CHAPTER 1

Function, Limit, Continuity

1. Numbers

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1.1. Integers. Following historical development, the earliest numbers were the *counting numbers* $1, 2, 3, \ldots, n, \ldots$. Introducing the number zero, one obtain the numbers 0 $1, 2, \ldots, n, \ldots$ called the *natural numbers*. The natural numbers, except 0, that is, the counting numbers are all positive and are referred to as *positive integers*. Assigning "-" sign to these numbers one gets the *negatives integers*, namely, $-1, -2, -3, \ldots$ A positive integer, a negative integer or zero is called an *integer*.

1.2. Rational Numbers. Any number in the form of a ratio p/q of two integers ($p \neq 0$) is called a *rational number* or a *fraction*. Any integer is a rational number ($p = p/1$). Thus 3/4, 17/5, -11/7, 6, -9 are rational numbers.

The decimal expansion of any rational number p/q obtained by ordinary division is either finite or else infinite. It is know from Arithmetic that an infinite expansion of a rational number contains a repeating block as given in the following examples:

$$0,19771977...1977... \quad (= 0,\overline{1977})$$
$$-5,112323...23.. \quad (= -5,11\overline{23})$$

A finite expansion can be considered as an infinite expansion with "0" as repeating block:

$$12,75 \quad (= 1275,\overline{0})$$

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Example 1.1. Find the (repeating) decimal expansion of the rational number 152/55.

Dividing 152 by 55 one gets [1]

$$\begin{array}{l|l} 152 & 55 \\ \cline{2-2} & 2,76363\cdots63\cdots=2,76\overline{\overline{63}} \\ 110 & \\ \hline \ \ 420 & \\ \ \ 385 & \\ \hline \ \ 150 & \end{array}$$

[1] @HB needs correction

Conversely, any decimal expansion with repeating block (*cyclic expansion*) represents a rational number.

EXAMPLE 1.2. Express the repeating decimal expansion $3,71\overline{05}$ as a ratio of two integers.
Solution.
Set $r = 3,71\overline{05}$. Multiply each side by 1000 to bring "," just after the repeating block, and also multiply each side by 100 to bring "," just before the repeating block:

$$\begin{aligned} 10000r &= 37105,\overline{05} \\ 100r &= 371,\overline{05} \end{aligned}$$

Subtraction gives

$$9900r = 36734$$
$$r = \frac{36734}{9900}$$

PROPERTY 1.1. If $r_1 (= p_1/q_1)$, $r_2 (= p_2/q_2)$ are two rational numbers, then the numbers page=b1p1/3

i. $r_1 + r_2 \left(\dfrac{p_1}{q_1} + \dfrac{p_2}{q_2} = \dfrac{p_1 q_2 + p_2 q_1}{q_1 q_2} \right)$,

ii. $r_1 - r_2 \left(\dfrac{p_1}{q_1} - \dfrac{p_2}{q_2} = \dfrac{p_1 q_2 - p_2 q_1}{q_1 q_2} \right)$,

iii. $r_1 \cdot r_2 \left(\dfrac{p_1}{q_1} \cdot \dfrac{p_2}{q_2} = \dfrac{p_1 p_2}{q_1 q_2} \right)$,

iv. $r_1 : r_2 \left(\dfrac{p_1}{q_1} : \dfrac{p_2}{q_2} = \dfrac{p_1 q_2}{q_1 p_2} \right)$

are all rational.

COROLLARY 1.1. *Between any two distinct rational numbers there exists at least one rational number, hence infinitely many.*

PROOF. Let the given rational numbers be r_1 and r_2 : $r_1 + r_2$ rational $\Longrightarrow \dfrac{1}{2}(r_1 + r_2)$ is rational. (why this arithmetic mean is between r_1 and r_2?)
This process can be continued indefinitely. □

1.3. Irrational numbers. A number which is not rational is called an *irrational number*. Since any cyclic decimal expansion is a rational number, then non cyclic ones represent irrational numbers:

$0,81881888188881\cdots$ (Number of 8's increases by 1 in each step)

$4,303003000300003\cdots$

The existence of irrational numbers may also be shown by the following theorem: page=b1p1/4

THEOREM 1.2. *If n is a positive prime number, then $\sqrt{n}$ is irrational.*

Proof. Suppose $\sqrt{n} = p/q$ where the integers p, q have no common factor (divisor) other than 1. Any fraction can be reduced into this form by simplification.

$$\sqrt{n} = p/q \Longrightarrow q^2 n = p^2$$

Since $n \mid q^2 n$ (n divides $q^2 n$), then $n \mid pp$ implying $n \mid p$. Therefore for some integer k we have $p = kn$.

$$q^2 n = k^2 n^2 \Longrightarrow k^2 n = q^2 \Longrightarrow n \mid q.$$

The results $n \mid p$, $n \mid q$ show that p, q have a common factor $n (> 1)$, contradicting the assumption that p, q had no common factor. □

Some irrational numbers of this form are

$$\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7} \quad \text{(Why } \sqrt{4} \text{ is not irrational?)}$$

Property 1.2. Let r be a rational and α be an irrational number. Then

$$1)\ r + \alpha \quad 2)\ r - \alpha \quad 3)\ r\alpha \quad 4)\ r/\alpha$$

are all irrational.

Proof of i. Suppose that $r + \alpha$ is equal to a rational number s. Then, $r + \alpha = s \Longrightarrow \alpha = s - r \Longrightarrow \alpha$ is a rational number, since $s - r$ is rational. This contradicts the hypothesis. Hence $r + \alpha$ is irrational.

□

The proofs of other cases can be done similarly.

Remark 1.1. The sum, difference, product and the ratio of two irrational numbers may not be an irrational number:

$(3 + \sqrt{2}) + (5 - \sqrt{2}) = 8,$ $\quad (3 + \sqrt{2}) - (5 + \sqrt{2}) = -2,$

$\left(\frac{2}{3} + \sqrt{5}\right)\left(\frac{2}{3} - \sqrt{5}\right) = -\frac{41}{9},$ $\quad \sqrt{18}/\sqrt{2} = 3.$

Corollary 1.3. *Between any two distinct rational numbers, there exists at least one irrational number, and hence infinitely many.*

Proof. Let the given rational numbers be r_1 and r_2 ($r_1 < r_2$). $\sqrt{2}$ being irrational, for a sufficiently large positive integer m, the irrational number $\sqrt{2}/m$ is less than the difference $r_2 - r_1$. Then $r_1 + (\sqrt{2}/m)$ is irrational and lies between r_1 and r_2.

For all integers $n > m$ the irrational numbers $r_1 + (\sqrt{2}/n)$ lie between r_1 and r_2. □

1.4. Real numbers. A rational or an irrational number is called a *real number*.

The four arithmetic operations (rational operations) for any two real numbers will always yield real numbers (excluding the case a/b where $b = 0$).

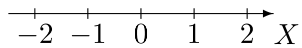

FIGURE 1.1. Number axis

FIGURE 1.2. Construction of a rational number

The above definition provides a classification of real numbers as rational and irrational. Real numbers can also be classified as algebraic and non-algebraic (transcendental) numbers: The roots of polynomial equation

$$a_0x^n + \cdots + a_{n-1}x + a_n = 0$$

with rational coefficients are called *algebraic numbers*, and non algebraic real numbers are called *transcendental numbers*.

According to this definition all rational numbers are algebraic ($x-p/q=0$). Some irrational numbers which are algebraic are $\sqrt{2}$, $5-\sqrt{3}$; for $x=\sqrt{2} \Longrightarrow x^2-2=0$ and

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$x = 5-\sqrt{3} \Longrightarrow (x-5)^2 = 3 \Longrightarrow x^2-10x+22=0$. Some irrational numbers which are transcendental are the well known numbers π and the base e of natural logarithm.

1.4.1. *Real number axis.* A line (straight line) on which real numbers are represented in some manner is called a *real number axis* or shortly a *number axis*. In general a representation is done by choosing on the axis a fixed point 0 as origin corresponding to zero, a positive sense, and a unit length to locate first, integers in succession as seen in Fig 1.1.

By the use of Thales Theorem, a rational number p/q can be constructed on the number axis. To find the point on the number axis corresponding to the number p/q, a ray OT (non parallel to $0X$) is drawn on which line segments $[OP]$, $[OQ]$ of lengths p, q units are taken (Fig 1.2). Then Q is joined to the point represented by 1. The line passing through P and parallel to $[Q1]$ intersects the number axis at the required point.

When $p<q<0$, the point Q is joined to the point representing -1 instead of 1.

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The positive square root of $a(>0)$ is denoted by $\sqrt{a}$ and the negative one by $-\sqrt{a}$. Thus,

$$\sqrt{4}=2, \quad -\sqrt{4}=-2, \quad \sqrt{(-3)^2}=\sqrt{9}=3.$$

The number 0 which is neither positive nor negative has only one square root, namely 0, as a double root of $x^2=0$.

1.4.2. *Absolute Value.* The *absolute value* of a real number a is a non-negative real number, denoted by $|\,a\,|$ and defined by

$$|\,a\,| = \sqrt{a^2} \quad (\geq 0)$$

or equivalently, by

$$|\,a\,| = \begin{cases} a, & a > 0, \\ 0, & a = 0, \\ -a, & a < 0. \end{cases}$$

The equivalency of two definitions can be seen by considering three cases $a > 0,\ a = 0,\ a < 0$ separately.

$|\,5\,| = \sqrt{5^2} = 5,$ $\quad |\,-3\,| = \sqrt{(-3)^2} = \sqrt{9} = 3$

$|\,-2\,| = -(-2) = 2,$ $\quad |\,2\,| = 2$

As an immediate corollary we have

Corollary 1.4. *1.* $|\,a\,|^2 = a^2$ *2.* $-|\,a\,| \leq a \leq |a|$.

Some other properties are stated in the next theorem.

Theorem 1.5. *If a, b are real numbers, then*

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1. $|\,ab\,| = |\,a\,|\,|\,b\,|$ *2.* $\left|\,\frac{a}{b}\,\right| = \frac{|\,a\,|}{|\,b\,|}$

3. $|\,a + b\,| \leq |\,a\,| + |\,b\,|$

Proof. 1. $|\,ab\,| = \sqrt{(ab)^2} = \sqrt{a^2}\sqrt{b^2} = |\,a\,|\,|\,b\,|$

2. Proved similarly.

3.

$$\begin{aligned} |\,a+b\,|^2 &= (a+b)^2 \\ &= a^2 + 2ab + b^2 \\ &= |\,a\,|^2 + 2ab + |\,b\,|^2 \\ &\leq |\,a\,|^2 + 2\,|\,a\,|\,|\,b\,| + |\,b\,|^2 \\ &= (|\,a\,| + |\,b\,|)^2 \\ |\,a+b\,|^2 &\leq (|\,a\,| + |\,b\,|)^2 \end{aligned}$$

where $|\,a+b\,|$, $|\,a\,| + |\,b\,|$ being non-negative, taking positive square roots of each side,

$$|\,a+b\,| \leq |\,a\,| + |\,b\,|$$

follows.

