t < \pi$.

b) $x = e^t \cos t$, $y = e^t \sin t$, $0 \leqslant t \leqslant 2$.

76. Find the area of the surface obtained by revolving the given arc about x-axis.

$y = x^{3/2} - x^{1/2}/3$, $1 \leqslant x \leqslant 2$, b) $y = x^3/6 + 1/(2x)$, $1 \leqslant x \leqslant 2$

77. Find the area of the surface of revolution when the curve $r^2 = 2a^2 \cos 2\Theta$ is revolved about the polar axis.

78. Find the area of the surface of revolution when the curve $r = 2a \cos\Theta$ is revolved about copolar axis.

79. Find the volume of revolution V_{0x} of the following region bounded by the following curves:

a) $y = 1/\cos x$, $y = 0$, $x = 0$, $x = \pi/4$.

b) $y = xe^{x/2}$, $y = 0$, $x = 0$, $x = 1$.

80. Let $b (> 1)$ be a number. What is V_{0x} for the region

bounded by $y = e^{-x}$, $y = 0$, $x = 1$, $x = b$? Does this volume approach a limit as $b \to \infty$? If so what is the limit?

81 The region bounded by a parabola and its latus rectum is rotated about a line throught its vertex and perpendicular to the axis. Find the volume of the solid generated (letting p be the distance between the focus and the vertex)

82. Find V_{ox}, V_{oy} of the solid of revolutions for the region bounded by the given curve:

a) $y = x^2$, $y = 4$, $x = 0$, b) $y^2 = 4x$, $y = 0$, $x = 9$

83. If $R_{xy} = (1, \infty;\ 0, 1/x)$

a) find V_{0x} b) find lateral surface area of the solid of revolution.

84. Compute the volume of the solid obtained by rotating the region

$$R_{xy} = \left[0, 2;\ y_1 = 1,\ y_2 = x^2/4\right]$$

about

a) x-axis b) y-axis

c) vertical line passing through (2, 0)

d) horizontal line passing through (0, 2)

85. Use the method of cylindrical disc to find the volume of the solid obtained by revolving the given region about the x-axis.

a) $R_{yx} = \left[0, 2;\ 2, y^2\right]$, b) $R_{xy} = \left[1, 4;\ 0, x-2\right]$

c) $R_{xy} = \left[0, 6;\ 0, x-|x|\right]$, d) bounded by $y = x^2 + x$, $y = 4x$.

86. Use the method of cylindrical shells to find the volume of the solid obtained by revolving about the y-axis of the

region bounded by

a) $y = |x-2|$, $1 \leqslant x \leqslant 4$, b) $y = x - [x]$, $0 \leqslant x \leqslant 4$

c) $y = 1/x$, $y = -x/2 + 3/2$, d) $y = 9 - x^2$, $y = 9 - 3x$

87. Find the volume of the solid generated when the given polar region is revolved about the polar axis.

a) $r = \cos\theta$, $\theta = 0$, $\theta = \pi/4$

b) $r\cos(\theta - \frac{\pi}{6}) - 5 = 0$, $\theta = 0$, $\theta = \pi/2$

88. Use the method of cylindrical shells to find the volume of the solid obtained by revolving about the x-axis of the given region bounded by:

a) $x = y^2$, $x = y^4$, $y \geqslant 0$, b) $y = 1/x$, $y = -x/2 + 3/2$

c) $x = 3y - y^2$, $x = -2y$ d) $y = x$, $y = \sqrt{x}$

89. Find V_{0x} of the solid generated by the region bounded by

a) $x = a\cos^3 t$, $y = a\sin^3 t$, $a>0$ b) $x = t + \frac{1}{t}$, $y = t - \frac{1}{t}$, $t \in [1,2]$

90. The base of a certain solid is the parabolic segment enclosed between the parabola $y^2 = 4x$ and the line $x = 4$. Every section of the solid perpendicular to the x-axis is an isoceles right triangle with its hypotenese in the plane of the base. Find the volume of the solid.