□

Changing b to $-b$ in the last inequality the latter is seen to include the inequality:

$$|\,a+b\,| \leq |\,a\,| + |\,b\,|.$$

1.4.3. *Distance.* The *distance* between two points A and B with coordinates a, b on the number axis, denoted by

$$d(A,B) = d(a,b) = |\,AB\,|,$$

is defined as the non negative real number $|\,b-a\,|$.

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EXAMPLE 1.3.
$$\begin{array}{ll} d(3,5) = |\,5-3\,| = 2 & d(3,5) = |\,3-5\,| = 2 \\ d(3,-5) = 3+5 = 8 & d(-2,7) = |\,7+2\,| = 9 \end{array}$$

1.5. Complex numbers. The roots of the quadratic equation

$$ax^2 + bx + c = 0 \quad (a \neq 0)$$

with real coefficients, are given by

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

They are real if and only if (iff) the discriminant $\Delta = b^2 - 4ac$ is non negative. Then for a real k, if $\Delta = -k^2 < 0$ the roots become non real and have the form

$$x_{1,2} = \frac{-b \pm ki}{2a} = u + iv$$

where u and v are real numbers and $i = \sqrt{-1}$, unit imaginary number, with i^2 = -1.

Hence in general case for any Δ the roots of a quadratic equation are numbers of the form

$$u + iv$$

which is called a *complex number*.

A complex number

$$z = a + ib$$

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is real or imaginary according as $b = 0$ or $b \neq 0$. The real numbers a and b are called, respectively, the *real part* and *imaginary part* of z, written

$$a = \text{Re } z, \quad b = \text{Im } z.$$

DEFINITION 1.1 (Equality). Two complex numbers are equal iff their real parts are equal and imaginary parts equal:

$$a + ib = c + id \iff a = c, b = d.$$

Hence $a + ib = 0 \iff a = 0, b = 0$.

DEFINITION 1.2 (Conjugation). If $z = a + ib$, then the number $a - ib$ is called the *complex conjugate* or simply *conjugate* of z, written $\overline{z} = a - ib$.

From $a + ib = a - ib \implies b = 0$, it follows that a complex number is real iff it is equal to conjugate:

$$z = \overline{z} \iff z \text{ is real.}$$

DEFINITION 1.3 (Addition and subtraction). If $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$, then their *sum* and *difference* are defined as follows:

1. $z_1 + z_2 = a_1 + a_2 + i(b_1 + b_2)$,
2. $z_1 - z_2 = a_1 - a_2 + i(b_1 - b_2)$.

One concludes that

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}, \quad \overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}.$$

In words: The conjugate of a sum (difference) is the sum (difference) of conjugates.

A complex number is multiplied by a real scalar k by multiplying its real and imaginary parts by k:

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$$k(a + ib) = ka + ikb.$$

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EXAMPLE 1.4. Simplify
a) $u = \overline{(2 - 3i)} - 2(4 + 2i)$,
b) $v = \overline{2(3 - 2i) + 3i}$.

Solution.

a) $u = 2 - 3i - 8 - 4i = 2 - 8 - (3i + 4i) = -6 - 7i$.
b) $v = \overline{6 - 4i + 3i} = \overline{6 - i} = 6 + i$.

DEFINITION 1.4 (Multiplication). The *product* of two complex numbers is obtained as follows:

$$\begin{aligned}(a + ib)(c + id) &= ac + iad + ibc + i^2bd \\ &= ac + i(ad + bc) - bd \quad \text{(Note that } i^2 = -1) \\ &= (ac - bd) + i(ad + bc)\end{aligned}$$

COROLLARY 1.6. $z = a + ib \implies z\overline{z} = a^2 + b^2$.

EXAMPLE 1.5. Perform multiplications:
a) $u = (2 - 3i)(5 + i)$,
b) $v = (2 - 3i)(2 + 3i)$.

Solution.

a) $u = 10 + 2i - 15i - 3i^2 = 10 - 13i + 3 = 13 - 13i$.
b) $v = (2 - 3i)(2 + 3i) = 2^2 + 3^2 = 4 + 9 = 13$.

DEFINITION 1.5. In view of above corollary, *division u/v* is carried out by multiplying the numerator and denominator by the conjugate $\overline{v}$ of the denominator:

$$\frac{u}{v} = \frac{u}{v}\frac{\overline{v}}{\overline{v}} = \frac{1}{v\overline{v}}u\overline{v}.$$

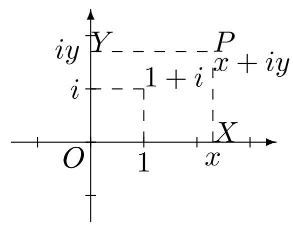

FIGURE 1.3.

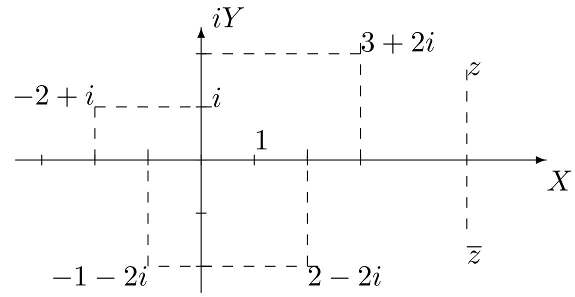

FIGURE 1.4.

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1.5.1. *Geometric Representation.* By taking two perpendicular axes with a common origin 0, and considering the horizontal axis as the *real axis* and the vertical axis as the *imaginary axis* containing pure imaginary numbers (See Fig. 1.3 and Fig. 1.4), any complex numbers $z = x + iy$ will be represented by a point P as the vertex of the rectangle $OXPY$ where X is on the real axis with abscissa x, and Y is on the imaginary axis iy. The plane in which complex numbers represented this way is called *complex plane* (*z-plane* or *ARGAND plane*).

On the accompanying Fig. 1.4, the numbers 1, i, $3+2i$, $-2+i$, $-1-2i$, $2-2i$ are plotted.

The conjugate numbers $z = x + iy$ and $z = x - iy$ will symmetrically placed with respect to real axis.

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EXAMPLE 1.6. $\dfrac{2+3i}{1-i} = \dfrac{2+3i}{1-i}\dfrac{1+i}{1+i} = \dfrac{-1+5i}{2} = -\dfrac{1}{2} + \dfrac{5}{2}i$

One may show that

$$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}, \quad \overline{z_1/z_2} = \overline{z_1}/\overline{z_2}$$

In words: The conjugate of a product (ratio) is the product (ratio) of conjugates.

THEOREM 1.7 (The Fundamental Theorem of Algebra). *A polynomial equation with real coefficient of degree n has at least one root, real or imaginary, and hence n roots, real or imaginary, simple or repeated.*

PROOF. Omitted. □

COROLLARY 1.8. *If a polynomial equation with real coefficients has an imaginary root it admits its conjugate as another root.*

PROOF. The proof is an applications of conjugation: Let the equation $P(x) = a_0 + a_1x + \cdots + a_nx^n = 0$ which can be represented as

$$P(x) = \sum_{k=0}^{n} a_k x^k = 0$$

admit the imaginary number z as root.Then

$$\begin{aligned} 0 = P(z) &= \sum a_k z^k \\ \Longrightarrow 0 &= \overline{\sum a_k z^k} = \sum \overline{a_k z^k} = \sum \overline{a_k}\,\overline{z^k} \\ &= \sum a_k (\overline{z})^k = P(\overline{z}) \Longrightarrow P(\overline{z}) = 0. \end{aligned}$$

□

COROLLARY 1.9. *A polynomial equation with real coefficients of odd degree has at least one real root.*

Polar form of complex numbers and related properties will be treated in Chapter 4. [2] page=b1p1/15

We conclude this section by two classification of numbers in Fig. 1.5 and Fig. 1.6.

1.6. Exercises (1).

1.1. Construct the following numbers on the number axis:

a) $3/5$ b) $-7/3$ (use Thales Theorem)
c) $\sqrt{8}$ $\sqrt{12}$ (use Pythagorean Theorem).

1.2. Give examples of two irrational numbers such that their page=b1p1/16

a) sum b) difference c) product d) ratio

is a rational number.

1.3. Let e_1, e_2 be two even and o_1, o_2 be two odd numbers. Then prove the following:
a) $e_1 + e_2$, e_1e_2, $o_1 + o_2$ are even numbers
b) $e_1 + o_1$, o_1o_2 are odd numbers

1.4. If the product of two consecutive
a) even numbers is 624,
b) odd numbers is 1155
find them. [a) $\pm 24, \pm 26$, b) $\pm 33, \pm 35$.]

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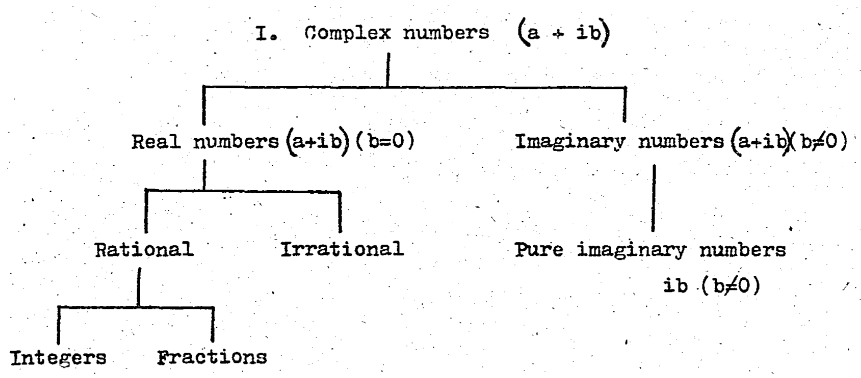

Figure 1.5. Classification of complex numbers

1.5. If the sum of three consecutive a) integers is 294,
b) even integers is 288,
c) odd integers is 327
find them. [Hint: Take the middle number as a variable.]

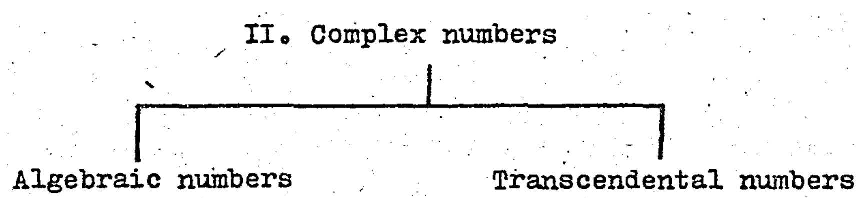

FIGURE 1.6. Classification of complex numbers

1.6. Prove that the square
a) of an even number is an even number.
b) of an odd number is an odd number.

1.7. Prove the irrationality of the numbers
a) $\sqrt{7}$ b) $3+\sqrt{2}$

1.8. Find the value of $|\,2x+15\,|$ for

a) $x = -9$ b) $x = -7, 8$

1.9. Show the following properties of absolute value:

a) $|\, a\,|^2 = a^2$ b) $-\,|\, a\,| \leq a \leq |\, a\,|$
c) $|\, a - b\,| = |\, b - a\,|$ d) $|a| = 0 \Leftrightarrow a = 0$
e) $|\, ab\,| = |\, a\,|\,|\, b\,|$ f) $|\, a/b\,| = |\, a\,| \,/\, |\, b\,|$
g) $|\, a + b\,| \leq |\, a\,| + |\, b\,|$ h) $|\,|\, a\,| - |\, b\,|\,| \leq |\, a - b\,|$

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1.10. Find the distance between the given points. First express them as absolute value, and then compute.

a) 2.72 and 5.16 b) 3.86 and −7.28
c) −3.86 and 7.28· d) −1.23 and −12.35

1.11. $(3 + i)^3 =?$ [Ans. $18 + 26i$].

1.12. $\frac{2+i}{3-2i} =?$ [Ans. $(4 + 7i)/13$].

1.13. Write a polynomial of least degree with real coefficients having the roots 3, $1 - 2i$. [Ans. $x^3 - 5x^2 + llx - 15$].

1.14. Solve for real x and y:

$$\frac{2 - i}{3 + iy} = \frac{2x - 3iy}{2 + i} \quad \text{[Ans. } x = 5/6, y = 0].$$

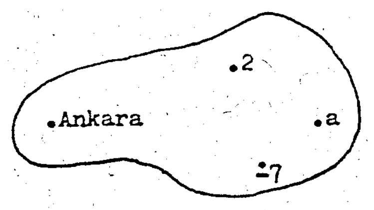
1.15. If $z = 5 + 4i$ find $z^2 - 2z + z\overline{z}$ [Ans. $60 + 32i$].

2. SETS

2.1. Definitions.

Definition 2.1. Any collection of objects (concrete or abstract) is called a *set,* and the objects in the set are its *elements* or *members*. The sets are usually represented by capital letters A, B, $\cdots$. Two sets formed by the same elements are said to be *equal*.

The set A consisting of elements, say, 2, a, Ankara, −7, is denoted either by listing the elements within two braces, or by a diagram (Venn diagram) in which the elements page=b1p1/18
are marked arbitrarily in the plane and enclosed by a closed curve:



$A = \{2, a, \text{Ankara}, -7\}$
$A = \{\text{Ankara}, 2, -7, a\}$

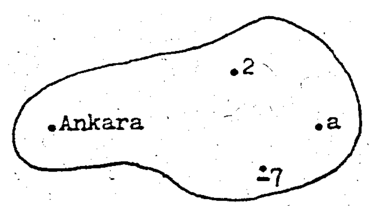


FIGURE 2.1. Set

The symbol $\in$ is used to mean "is an element of" or "belongs to", and $\notin$ is used otherwise. Then

$$2 \in A, \quad \text{Ankara} \in A, \quad 7 \notin A, \quad \text{Anka} \notin A.$$

A set having finitely many elements is said to be a *finite set*, and one having infinity of distinct elements an *infinite set*. Thus $\{2, a, \text{Ankara}, -7\}$ is finite, while the set $\{1, 2, 3, \cdots, n, \cdots\}$ of natural numbers is infinite.

If S is a finite set, the number of its distinct elements is denoted by $n(S)$. [3]

Example 2.1. 1. For the set $D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ of digits (*numerals*), $n(D) = 10$.
2. For $E = \{\text{Venus}, \text{ Earth}, \text{ Izmir}, 3, \text{ Earth}, 3, -5\}$, $n(E) = 5$.

Another way of representing the sets is by the use of a property common to all elements. If such a property is expressed by a true statement $p(x)$, then the symbol

$$\{x : p(x)\} \quad \text{or} \quad \{x \mid p(x)\}$$

represents the set of all objects having the property $p(x)$.
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The meanings of the symbols $\{x : p(x) \text{ and } q(x)\}$ and $\{x : p(x) \text{ or } q(x)\}$ are clear.

Example 2.2. (for finite sets):
1. $D = \{n : n \text{ is a digit }\} = \{0, 1, 2, \cdots, 9\}$
2. $\{n : n \in D, \quad n \text{ is prime}\} = \{2, 3, 5, 7\}$
3. $\{n : n \in D, \quad 1 \leq n < 7\ \} = \{\ 1,\ 2,\ 3,\ 4,\ 5,\ 6\ \}$

Example 2.3. The following infinite sets of numbers are used frequently in mathematics:
1. $\mathbb{N} = \{n : n \text{ is a natural number }\} = \{0, 1, 2, \cdots, n, \cdots\}$
2. $\mathbb{Z} = \{n : n \text{ is an integer }\} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$
3. $\mathbb{Q} = \{r : r \text{ is a rational number }\} = \{\frac{p}{q} : p, q \in Z, q \neq 0\}$
4. $\mathbb{Q}' = \{r' : r' \text{ is an irrational number }\}$
5. $\mathbb{R} = \{x : x \text{ is a real number}\} = \{x : x \in Q \text{ or } x \in Q'\}$
6. $\mathbb{C} = \{z : z \text{ is a complex number }\} = \{a + ib : a, b \in R, i^2 = -1\}$

A set worth of mentioning is the one having no element at all. It is called the *empty set* (*null set*) and denoted by $\emptyset$, so that $n(\emptyset) = 0$.

Example 2.4. Each one of the following is the null set $\emptyset$:
1. $\{x : x^2 - 1 = 0,\ x \in \mathbb{R}\}$
2. $\{x : |\,x\,| < 0,\ x \in \mathbb{R}\}$
3. $\{x : x \text{ is a box}, x \text{ is open and } x \text{ is closed }\}$

In any particular discussion, a set that contains all the objects that enter into that discussion is called *the universal set*.
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Clearly numerous universal sets exist corresponding to numerous particular discussions. A universal set is denoted by U .

If real numbers are taken into consideration, $\mathbb{R}$ is the universal set.

[3]@HB —S—?

2.2. Subsets. A set A is said to be a subset of a set B, if every element of A is also an element of B, and one writes

$$A \subseteq B \quad \text{(Read: } A \text{ is a subset of } B\text{)}$$

where B is said to be a *superset* of A.

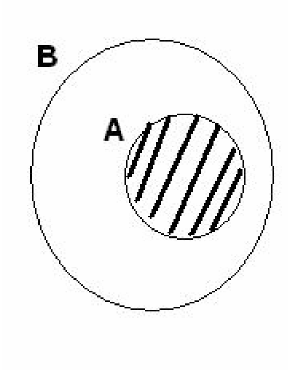

FIGURE 2.2. $A \subseteq B$

It follows that any set is a subset of itself, and we agree that the empty set is a subset of any set. Thus

$$\emptyset \subseteq \emptyset \subseteq \{1\} \subseteq \{1\} \subseteq \{1, 2, 3\} \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}.$$

If $A \subseteq B$, but $A \neq B$ one uses the notation

$$A \subset B \quad \text{(Read: } A \text{ is a \textit{proper subset} of } B\text{)}$$

where B contains at least one element not contained in A. With this notation the above relations can be written in the form

$$\emptyset \subseteq \emptyset \subset \{1\} \subseteq \{1\} \subset \{1, 2, 3\} \subset \mathbb{N} \subset \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}.$$

EXAMPLE 2.5. Write all subsets of $\{1, 2, 3\}$.

Solution.
$\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1, 2, 3\}$.

If each of two sets is a subset of the other, then clearly they are equal, and vice versa:

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[4]

3. Induction

Some theorems $p(n)$ in mathematics which involve the integer n as a variable are usually proved by a method called *induction*. These theorems are very often expressed by the use of some notations which we define below.

Let $a_m, \cdots, a_i, \cdots, a_n$ be any numbers with a_i as the general term where the integer "i" is called the *index variable* or simply the *index*. $(m \leq i \leq n)$

The sum $a_m + \cdots + a_i + \cdots + a_n$ where i runs from m up to n is denoted by the use of capital Greek letter Σ (sigma) as

$$\sum_{i=m}^{n} a_i = a_m + \cdots + a_n \quad \text{(summation of } a_i \text{ from } m \text{ to } n) \ ,$$

Σ being called the *summation notation* and the product $a_1 \cdots a_i \cdots a_n$ is represented by the use capital letter Π (pi) as

$$\prod_{i=m}^{n} a_i = a_m \cdots a_n \quad \text{(product of } a_i \text{ from } m \text{ to } n) \ ,$$

Π being called the *product notation*.

Example 3.1.

1.

$$\begin{aligned}\sum_{i=3}^{i=6}(2i^2+5) &= (2\cdot 3^2+5)+(2\cdot 4^2+5)+(2\cdot 5^2+5)+(2\cdot 6^2+5)\\ &= 2(3^2+4^2+5^2+6^2)+4\cdot 5\\ &= 2\cdot 86+20 = 192.\end{aligned}$$

2.

$$\begin{aligned}\prod_{i=2}^{4}(2i^2+5) &= (2\cdot 2^2+5)(2\cdot 3^2+5)(2\cdot 4^2+5)\\ &= 13\cdot 23\cdot 37.\end{aligned}$$

3.

$$\prod_{i=1}^{n} = 1 \cdots n.$$

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[4] 21-28 missing

The last example gives the product of all positive integers from 1 up to n. This particular product is abbreviated by the use of notation "!" called the *factorial notation*:

$$1 \cdots m = m! \quad \text{(read: } m \text{ factorial, or factorial } m\text{)}$$

Defining in addition 0! as 1 we have

$$0! = 1, \quad 1! = 1, \quad 2! = 1 \cdot 2, \quad 3! = 1 \cdot 2 \cdot 3$$

$$4! = 1 \cdot 2 \cdot 3 \cdot 4, 5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 4!.5$$

$$(n+1)! = 1 \cdots n \cdot (n+1) = n! \cdot (n+1)$$

Another symbol is "|" which is put between two integers or between two polynomials to mean that the left quantity divides the right one:

$$5 \mid 25, \quad 9 \mid 27, \quad -7 \mid 91, \quad x - 2 \mid x^2 - 4^2.$$

Some statements to be proved by induction are the following:

$$p(n) : \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \text{ for all } n \in \mathbb{N}$$

$$q(n) : n! > 2^n \text{ for all } n \in \mathbb{N}$$

$$r(n) : x - y \mid x^n - y^n \text{ for all } n \in \mathbb{N}_1$$

where the sets $\mathbb{N}_1$, $\mathbb{N}_4$ or in general $\mathbb{N}_m$ means

$$\mathbb{N}_m = \{m, m+1, m+2, \cdots\}$$

which consists of all successive integers, smallest of which is the integer $m \in \mathbb{N}$.