91. Find the volume of the solid whose cross section made by a plane perpendicular to the x-axis has $ax^2 + bx + c$ as boundary for each $x \in [0, h]$.

92. A solid has a base in the xy-plane which is a circular disc of radius $a(>0)$ and every section of the solid by a plane perpendicular to the x-axis is a triangle as described. Find the volume of the solid.

a) isoceles, altitude is twice the base.

b) equilateral.

93. Let a sphere of radius r be cut by a plane, thereby forming a segment of the sphere of heigh h. Prove that the volume of the segment is $h^2(r - h/3)$.

94. If the current $i(t)$ at time t is given by $t + 1/t^2$ find the total charge entering to a capacitor during the time interval [1, 4].

95. Find the natural length of a metal spring, given that the work done in stretching it from a length of 2 ft to a length of 3 ft is one-half the work done in stretching it from a length of 3 ft to a length of 4 ft.

96. A hemispherical water tank of radius 10 m is being pumped out. Find the work done in lowering the water level from 2m below the top of the tank to 4 m below the top of the tank.

a) qiven that the pump is placed rihqt on top of the tank.

b) qiven that the pump is placed 3 m above the tank.

97. A particle on the x-axis is attracted toward the origin by a force of maqnitude

$$F = k \frac{x}{(x^2 + a^2)^{3/2}}$$

Find the work done by the force if it moves the particle from a distance 2a to a distance a from the origin.

98. The given surfaces are submerged vertically in a fluid of specific weight Find the force on one side of the surface:

a) an isoceles right triangle with legs 6 ft long and one lying in the surface of liquid.

b) an isoceles trapezoid of height 4 ft and bases 6 ft and 12 ft with the smaller base lying on the surface of fluid.

99. Find the indicated moments of inertia of the area bounded by the given curves:

a) $y = x$; $y = 2x$, $x + y = 12$, I_{oy}.

b) $y = 2x^3$, $y + x^3 = 0$, $2y = x + 3$, I_{ox}.

c) $y = x^2$, $y = 2x - 1$, $4x + y = -4$, $I_{x=1}$.

100. Find the center of gravity of the region bounded by $\sqrt{x} + \sqrt{y} = \sqrt{a}$, $x = 0$; $y = 0$.

101. Prove that the moment of an arc (or of a region) with respect to the center of gravity is zero.

102. Show that $I_o = I_{ox} + I_{oy}$.

103. Find the moment of the region

$$R_{xy} = \left[-a, a;\ y_1 = 0,\ y_2 = a^2 - x^2\right]$$

with respect to x- and y-axes, and find the centroid of the region ($\delta = 1$).

104. Find the moments M_{0x}, M_{0y} of the arc $r = a\cos\theta$ from $\theta = 0$ to $\theta = \pi/6$ if the density $= \sin\theta$.

105. Find the centroid of the region enclosed by

a) $x = 2\cos t - \sin t$, $y = 2\sin t$, $0 \leqslant t \leqslant 2$

b) $x = t^2$, $y = t^3$, $-1 \leqslant t \leqslant 1$, $x = 1$.

106. Use the PAPPUS' Theorem to compute lateral area of a frustum of cone with radiu 6 and 8 and altitude 16 cm.

ANSWERS TO EVEN NUMBERED EXERCISES

62. $\pi a^2/2$

64. a) $(\pi - 2)a^2/2$, b) $(3\sqrt{3} - \pi)a^2/3$ c) $\pi a^2/3$

66. a) $[\pi/6, \pi/2; \sec\theta, 1,]$ b) $[1/2, 1, \text{arcsec } r, \pi/2]$

68. $a^2/2$

70. a) 8, b) $\pi\sqrt{2}/2$

72. $\frac{1}{2}\ \frac{a}{b}\ \frac{\lambda^{1+p}}{1+p} + \frac{b}{a}\ \frac{\lambda^{1-p}}{1-p}$