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The proof of a theorem

$$\text{“}p(n), \text{ for all } n \in \mathbb{Z}_m = \{m, m+1, m+2, \cdots\}\text{''}; \quad m \in \mathbb{Z}$$

by induction is done in four steps:

(1) Verifying the truth of $p(m)$, or verifying $p(n)$ for the first integer m in $\mathbb{Z}_m$,
(2) Assuming the truth of $p(k)$ for a number $k \in \mathbb{Z}_m$
(3) Proving $p(k+1)$ using (2)
(4) Arguing as follows:
$p(m)$ is true by (1). Since $p(m)$ is true, then $p(m+1)$ must be true by (3). Since $p(m+1)$ is true, then $p(m+2)$ must be true again by (3). Continuing this way $p(n)$ must be true for all $n \in \mathbb{Z}_m$

Example 3.2. Prove by induction:

$$p(n) : \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \text{ for } n \in \mathbb{Z}_1$$

PROOF. Here $\mathbb{Z}_m$ is $\mathbb{Z}_1$, since 1 is the least value taken by n.

□

$$p(n) : \sum_{(i=1)}^{n} i^2 = \frac{1(1+1)(2+1)}{6} \iff 1 = 1(\text{ true})$$

(In case p(m) is false the statement is disproved and hence there is no need to go further.)

2)Suppose $p(k)$ is true for some $k \in Z_1$, that is suppose

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

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CORRECTION UP TO HERE

3) We need to prove

$$p(k+1) : \qquad \sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

under the hypothesis (2). Indeed,

$$\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= \left[\sum_{i=1}^{k} i^2\right] + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \qquad \text{(by (2))} \\
&= (k+1)\left[\frac{k(2k+1)}{6} + k + 1\right] \\
&= (k+1)\frac{k(2k+1)+6(k+1)}{6} \\
&= (k+1)\frac{2k^2+7k+6}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6}
\end{aligned}$$

which is $p(k+1)$.

4) Then $p(n)$ is true for all $n \in \mathbb{Z}_1$

EXAMPLE 3.3. Prove $n! > 2^n$ for all $n \in \mathbb{Z}_4$

PROOF. 1) For $m = 4$, $4! > 2^4$ (true).

2) Suppose $k! > 2^k$ is true for $k \in \mathbb{Z}_k$.

3) To prove $(k+1)! > 2^{k+1}$, having

$$(k+1)! = k!(k+1) > 2^k(k+1) \qquad \text{(by (2))}$$

it will suffice to show

$$2^k(k+1) > 2^{(k+1)}$$

or $k+1 > 2$ which is true since k $\in \mathbb{Z}_4$.

4) $a! > 2^n$ is true for all n $\in \mathbb{Z}_4$. □

Example 3.4. Prove $x-y|x^n-y^n$, for all n $\in \mathbb{Z}_1$.

Proof. i. For n=1, $x-y|x-y$ (true)

ii. Suppose $x-y|x^k-y^k$ for some k $\in \mathbb{Z}_1$.

We have supposed divisibility of x^k-y^k by x-y, that is, the existence of a polynomial B(x,y) such that

$$x^k - y^k = B(x,y).(x-y)$$

iii. We prove $x-y|x^{k+1}-y^{k+1}$ using (ii).

To use (ii) we express $x^{k+1}-y^{k+1}$ in terms of x^k-y^k:

$$x^{k+1} - y^{k+1} = x^{k+1} - x^k y + x^k y - y^{k+1} \tag{3.1}$$

$$= x^k(x-y) + y(x^k - y^k) \tag{3.2}$$

$$= x^k(x-y) + y.B(x,y)(x-y) by (2) \tag{3.3}$$

$$= [x^k + y.B(x,y)](x-y). \tag{3.4}$$

$$= C(x,y).(x-y) \tag{3.5}$$

meaning that

$$x-y|x^{k+1}-y^{k+1}.$$

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iv. The divisibility holds for all n $\in \mathbb{Z}_1$.

□

EXERCISES (1.3)

3.1. Evaluate

$$a) \sum_{i=2}^{6} i^2 \quad b) \prod_{i=2}^{4} i^2$$

$$c) \prod_{j=1}^{7} \frac{j}{i} \quad d) \sum_{i=2}^{7} \frac{j^2}{i}$$

3.2. Write the following by the use of $\sum, \prod$ or ! .

a) $2^2+3^2+4^2+5^2+6^2$ b) $2^2.3^2.4^2.5^2.6^2$ c) 3+6+9+12+15

d) 3 . 6 . 9 . 12 . 15 e) 5 . 10 . 15 . 20 . 25 . 30 f) 5+10+15+20+25+30

a) 2! b) 10! c) 32! d) 50!

e) 12! f) 100! g) 8! f) 5!

3.3. Write the following int the forms (n-1)!n and (n-2)!(n-1)n .

3.4. The symbol $\overline{a_n...a_0}$ represents a positive number with n+1 digits (for instance $\overline{1977} = 1977$). A mathematician proved that the equality

$$\sum_{k=0}^{n} a_k! = \overline{a_n....a_0}$$

holds only for numbers 1, 145 and 40585. Verify the equality for these numbers.

3.5. Simplify the following

$$\text{a) } \frac{9!}{8!} \quad \text{b) } \frac{10!}{11!} \quad \text{c) } \frac{12!}{14!} \quad \text{d) } \frac{27!}{25!}$$

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Solution. $\|7x+3\| = 5 \;\Rightarrow\; 5 \leqslant 7x+3 < 6$
$\Rightarrow\; 2 \leqslant 7x < 3 \;\Rightarrow\; 2/7 \leqslant x < 3/7$

TYPES OF FUNCTIONS

Polynomial functions:
A function

$$P : \Re \quad \Re, \quad P(x) = \sum_{k=0}^{n} a_k x^k = a_n x^n + ... + a_0 \quad (a_i \in \Re)$$

is called a polynomial function where the rule

$$a_n x^n + ... + a_0$$

for P is a polynomial of degree n (if $a_n \neq 0$). The only polynomial without degree is the zero polynomial where all coefficients are zero.

The polynomials of degree 0 are constant, and $P:\Re\to\Re$, $P(x)=c$ is called a constant function whose graph is a horizontal line. A polynomial

$$P:\Re\to\Re,\quad P(x)=ax+b,\quad (a\neq 0)$$

of degree 1 is called a linear function of which the particular case

$$I=\Re\to\Re,\quad I(x)=x$$

is called the identity function whose graph is the line $y=x$.

Rational and irrational functions:

A function

$$R:\Re\to\Re,\quad R(x)=\frac{P(x)}{Q(x)}$$

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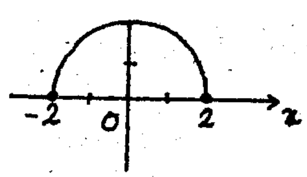
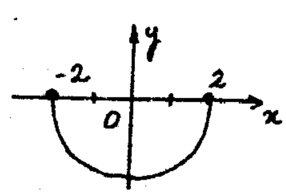
of degree n in y, defines at most n algebraic functions which we call implicitly
defined functions.

Example. The relation $\{(x,y): x\in\mathbb{R}, x^2+y^2=4\}$, where $x^2+y^2=4$ is of second degree in y, defines two functions whose rules are obtained by solving $x^2+y^2=4$ for y:

$$y=\sqrt{4-y^2} \qquad\qquad y=-\sqrt{4-y^2}$$



Graph of the function $y=\sqrt{4-y^2}$



Graph of the function $y=-\sqrt{4-y^2}$

More generally a function defined by a relation $f(x,y)=0$ is said to be an implicitly defined function. For instance $xy^2-(x+1)y+1=0$, y cos $y+x^3+x=0$ define some implicitly defined functions.

d. Trigonometric Functions.

A function which is not algebraic is called a transcendental function. As some examples for transcendental functions we will give trigonometric functions which we will represent simply by their rules:

Rules for trig. fn.	Domain	Range	Period = T
$y = \sin x$	$\mathbb{R}$	$[-1, 1]$	2π
$y = \cos x$	$\mathbb{R}$	$[-1, 1]$	2π
$y = \tan x$	$\mathbb{R} - \{x : x = (2k+1)\frac{\pi}{2}, k \in \mathbb{Z}\}$	$\mathbb{R}$	π
$y = \cot x$	$\mathbb{R} - \{x : x = k\pi, k \in \mathbb{Z}\}$	$\mathbb{R}$	π
$y = \sec x$	D_{tan}	$\mathbb{R} - (-1, 1)$	2π

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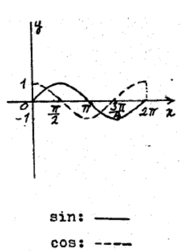
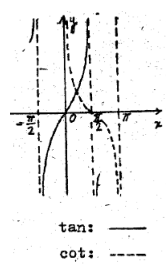
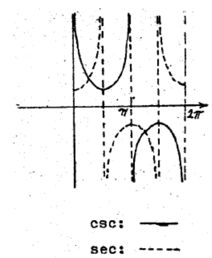
$y = \csc x \quad D_{\cot x} \quad R - (-1, 1) \quad 2\pi$

their graphs are given in an interval of length T:







Identities:

$$\cos^2 x + \sin^2 x = 1, \quad 1 + \tan^2 x = \sec^2 x, \quad 1 + \cot^2 x = \csc^2 x$$
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

$$\left.\begin{array}{l} \sin 2x = 2 \sin x \cos x \\ \cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x \\ \tan 2x = \frac{2\tan x}{1 - \tan^2 x} \end{array}\right\} \text{Double Angle Formulas}$$

$$\left.\begin{array}{l} \sin^2 \frac{x}{2} = \frac{1 - \cos x}{2} \\ \cos^2 \frac{x}{2} = \frac{1 + \cos x}{2} \end{array}\right\} \text{Half Angle Formulas}$$

$$\left.\begin{aligned} sinx + siny &= 2sin\tfrac{x+y}{2}cos\tfrac{x-y}{2} \\ sinx - siny &= 2sin\tfrac{x-y}{2}cos\tfrac{x+y}{2} \\ cosx + cosy &= 2cos\tfrac{x+y}{2}cos\tfrac{x-y}{2} \\ cosx - cosy &= -2sin\tfrac{x+y}{2}sin\tfrac{x-y}{2} \end{aligned}\right\} (Factor form)$$

C. Monotonic increasing (decreasing) functions:

A function f : D → R is said to be an increasing function on an open interval I which is a subset of the domain D, if

$$f(x_2) > f(x_1) \text{ or } f(x_2) - f(x_1) > 0$$

for any two numbers x_1, $x_2 \epsilon I$ for which $x_1 < x_2$.

The graph of such a function rises as x increases on I, and we say that f increases on I.

Under the same condition for x_1, x_2 if

$$f(x_2) < f(x_1) \text{ or } f(x_2) - f(x_1) < 0,$$

than f is called a decreasing function on I.

The graph of a decreasing function falls as x increases on I, and we say that f decreases on I.

Example. Show that $y = 4 - x^2$ increases on the interval R_0^-. and decreases on R_0^+.

Solution. For x_1, $x_2 \epsilon D = R$ with $x_1 < x_2$, we have

$$f(x_2) - f(x_1) = (4 - x_2^2) - (4 - x_1^2) = x_1^2 - x_2^2$$

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$$= (x_1 - x_2)(x_1 + x_2) \begin{cases} > 0 when x_1, x_2 \epsilon R_o^- \\ < 0 when x_1, x_2 \epsilon R_o^+ \end{cases}$$

If f is an increasing (or decreasing) function on an interval $I \subseteq D$, then f is said to be a monotonic increasing (or monotonic decreasing) function in the interval I.

The function given in the above example, is monotonic increasing in R_o^- and monotonic decreasing in R_o^+.

A monotonic increasing (or decreasing) function f an interval is expressed usually by saying that f is one-to- one (or simply 1-1) in I to mean that to distinct numbers x_1, x_2 in I correspond distinct images $f(x_1), f(x_2)$.

D. Inverse of a function

A function

$$f : D \to \mathbb{R}, y = f(x) or f = \{(x, y) : x \in D, y = f(x)\} \tag{3.6}$$

with D as the domain and R as the range, being a relation from D $\to$ R, its inverse

$$f^{-1} = \{(x, y) : x \in R, x = f(y)\} \tag{3.7}$$

is a relation from R to D.If the relation f^{-1} is function we call f^{-1} the inverse function of f, and f is said to be an invertible on the set D.

Since f is a function it maps an x in D into a image y in R, and since f^{-1} is a function from R to D it maps y backward to the single image x in D. This means that f is an one-to-one function and consequently f^{-1} is one-to-one function.

The graphs of f and f^{-1} are symmetric with respect

to the line y=x.(The pairs (x,y) of f and (y,x) of f^{-1} are symmetrical in y=x)

Example. Show that f:R$\to$ R,y=2x-1 is invertible on R. and find its inverse g.

$$\begin{gathered} f=\{(x,y):\ x\epsilon R,\ y=2x-1\ \} \\ f^{-1} =\{(x,y):\ x\epsilon R\ ,\ x=2y-1\ \} \\ =\{(x,y):\ x\epsilon R,\ y= \tfrac{x+1}{2}\ \} \\ g:R\to R,\ g(x)=\tfrac{x+1}{2} \end{gathered}$$

Corollary.If f: $D \to R$, y=f(x) is monotone increasing (or decreasing) on a interval $I \subseteq D$,then f is invertible on that interval I.

Proof . It will suffice to give the proof for the case where f is monotone increasing on I.

Since f is monotone increasing it maps distinct numbers in I to distinct numbers in R.

If the relation f^{-1} is not a function then some distinct numbers y1,y2 ϵ R are mapped to the same number x in D, contradicting that f is monotone on I.

FIGURE 3.1. Missing Figure: Pg. 53

Let f:R→R be a function with a domain $D \subseteq R$. If I in a subset of D, then f:$I \to J$ is said to be a restricted function in the restricted domain I.

If there are some intervals on which a function f satisfies required conditions, then f is said to be restricted on each interval or a subset of it, and the interval itself is the largest.

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Example.Find a restriction on the domain D of the function given by the rule y= $\mid x-1 \mid -2 \mid x \mid +x$ to be

a) a constant function,

b) an inveritble function.

Solution. The given function is the piecewisely defined function:

$$\text{y} = \begin{cases} 1+2x & \text{if } x\epsilon(-\infty,0) \\ 1-2x & \text{if } x\epsilon(0,1] \\ -1 & \text{if x } (1,\infty). \end{cases}$$

a) A domain of restriction is (1,∞).

b) A domain of restriction is (-∞,0] on which the function is increasing or (0,1] on which it is decreasing.

E.Operation with functions:

Let

$$\text{f:}I \to R,\ \text{y=f(x)}$$

be a function with domain I.If cϵR, then the function

$$\text{cf: } I \to R,\ \text{y=(cf)(x)=cf(x) (0)}$$

is called a scalar multiple of f.

Let now be given two functions

f:$I \to R$, y=f(x)
g:$J \to R$, Y=g(x)

with non disjoint domain I and J, then f+ g,f-g, fg ,

f/g called the *sum*, $difference$, $product$ and *ratio* of f and g, are defined as follows:

Domain
f+g: $I \cap J$, y=(f+g)(x) = f(x)+g(x) (1)
f-g: $I \cap I$, y=(f-g)(x) = f(x)-g(x) (2)
f*g: $I \cap J$, y=(fg)(x) = f(x)g(x) (3)
f/g: D, y=(f/g)(x) = f(x)/g(x) (4) where D = $(I \cap J) - x : g(x) = 0$.
Another function is gof, called $compositefunction$ which is defined as
gof:D, v=(gof)(x) = g(f(x))
where the domain D is the largest possible subset of $\Re$ on which g(f(x)), f(x) and g(x) are defined.
Because of the rule g(f(x)) we call also a $functionoffunction$ or a $chainfunction$.
$Example$. let f(x)= $\frac{|x|}{x}$ and g(x) = x$\sqrt{(1-x)}$ be two functions. We have D$_f$ = R^* = R 0, D$_g$ = $(-\infty, 1]$
and
0) $3f(x) = 3f(x) = 3\frac{|x|}{x}$
1) $(f+g)(x) = f(x) + g(x) = \frac{|x|}{x} + x\sqrt{(1-x)}$

2). $(f-g)(x) = f(x) - g(x) = \frac{|x|}{x} - x\sqrt{(1-x)}$

3). $(fg)(x) = f(x)g(x) = \frac{|x|}{x}x\sqrt{(1-x)} = |x|\sqrt{(1-x)}$

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where cancelation by x is permissible under x/neq and this condition is jointly written with the rule. 4) $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}) = \frac{|x|}{x^2\sqrt{(1-x)}}$
As to compositions gof and fog we have
5) $(gof)(x) = f(g(x)) = g(\frac{|x|}{x}) = \frac{|x|}{x}\sqrt{1 - \frac{|x|}{x}}$

$$fog(x) = f(g(x)) = f(x\sqrt{(1-x)}) = \frac{|x\sqrt{(1-x)}|}{x\sqrt{(1-x)}} = \frac{|x|\sqrt{(1-x)}}{x\sqrt{(1-x)}}$$

$$= \frac{|x|}{x} \quad (x \neq 1)$$

and
D$_{gof}$=$(-\infty, 1] - 0$, D$_{fog}$ = $(-\infty, 1] - 0, 1 = (-\infty, 1) - 0 = (-\infty, 1)^*$
$Example$. Given the functions
f:$\Re \to \Re$ f(x)= $\frac{x}{x-2}$; g:$\Re \to \Re$, g(x)= $x^2 - x$
find the rules for composite functions gof fog, and then determine their domains. *Solution*. 1.$(gof)(x) = g(f(x)) = f^2(x) - f(x) = \frac{x^2}{(x-2)^2} - \frac{x}{x-2} =$

$\frac{x^2-x(x-2)}{(x-2)^2} = \frac{2x}{(x-2)^2}$
2.$(fog)(x) = f(g(x)) = \frac{g(x)}{g(x)-2} = \frac{x(x-1)}{(x+1)(x-2)}$
$D_{gof} = \Re - 2$, $\quad D_{fog} = \Re - -1, 2$

Corollary: If f is an invertible function, then

$$f^{-1} \circ f = f \circ f^{-1} = I$$

where I is the identity function under a necessity restriction.

Proof: Let $f : D \to R$ $y = f(x)$ with $x = f^{-1}(y)$ then

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x = I(x)$$

$(f^{-1} \circ f)(x) = I(x)$ for all x implying that $f \circ f^{-1} = I$
Also

$$(f \circ f^{-1})(y) = f(f^{-1}(y)) = f(x) = y = I(y) \Rightarrow f \circ f^{-1} = I \ \square$$

Corollary: $(h \circ g) \circ f = h \circ (g \circ f)$

$$\begin{aligned} For, ((h \circ g) \circ f)(x) &= (h \circ g)(f(x)) \\ &= h(g(f(x))) = h \circ ((g \circ f)(x)) = (h \circ (g \circ f))(x) \end{aligned}$$

for all x. $\square$
Corollary: If f and g are invertible functions, then

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

Proof: We need to show that $(g \circ f) \circ (f^{-1} \circ g^{-1}) = I$
Indeed

$$\begin{aligned} (g \circ f) \circ (f^{-1} \circ g^{-1}) &= g \circ (f \circ f^{-1}) \circ g^{-1} \\ &= g \circ I \circ g^{-1} = g \circ (I \circ g^{-1}) = g \circ g^{-1} = I \ \square \end{aligned}$$

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F.Even and odd functions

Let f : be $D \to \mathbb{R}$ be a function with $x \in D \Rightarrow -x \in D$.Then
f is called

1)an even function if $f(-x) = f(x)$ for all $x \in D$,
2)an odd function if $f(-x) = -f(x)$ for all $x \in D$.

Example 3.5. for $n \in N$

1) $f(x) = x^{2n}$ is an even function.

2) $f(x) = x^{2n+1}$ is an odd function.

Solution.
1) $f(-x) = (-x)^{2n} = x^{2n} = f(x)$ for all $x \in \mathbb{R}$
2) $f(-x) = (-x)^{2n} = -x^{2n} = -f(x)$ for all $x \in \mathbb{R}$

The reader can show that the function $f(x) = x^3 - x^2$ is neither even nor odd, and that the zero function $0(x) = 0$ is both even and odd.
Why the graph of an even (odd) function is sym. w.r.to y-axis (origin)?

G.Periodic Functions

A function $f : \mathbb{R} \to \mathbb{R}$ with domain $\mathbb{R}$ is said to be periodic if there exist a number $T(\neq 0)$ such that

$f(x + T) = f(x)$ for all $x \in \mathbb{R}$

where T is called a period of x.
If T is a period, certainly, all integral multiples of T are also periods.
The smallest of all positive periods is called the fundamental period or the least period or the period of f, written T_f. As a period of constant function may be taken any real number.

3.1. ***Examples.*** **1**.sin(x),cos(x) $(T_F = (2\pi))$;

2.tan(x) , cot(x) $(T_F = (\pi))$;

3.x - [x], ($T_F = 1)$;

The graph of a perioadic function is obtained with the repetition of the graph of f in the interval of length T_F.

3.2. ***Corrolaries.*** **1**.f(x +t) = f(x) $\Rightarrow$ f(x +a + t)=f(x + a)

2. $T_{cf} = T_f$ (c$\in R$)

3. If the period of f is T_f ,then the period of f(ax + b) is T_f/a: Suppose f(ax +b) is periodic with period T' . Then

f(a(x +T') + b) = f(ax + b) holds implying f(ax + b+ aT') = f(ax + b) ⇒ aT'=T_f $=> T' = T_f/a$.

Example. Find the periods of cos(3x + 2) and tan ($\frac{x}{5}$).

Answer. $\frac{2\pi}{5}, 5\pi$

4.If the periods of f , g are T_f, T_g ,respectively,

then f +g ,f-g, fg ,f/g are periodic and positive

period T is in interval of length T such that

T/$T_f, T/T_g$ are positive integers .