74. a) 53/9, b) 47/16

78. $4\pi^2\ a^2$

80. $\pi(e^{-2} - e^{-2b})/2$, yes, $\pi\ e^{-2}/2$

82. a) $128\pi/5$, 8π, b) 162π, $1944\pi/5$

84. a) 2π, b) $18\pi/3$, c) $16\pi/15$

86. a) $44\pi/3$, b) $26\pi/3$, c) $\pi/6$, d) $27\pi/2$

88. a) $\pi/6$, b) $\pi/12$, c) $625\pi/6$, d) $\pi/6$

90. 32 unit^3.

92. a) $16a^3/3$, b) $4\sqrt{3}\ a^3/3$

94. 33/4

96. 33750π ton-meter

98. 368δ.

100. (a/5, a/5)

104. $a^2/24$, $a^2/3 - a^2\sqrt{3}/8$

106. $28\pi\sqrt{65}$.

APPENDIX

Proof of the theorem on the decomposition of a proper rational function into partial fractions

(See Chapter 7, p. 435)

<u>Proof</u>. Let $r(x)/Q(x)$ be a proper rational function so that $\deg r(x) < \deg Q(x)$. We suppose also that $r(x)/Q(x)$ is a reduced one, that is, $r(x)$ and $Q(x)$ have no common factor.

Let $Q(x)$ have the factorization

$$Q(x) = k.(x-a)^{\alpha}\ldots(x-b)^{\beta}\ldots(x^2+px+q)^{\lambda}\ldots(x^2+rx+s)^{\mu}\ldots$$

where k is a constant, $a, b, \ldots$ are real roots with multiplicities $\alpha, \beta, \ldots$ respectively, and x^2+px+q, $x^2+rx+s, \ldots$ have pair of imaginary roots of common multiplicities $\lambda, \mu, \ldots$ respectively.

Setting $Q(x) = (x-a)^{\alpha}\; Q_1(x) \qquad (\alpha \in N_1)$

with $Q_1(a) \neq 0$, first we show the decomposition

$$\frac{r(x)}{Q(x)} = \frac{A_{\alpha}}{(x-a)^{\alpha}} + \frac{r_1(x)}{(x-a)^{\alpha-1}\, Q_1(x)} \tag{1}$$

where A_{α} is a constant and $r_1(x)/(x-a)^{\alpha-1}\, Q_1(x)$ is a proper fraction. Indeed consider the difference

$$\frac{r(x)}{Q(x)} - \frac{A_{\alpha}}{(x-a)^{\alpha}} = \frac{r(x) - A_{\alpha}\;.\;Q_1(x)}{(x-a)^{\alpha}\; Q_1(x)}$$

Since $\deg r(x) < \deg Q(x)$, $A_{\alpha} \in R$, $\deg\; Q_1(x) < \deg Q(x)$, the degree of numerator is less than that of denominator and then the fraction is a proper one. Now we determine A_{α} such that the numerator is divisible by $x-a$ or that

$r(a) - A_\alpha \cdot Q_1(a) = 0$ implying $A_\alpha = r(a)/Q_1(a)$ since $Q_1(a) \neq 0$. Also since $r(x)/Q(x)$ is a reduced one, $r(a) \neq 0$ and $A_\alpha \neq 0$ follow. For this A_α, the simplified fraction

$$\frac{r_1(x)}{(x - a)^{\alpha-1} \; Q_1(x)}$$

is again a proper fraction.

Repeating the process for this new proper fraction one can separate from this the partial fractions $A_{\alpha-1}/(x-a)^{\alpha-1}$, ... $A_1/(x-a)$ successively, all corresponding to the real root "a", and gets the decomposition

$$\frac{r(x)}{Q(x)} = \frac{A_\alpha}{(x - a)^\alpha} + \dots + \frac{A_1}{x - a} + \frac{s_1(x)}{Q_1(x)}$$

Applying the process to the proper fraction $s_1(x)/Q_1(x)$ for the other real roots, at the end one arrives at a proper fraction $s(x)/q(x)$ where $q(x)$ contains only imaginary roots.

$$q(x) = (x^2 + px + q)^\lambda \; q_1(x)$$

where $q_1(x)$ has no further factor $x^2 + px + q$.