Example. Find a period of cos(x) + cos(3x)

Solution. Let f(x) =cosx and g(x) = cos 3x. Then we have T$_f$ = 2π and T$_g$ = $2\pi/3$ implying that T=2π since T/T$_f$ = 1, T/T$_g$ = 3.

Example. Find a period of 2 sin(x) cos(x).

Solution. Period of sin(x),cos(x) being $2\pi, 2\pi, a$

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period is $T = 2\pi$, but this not the least period, because $2\sin x\cos x = \sin 2x$ has period $2\pi/2 = \pi$. **5.**gof is periodic if f is periodic:

$(gof)(x+T_f) = g(f(x+Tf)) = g(f(x)) = (gof)(x)$. **Inverse Trigonometric Functions**

Each of the six trigonometric functions has an inverse in an interval in which it is increasing or decreasing. For each one, a fundamental restricted interval is selected. This interval for a particular function will be the fundamental range of the inverse of that function.

Trigonometric functions,
their intervals of increase or decrease,
and chosen fundamental intervals

f	Intervals of increase or decrease of f	fundamental interval
$y = \sin x$	$[(2k-1)\frac{\pi}{2}, (2k+1)\frac{\pi}{2}]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$v = \cos x$	$[k\pi, (k+1)\pi]$	$[0, \pi]$
$y = \tan x$	$((2k-1)\frac{\pi}{2}, (2k+1)\frac{\pi}{2})$	$(-\frac{\pi}{2}, \frac{\pi}{2})$
$y = \cot x$	$(k\pi, (k+1)\pi)$	$(0, \pi)$
$y = \csc x$	$((2k-1)\frac{\pi}{2}, (2k+1)\frac{\pi}{2})$	$(-\frac{\pi}{2}, \frac{\pi}{2})$
$y = \sec x$	$(k\pi, (k+1)\pi)$	$(0, \pi)$

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a) x b) $|x|$ c) $x^2 - 1$ d) $4 - x^2$
e) $\cos x$ f) $\sin x$ (for (e),(f), $x \in [0, 2\pi]$)

(1) Prove
   a) $(f + g) \circ h = f \circ h + g \circ h$
   b) $(f - g) \circ h = f \circ h - g \circ h$
   c) $(fg) \circ h = (f \circ h)(g \circ h)$
   d) $(f/g) \circ h = (f \circ h)/(g \circ h)$
(2) Write the intervals in which the following functions are monotone(you may use graph):
   a) $y = \frac{1}{x+3}$ b) $y = \sin x + \cos x$ c) $y = |x^2 - 4| + 4$
(3) Find the inverse of the function given in Exercise 74 choosing one proper interval.
(4) Find the inverse of the function

$$f(x) = \begin{cases} 3x - 1 \; when & x \leq -1 \\ \frac{3x}{x+2} \; when & x > -1 \end{cases}$$

(5) Find the points of intersection, if any, of the given pairs of functions:
   a) $y = \frac{x+2}{x-1}$, $y = \frac{x-2}{x+1}$
   b) $y = \frac{2x-1}{x+3}$, $y = \frac{3x+1}{2-x}$
(6) If $f(x) = \sin x$ and $g(x) = x^2 + 2$, then find
   a) $f(\frac{x}{2} + \pi)g(2x - 1)$ b)$f(3a)g(\sin a)$
(7) Find the ranges of the following functions(Hint: Solve for x!)

- $y = \frac{x^2-3x}{x+1}$
- $y = \frac{x^2}{x^2-2x-3}$

Question 3.1. Find the periods of

i. $\cos(2x + 3)$
ii. $\sin\left(\frac{x}{3} - 2\right)$
iii. $\tan\left(\frac{x}{2} + \Pi\right)$
iv. $\cot(3x - \Pi)$
v. $\cos(\Pi x - \Pi)$
vi. $\sin\left(2\Pi x - \Pi^2\right)$
vii. $\sin x \cos x$
viii. $\tan^2 x$

Question 3.2. Examine the following functions for evenness and oddness

i. $|x|$
ii. $3 - x$
iii. $x + 2x^3$
iv. $x\,|x|$
v. $|x| - x^2$
vi. -3
vii. $\sin^3 2x$
viii. $\frac{\sin 2x}{\sin 3x}$

Question 3.3. Find $f \circ g$ and $g \circ f$ if

$$f(x) = \sqrt{x + 1}$$

$$and$$
$$g(x) = \frac{x}{x^2 - 4x + 3}$$
and determine the domain of each of these composite functions.

Question 3.4. Express the area of

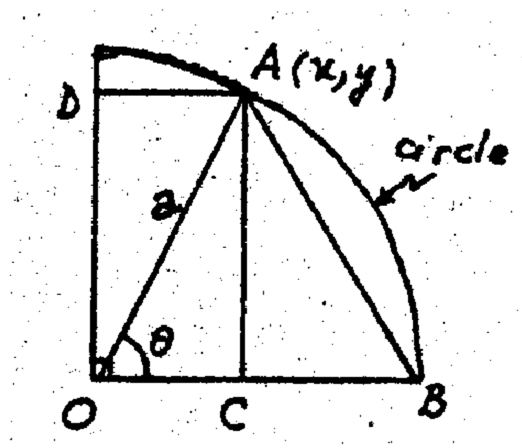

Figure 3.2. AOB Triangle and ACOD Rectangle

i. the triangle AOB in terms of Θ
ii. the triangle AOB in terms of x
iii. the rectangle ACOD in terms of Θ
iv. the rectangle ACOD in terms of x

Question 3.5. Find the domain of restriction in which the relation $| \, x + y \, | - y + 2 = 0$ is a function.

Question 3.6. Given the relation $9x^2 - 36x + 16y^2 + 96y + 36 = 0$. Write two functions equivalent to this relation.

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Answers to even numbered exercises

56.Only a). 58. Polynomials: F,H; Rational functions: f,g,F,H; Irrational function:h,i; Algebraic functions: f,g,h,F,G,H; Trans. functions: i.
60. a) R - {-1, 1}, R - {0, 1/2}.
b) [1, 2], [0, ∞) .
c)R - [2, 4], R - [2 , 16] .
62. y = 1/x , y = x .
64. a) [2kπ-π/2 , 2kπ+π/2], increasing; [2kπ+π/2 , 2kπ+3π/2], decreasing.
b)[kπ-π/2 , kπ+π/2], increasing.
c)[2kπ , (2k+1)π] increasing; [(2k+1)π , (2k + 2)π] decreasing.
d)[kπ , kπ+π/2] decreasing; [kπ+π/2 , kπ+π] increasing.
66. Missing figure

70. a) y = x + 5 b) y = -1 c) x = 3, not a function.
d)$x = y^2 - 1$, not a function e) y = cos x f) y = arcsin x
72. Missing figure
74.a) (-∞, -3) , (-3, ∞) b) [3π/4, 5π/4] , [5π/4, 7π/4]
c) (-∞, -2) , [-2, 0] , [0, 2] , [2, ∞).

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Left and right limits:

The limit of a function f at a point x_0 under the conditions

$$x < x_0 \quad , \quad 0 < |x - x_0| < \delta$$

is called the left limit of f at x_0, and the limit of f at x_0 under the conditions

$$x > x_0 \quad , \quad 0 < |x - x_0| < \delta$$

is called the right limit of f at x_0.

The notations for left limit are

$$\lim_{\substack{x\to x_0\\ x<x_0}} f(x), \qquad \lim_{x\uparrow x_0} f(x), \qquad \lim_{x\nearrow x_0} f(x), \qquad \lim_{x\to x_0^-} f(x), \qquad \lim_{x\to x_0-} f(x)$$

and those for right one are:

$$\lim_{\substack{x\to x_0\\ x>x_0}} f(x), \qquad \lim_{x\downarrow x_0} f(x), \qquad \lim_{x\searrow x_0} f(x), \qquad \lim_{x\to x_0^+} f(x), \qquad \lim_{x\to x_0+} f(x)$$

At a given point x_0 some functions have both the left and right limit, some others have only one, and still others have none.

If both the left and right limit exist at x_0 for a function f, and are equal to each other (=ℓ), then we say that f(x) has the limit ℓ, and one writes

$$\lim_{x\to x_0} f(x) = \ell$$

If f: I → R, where I is an interval with end points a

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$$(3.8) \qquad |f(x) - \ell| < \varepsilon (\varepsilon < |\ell| \, is \ taken)$$

$$(3.9) \qquad \Longrightarrow ||f(x)| - |\ell|| \leq |f(x) - \ell| < \varepsilon (From \, ||a| - |b|| \leq |a - b|)$$

$$(3.10) \qquad \Longrightarrow ||f(x)| - |\ell|| < \varepsilon$$

$$(3.11) \qquad \Longrightarrow -\varepsilon < |f(x)| - |\ell| < \varepsilon$$

$$(3.12) \qquad \Longrightarrow 0 < |\ell| - \varepsilon < |f(x)| < |\ell| + \varepsilon \qquad ...(i)$$

Now for $x \in N(x_0)$

$$\left|\frac{1}{f(x)}-\frac{1}{\ell}\right|=\frac{|f(x)-\ell|}{|f(x)|\,|\ell|}<\frac{\varepsilon}{|f(x)|\,|\ell|}<\frac{\varepsilon}{(|\ell|-\varepsilon)\,|\ell|}$$

iv. Since f is invertible we have $y=f(x)\Longleftrightarrow x=f^{-1}(y)$ so that

$$\lim_{x\to x_0} f(x)=y_0 \Longleftrightarrow \lim_{y\to y_0} f^{-1}(y)=x_0 \Longleftrightarrow \lim_{x\to y_0} f^{-1}(x)=x_0 \blacksquare$$

THEOREM 3.1. *If the functions f, g have limits at a point x_0, then*

i. $\lim_{x\to x_0}[f(x)+g(x)]=\lim_{x\to x_0} f(x)+\lim_{x\to x_0} g(x)$

ii. $\lim_{x\to x_0}[f(x)-g(x)]=\lim_{x\to x_0} f(x)-\lim_{x\to x_0} g(x)$

iii. $\lim_{x\to x_0}[f(x).g(x)]=\lim_{x\to x_0} f(x).\lim_{x\to x_0} g(x)$

iv. $\lim_{x\to x_0}[f(x):g(x)]=\lim_{x\to x_0} f(x):\lim_{x\to x_0} g(x)$ *(if* $\lim_{x\to x_0} g(x)\neq 0$*)*

PROOF. Let

$$\lim_{x\to x_0} f(x)=\alpha, \qquad \lim_{x\to x_0} g(x)=\beta$$

Then given $\varepsilon>0$, there exist deleted neighbourhoods N_1, N_2 of x_0 such that

$$x\epsilon N_1 \Rightarrow |f(x)-\alpha|<\varepsilon, \qquad x\epsilon N_2 \Rightarrow |g(x)-\beta|<\varepsilon$$

Taking $N=N_1\cap N_2$, we have

$$x\epsilon N \Rightarrow |f(x)-\alpha|<\varepsilon, \quad |g(x)-\beta|<\varepsilon$$

a)

$$\begin{aligned} x\epsilon N &\Rightarrow |f(x)+g(x)-(\alpha+\beta)|=|f(x)-\alpha+g(x)-\beta)| \\ &\leq |f(x)-\alpha|+|g(x)-\beta)|<\varepsilon+\varepsilon=2\varepsilon \end{aligned}$$

Since $\varepsilon(>0)$ is arbitrary, then $f(x)+g(x)\to\alpha+\beta$ as $x\to x_0$.

b) Similarly proved.

c)

$$\begin{aligned}
x\epsilon N \quad &\Rightarrow \quad |f(x)g(x)-\alpha\beta| \\
&= \quad |f(x)g(x)-\alpha g(x)+\alpha g(x)-\alpha\beta| \\
&= \quad |(f(x)-\alpha)g(x)+\alpha(g(x)-\beta)| \\
&\leq \quad |(f(x)-\alpha)||g(x)|+|\alpha||(g(x)-\beta)| \\
&< \quad \varepsilon|g(x)|+|\alpha|\varepsilon \\
&< \quad \varepsilon(|\beta|+\varepsilon)+|\alpha|\varepsilon \\
x\epsilon N \quad &\Rightarrow \quad |f(x)g(x)-\alpha\beta| < (|\alpha|+|\beta|+\varepsilon)\varepsilon \to 0.
\end{aligned}$$

d)

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}[f(x).\frac{1}{g(x)}] = \lim_{x\to x_0} f(x).\lim_{x\to x_0}\frac{1}{g(x)} \tag{3.13}$$

□

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$$= \alpha\cdot\frac{1}{\beta} = \frac{\alpha}{\beta} \quad \text{(Theorem 1 c)}$$

COROLLARY 3.2. *Let a composite function $g\circ f$ be given. Then* $\lim_{x\to x_0} f(x)=\alpha$ *and* $\lim_{x\to\alpha} g(x)$ *exists* $\Rightarrow \lim_{x\to x_0}(g\circ f)(x)=g(\alpha)$.

THEOREM 3.3.

1) *If $f(x)<g(x)$ holds for all x in a deleted neighbourhood $N(x_0)$ and if f, g have limits α, β at x_0, then $\alpha\leq\beta$.*
2) *If $f(x)<u(x)<g(x)$ holds for all $x\in N(x_0)$ and if f, g have the same limit ℓ at x_0, then*

$$\lim_{x\to x_0} u(x)=\ell.$$

PROOF.

1)

$$g(x)-f(x)>0 \Rightarrow \lim_{x\to x_0}[g(x)-f(x)]\geq 0 \Rightarrow \lim_{x\to x_0} g(x)-\lim_{x\to x_0} f(x)\geq 0$$
$$\Rightarrow \beta-\alpha\geq 0 \Rightarrow \alpha\leq\beta$$

2) Since f, g have limits ℓ at $x_0\in D_f\cap D_g$ then there exist $N_1(x_0)$, $N_2(x_0)$ such that

$$x\in N_1(x_0)\Rightarrow|f(x)-\ell|<\mathcal{E}, \quad x\in N_2(x_0)\Rightarrow|g(x)-\ell|<\mathcal{E}$$

implying $\ell-\mathcal{E}<f(x)<\ell+\mathcal{E}$ and $\ell-\mathcal{E}<g(x)<\ell+\mathcal{E}$. Since $f(x)<u(x)<g(x)$ we have $\ell-\mathcal{E}<u(x)<\ell+\mathcal{E}$ which implies $|u(x)-\ell|<\mathcal{E}$ or that $\lim_{x\to x_0} u(x)=\ell$.

□

Corollary 1

$$P(x)=\sum_{k=0}^{n} a_k x^k \Rightarrow \lim_{x\to x_0} P(x)=P(x_0) \tag{3.14}$$

PROOF.

$$\begin{aligned}
\lim_{x\to x_0} P(x) &= \lim_{x\to x_0}\sum_{k=0}^{n} a_k x^k \\
&= \sum_{k=0}^{n} \lim_{x\to x_0}(a_k x^k) && ...(Theorem\ 2a) \\
&= \sum_{k=0}^{n} a_k \lim_{x\to x_0} x^k && ...(Theorem\ 1b) \\
&= \sum a_k (\lim_{x\to x_0} x)^k && ...(Theorem\ 2c) \\
&= \sum a_k x_0^k && (x\to x_0) \\
&= P(x_0)
\end{aligned}$$

□

Corollary 2

If $P(x)/Q(x)$ is a rational function with $Q(x_0)\neq 0$, then;

$$\lim_{x\to x_0}\frac{P(x)}{Q(x)}=\frac{P(x_0)}{Q(x_0)} \tag{3.15}$$

PROOF.

$$\begin{aligned}
\lim_{x\to x_0}\frac{P(x)}{Q(x)} &= \frac{\lim_{x\to x_0} P(x)}{\lim_{x\to x_0} Q(x)} && \text{(Theorem 2d)} \\
&= \frac{P(x_0)}{Q(x_0)} && \text{(Coroll.1)}
\end{aligned}$$

□

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3.3. Indeterminate forms. If $\lim f(x) = 0$, $\lim g(x) = 0$ when $x \to x_0$ or $x \to \infty$, the use of property

$$\lim \frac{f(x)}{g(x)} = \frac{\lim f(x)}{\lim g(x)}$$

does not help in getting the limit of $f(x)/g(x)$, since the form $0/0$ is not defined and may be taken as equal to any number k. Indeed, the equality $0/0 = k$ is equivalent to $0 = 0 \cdot k$ and the latter holds true for any $k \in \mathbb{R}$. For this reason $0/0$ is called an *indeterminate form*. The indeterminate forms that we encounter in this chapter are

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty \cdot 0, \quad \infty - \infty$$

There are also three other which arise in considering limit of a function of the form $f(x)^{g(x)}$, and are 0^0, $1^\infty, \infty^0$. These indeterminate forms will be taken up in a later chapter where, by the use logarithms, they will be reduced to above mentioned indeterminate forms.

3.3.1. *The indeterminate form* $0/0$*:* A remarkable example is the following

$$\lim_{\Theta \to \infty} \frac{\sin \Theta}{\Theta} = \left[\frac{0}{0}\right]$$

which we state as a theorem:

THEOREM 3.4. *If* Θ *is measured in radian, then*

$$\lim_{\Theta \to \infty} \frac{\sin \Theta}{\Theta} = 1 \quad or \quad \lim_{\Theta \to \infty} \frac{\Theta}{\sin \Theta} = 1$$

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(2) $f(n) = \begin{cases} 1 & \text{when } x = -1,\ x_0 = -1 \\ x & \text{when} x \geq 0 \end{cases}$

(3) $h(x) = \dfrac{x-2}{x-2}$, $x_0 = 2$

(4) $\dfrac{1}{x-1}$, $x_0 = 1$

Solution

(1) f is not defined at $x_0 = 0$ (finite jump)
(2) $x = -1$ is an isolated point of g.
(3) h is undefined at $x_0 = 2$. h having limit (=1) at $x_0 = 2$ the discontinuity is removable.
(4) k is undefined at $x_0 = 1$ (infinite jump)

EXAMPLE 3.6. Test the function $f(x) = |x|$ for continuity at the origin.

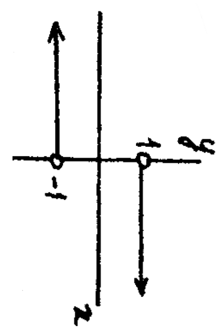


FIGURE 3.3.

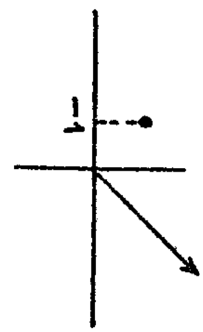


FIGURE 3.4.

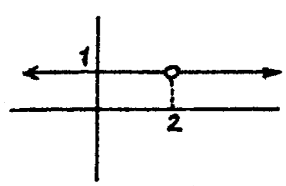


FIGURE 3.5.

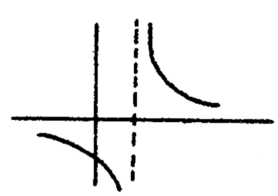


FIGURE 3.6.

Solution Since $\underset{x \to 0}{limit} |x| = 0$ and this limit is equal to $f(0)$, f is continuous at 0.

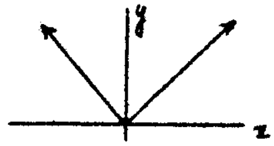


FIGURE 3.7.

EXAMPLE 3.7. Test the function $f(x) = [3x + 1]$ at $x_0 = \frac{1}{2}$

Solution $\lim_{x \to \frac{1}{2}} f(x) = 2 = f(\frac{1}{2})$

It is continuous. page=b1p1/98

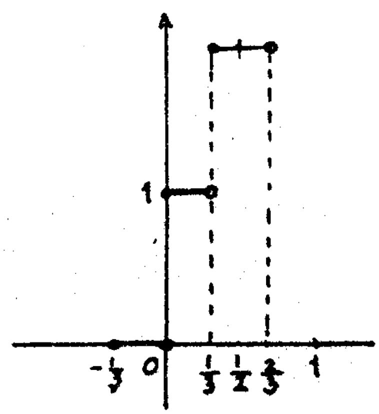
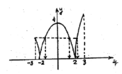


FIGURE 3.8.

a) Since f is increasing in [1,4], we have
m=f(1) = 1/4 , M = f(4) = 4/7. $\mu = \frac{1}{2} \in [\frac{1}{4}, \frac{4}{7}]$
Then $\frac{x}{x+3} = \frac{1}{2} \implies$ c = 3 ∈ [1 , 4]

b) Since g is increasing in [2 , 5] , we have
m=f(2) , M=f(5) = 23, $\nu = 14 \in$ [2,23]. Then
x^2 - 2 = 14 $\implies$ x = ± 4 , and c = 4 ∈ [2 , 5]

c) 
From the graph
m = f(-2) = f(2) = 0 ,
M = f(3) = 5 .
$\nu = \frac{9}{4} \in [0 , 5]$. Then

$$|x^2 - 4| \leq \frac{9}{4} \implies x^2 - 4 = \pm\frac{9}{4} \implies x^2 = \frac{16 \pm 9}{4} \implies$$
$$x_{1,2} = \pm\frac{5}{2}, x_{3,4} = \pm\frac{7}{2} \implies$$

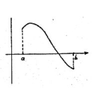
$c_1 = -5/2, \quad c_2 = -\sqrt{7}/2, \quad c_3 = \sqrt{7}/2, \quad c_4 = 5/2 \in [-5/2, 3].$

d) Since k is defined on an open interval there will be no smallest and no largest values,

$$but \quad \frac{1}{25} < k(x) < 1.$$

$\mu \in (1/25, 1) then \quad 1/x^2 = 4/9 \implies x = \pm 3/2 \implies c = 3/2 \in (1, 5).$

Corollery: If f $\in c[a, b]$ and $f(a)f(b) < 0$, then there exists at least one $c \in [a, b]$ such that $f(c) = 0$, in other words the equation $f(x) = 0$ has at least one root between a and b .