Writing

$$x^2 + px + q = (x + \tfrac{p}{2})^2 - \tfrac{p^2}{4} + q = (x + \tfrac{p}{2})^2 + (\tfrac{\sqrt{-\Delta}}{2})^2 \quad (-\Delta > 0)$$

and setting

$$x + \frac{p}{2} = \frac{\sqrt{-\Delta}}{2} u$$

we have

$$x^2 + px + q = (\frac{\sqrt{-\Delta}}{2} u)^2 + (\frac{\sqrt{-\Delta}}{2})^2 = \frac{-\Delta}{4} (u^2 + 1)$$

so that

$$\frac{s(x)}{q(x)} = \frac{f(u)}{F(u)} = \frac{f(u)}{(u^2 + 1)^\lambda \; F_1(u)}$$

To show the separation

$$\frac{f(u)}{F(u)} = \frac{C_\lambda u + D_\lambda}{(u^2+1)^\lambda} + \frac{f_1(u)}{(u^2+1)^{\lambda-1}\, F_1(u)} \tag{2}$$

with constant C_λ, D_λ, consider the difference

$$\frac{f(u)}{F(u)} - \frac{Cu + D}{(u^2+1)} = \frac{f(u) - (Cu+D)\, F_1(u)}{(u^2+1)^\lambda\, F_1(u)} \tag{2'}$$

where for simplicity $C = C_\lambda$, $D = D_\lambda$ are taken, is certainly a proper fraction.

Now, in the polynomials $f(u)$, $F_1(u)$ we separate the even and odd degree terms and put u as a common factor in the odd degree terms, having thus

$$f(u) = g(u^2) + u\, h(u^2)$$

$$F_1(u) = G(u^2) + u\, H(u^2),$$

and setting $u^2 = v$, the numerator in (2') becomes

$$\begin{aligned} f(u) - (Cu + D)\, F_1(u) &= \big(g(v) - uh(v)\big) - (Cu + D)\big(G(v) + uH(v)\big) \\ &= \big(g(v) - DG(v) - Cv\, H(v)\big) \\ &\quad -\big(h(v) + CG(v) + DH(v)\big)u \end{aligned}$$

We determine the constants C and D such that this numerator is divisible by $u^2 + 1 (= v + 1)$. This is possible if each brecket vanishes for $v = -1$:

$$g(-1) - DG(-1) + CH(-1) = 0$$

$$h(-1) + CG(-1) + DH(1) = 0$$

or

$$H(-1)C - G(-1)D = -g(-1)$$

$$G(-1)C + H(-1)D = -h(-1)$$

The determinant

$$\Delta' = \begin{vmatrix} H(-1) & -G(-1) \\ G(-1) & H(-1) \end{vmatrix} = H^2(-1) + G^2(-1)$$

of this system is not zero. Indeed, if were zero, then $H(-1) = 0$, $G(-1) = 0$ follow, meaning that $v + 1 (= u^2 + 1)$ divides both $H(u^2)$ and $F_1(u^2)$ or divides $F_1(u)$. But this contradicts the hypothesis that $u^2 + 1$ does not divide $F_1(u)$. Since $\Delta' \neq 0$, the system admits a unique solution for C, D. These are not both zero, for otherwise $g(-1) = 0$, $h(-1) = 0$ would follow. But in that case, by similar argument, $f(u)$ is divisible by $u^2 + 1$, which contradicts the hypothesis that $f(u)/F(u)$ was reduced.

So we have proved the separation (2).

Repeating the process, all partial fraction can be separated and the proof is completed.