To find an approximate root of an equation $f(x) = 0$, in the first step, one determines an interval $[a, b]$ on which f is continuous and $f(a)f(b) < 0$ and the

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113. Show that the following functions are continuous for all $x \in R$:

$$a)f_{(x)} = \begin{cases} x^2, & x < -1 \\ 1, & x = -1 \\ x+2, & x > -1 \end{cases} \quad b)g_{(x)} = \sqrt{\tfrac{x^2+2x+3}{x^2+x+1}}$$

114. Show that the following functions are continuous at all x in their domain of definition:

$$a)f_{(x)} = |x^2-2|\tfrac{x}{x} \quad b)g_{(x)} = \sqrt{x^2-5x+4}$$

$$c)h_{(x)} = \sqrt[3]{x+5} \qquad d)h_{(x)} = \sqrt[4]{x^2+2}$$

115. Find the points of discontinuity and identify their types of the following functions, if any:

$$a)f_{(x)} = \tfrac{x^2+3x-10}{x-2} \qquad b)g_{(x)} = \begin{cases} x^2+3, & x < -2 \\ 5-x, & x > -2 \end{cases}$$

$$c)F_{(x)} = \begin{cases} x+4, & x < 2 \\ 7, & x = 2 \\ 2x+2, & x > 2 \end{cases} \quad d)G_{(x)} = [x] - x$$

116. Same question for:

$$a)f_{(x)} = \begin{cases} x, & x < 0 \\ 1, & x = 0 \\ \tfrac{1}{1-x}, & x > 0 \end{cases} \quad b)g_{(x)} = \tfrac{x}{sinx}$$

$$c)F_{(x)} = xcotx \qquad d)G_{(x)} = \tfrac{tanx}{arctanx}$$

117. Find the points and type if discontinuity of the following functions in the indicated intervals, if any:

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a) f(x) $= \dfrac{x^2+3}{|x-2|-1}$, $[0,2]$

b) g(x)$= \dfrac{x}{[2x]-x}$, [0,5]

c) h(x) $= \dfrac{sinx+cosx}{sinx-cosx}$, $[0,\pi/2]$

d) k(x) $= \dfrac{sinx}{arcsinx}$, $[0, \pi/2]$

118. Find the points of discontinuity and identify their types of the following domain, if any;

a) F(x) $= [sinx]$

b) A function defined by

$$x^3y^2 - 2x^2y - xy^2 + 8xy + 5x - y + 3 = 0$$

c) GoG if $G(x) = [x^2 + 1]$

d) H^{-1} if$H(x) = \dfrac{1}{x+1}$

119. Find the points of discontinuity of f + g, fg,
f/g if

$$f(x) = 2x - \frac{1}{x^2}, g(x) = x^2 + \frac{1}{x^2}$$

120. Find the points of discontinuity of fog and gof
and determine their types, if any, where

a) $f(x) = x^2 - 1, g(x) = sinx$

b) $f(x) = cosx, g(x) = \dfrac{1}{x^2+1}$

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interval if they are continuous; then find x for the given value of f(x):

121. Find m, M of the following functions in the given

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140.Find the interval defined by

a) $|x - 3| \leqslant 2$

b) $|x + 2|$¡3

c) $|x + 7|$¡9

d) $|x + 9| \leqslant 7$

141.Express the given interval as an inequality involving an absolute value:

a) $(8, 8)$

b) $[5, -7]$

c) $[-4, 7]$

d) $(-2, 5)$

142.Find the set of solution of the following equation:

a) $|x^2 - 2x| - x - 1 = 0$

b) $|x + 3| - |2x - 1| - x = 0$

143.Same question for

a) $2|x+4|-|x-2|+x=0$

b) $|x-3|-|x+1|+4=0$

144.Prove by induction:

a) x^n+y^n is divisible by $x+y$ for $n \in \mathbb{Z}_1$

b) x^n-y^n is divisible by $x-y$ for $n \in \mathbb{Z}_1$

145.Prove by induction:

$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+-\ldots+\frac{1}{2n-1}-\frac{1}{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}, n \in \mathbb{Z}_1$

146.Prove by induction:

$\left\|\frac{\sin nx}{\sin x}\right\| \leqslant n$ for $n \in \mathbb{Z}_0$,$x \neq k\pi$

147.Given the relation $x-|y|=1$

a) Sketch it

b) write its inverse

148.Same question for $|x|+|y|=1$

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149. Find the inverse of the the relation

(x-y)(3y+x)+1=0.

150.Which ones of the following relations are symmetric?

a) {(x,y) : x^2 + y^2 > 4}

b) {(x,y) : x + y < 2}

c){(x,y) : x - y > 1}

d) {(x,y) : xy + 4 = 0 }

e) {(x,y) : |x - y| < 2}

f) {(x,y) : xy^2 - 1 = 0}

151.Sketch the graph of relations:

a) ρ = {(x,y) : |y| - x + 1 > 0}

b) ρ = {(x,y) : ||x| - y| - 3 < 0}

152.Same question for:

a) {(x,y) : ⌊x -2⌋ = 3, ⌊y-3⌋ = 2 }

b) {(x,y) : ⌊x⌋ + ⌊y⌋ = 1 }

153.Sketch:

a) {(x,y) : |x| + |x-1| = 3}

b) {(x,y) : |y| - |y-1| > 3}

c) {(x,y) : |x| + |y-1| < 3}

d) {(x,y) : |y| - |x-1| > 3}

154.Sketch:

a) {(x,y) : |x| =2} b) {(x,y) : ⌊x⌋ =2}

c) {(x,y) : |x-3| =2, y=1} d) {(x,y) : ⌊x-3⌋ =2, y=1}

155.Sketch:

a) {(x,y) : |2x+3| =5, ⌊y⌋ =2}

b) {(x,y) : ⌊2x+3⌋ =5, |y| =2}

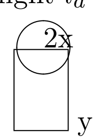
156.: Sketch the graphs of the relations;

a) $\{(x, y) : \lfloor x \rfloor \lfloor y \rfloor = 1\}$
b) $\{(x, y) : \lfloor x \rfloor \lfloor y \rfloor = -1\}$
c) $\{(x, y) : \lfloor x \rfloor \lfloor y \rfloor = 0\}$
d) $\{(x, y) : \lfloor x \rfloor \lfloor y \rfloor = 4\}$

157.: Prove

a) $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$ **b)** $(a) \Leftrightarrow 0 \leq x - \lfloor x \rfloor < 1$
c) $\lfloor x \rfloor + \lfloor x \rfloor \leq 0$ **d)** $0 \leq \lfloor x \rfloor - 2\lfloor x/2 \rfloor \leq 1$

158.: Given the Figure a window with constant area S. The glass in rectangular form permits the light half of that of semicircular form . Find the amountof light $l(x)$ passing through the window.(Glass in rectangular form permits amount of light l_a per unit area)



159.: Find the area A of an isocales triangle with equal sides a and angle between them is x; then discuss the contiunity of A as a function of x. Find m and M.

160.: Find the distance function $d(m)$ of the foot of the perpendicular from $(4, 0)$ to the line $y = mx$. Find the domain D and range of this fuctions.

161.: A variable point P on $(x - 2)^2 + y^2 = 4$ is given. Find the sum of the distance of P form the lines $y = x$ and $y = -x$ as a function of x.

162.: If $f(\sqrt{2}x + 3) = x^2 + x$, find $f(x)$.

163.: If $f(x) = \sqrt{x^2+1}$, $g(x) = x/(x^2+1)$, find

a) $(fog)(x)$ **b)** $(gof)(x)$

c) $f^{-1}(x)$ **d)** $g^{-1})(x)$

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(164) If 1/p + 1/q = 1 show that f(x) = x^{n-1} and g(x) = x^{q-1} are inverse functions.

(165) Using the data given in the Figure, compute the time t(x) for a man walking from A to B via C if the speed from A to C is 2km/hr and C to B is 3km/hr.

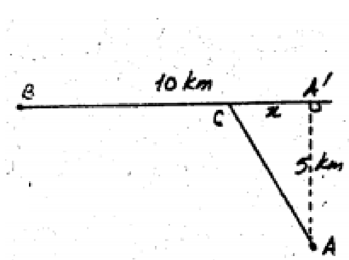

FIGURE 3.9.

(166) Let $e_1(x)$, $e_2(x)$ be two even and $o_1(x)$, $o_2(x)$ be two odd functions. What can be said about evenness or oddness of

(a) e_2 o e_1

(b) e_1 o o_1

(c) o_1 o e_1
(d) o_2 o o_1

(167) If F, G, H are three given invertible functions and f, g, h are unknown functions defined by f o $F = G$, F o $g = G$ and F o h o G $= H$ show that
(a) $f = G$ o F^{-1}
(b) $g = F^{-1}$ o G
(c) $h = F^{-1}$ o H o G^{-1}

(168) Given f(x) = $\sqrt{x+1}$, g(x) = $tan^2 x$ and h(x) = $4x^2$find the following:
(a) $(f$ o g o $h)(\frac{\sqrt{\Pi}}{4})$
(b) $(f$ o h o $g)(\frac{\Pi}{3})$
(c) $(g$ o h o $f)(3)$
(d) $(h$ o f o $g)(\frac{\Pi}{6})$

(169) Prove:

$$csc\frac{\Pi}{7} - csc\frac{2\Pi}{7} - csc\frac{3\Pi}{7} = 0$$

(170) Prove:

$$arctan\frac{1}{2} + arctan\frac{1}{5} + arctan\frac{1}{8} = \frac{\Pi}{4}$$

(171) Evaluate the following

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(158) $l(x) = \frac{1}{2}(S + \frac{\pi}{2}x^2)l_0$
(160) $d(m) = 4m/\sqrt{1+m^2}; D_d = \mathbb{R}, R_d = [0, 4]$.
(162) $f(x) = (x^4 - 12x^3 + 56x^2 - 120x + 99)/4$.
(164) (a) $\sqrt{2}$
(b) $\sqrt{37}$
(c) $tan^2 16$
(d) $16/3$
(166) (a) $\pi/2$
(b) 2π
(c) 3
(d) $\pi/3$
Hint: Transform first the given expression into linear form such as $3tan2x - sin5x$, and then find the period.
(168) (a) 0
(b) 0
(170) 5
(172) (a) 7
(b) 0
(c) No limit
(d) No limit

(174) (a) $\mathbb{R}$
(b) $\mathbb{R}$ - $\{x : x = k/2,\, k \in \mathbb{Z}\}$
(c) Yes
(d) $x = [2y - 1]$

(176) $x = -Rcos2\alpha$, $y = Rsin2\alpha$; $\alpha = 3\pi/8$

(178) $A(\alpha) = \frac{1}{2}(-sin\alpha)cos\alpha$; $m = 0$, $M = 1/2$