BOOK ONE
part one

FRESHMAN CALCULUS

B. SÜER & H. DEMİR

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Sayın
Haluk Bingöl'e
saygılarımın ve başarı
dileklerimle.

FRESHMAN CALCULUS

B. SÜER & H. DEMİR

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Yanlış Doğru Cetveli

Yanlış	Doğru
1- S.4 Theorem. ıt n is not the square of a positive integer, then $\sqrt{n}$ is irrational	1- S.4 Theorem. ıt n is a positive prime number, then $\sqrt{n}$ is irrational
2- S. 122	2- S. 123
3- S. 123	3- S. 122
4- S. 213 Exercises 125 $s = 0,05v^2 + 1,4v + 5$	4- S. 213 Exercises 125 $s = 0,05v^2 + 1,4v + 5$ (s is in feed)

F R E S H M A N   C A L C U L U S

B O O K I

Part I

Bedri SÜER   Hüseyin DEMİR

M.E.T.U.

1982

P R E F A C E

Being well aware of the existence of excellent textbooks of similar content, before adding another one to the market, we humbly feel that the nature of the challenge, which motivated us to prepare this work, needs a justifying explanation. Summing up we may state briefly the following facts:

a. This work is designed primarily for students who were - like the ones at METU - to utmost one year intensive language training in Egnlish,

b. Its content is closely related to the syllabus traditionally followed at METU and similar institutions,

c. It is a practical answer to the ever increasing demand, caused by contemporary currency fluctuations which effectively curb the availability of the textbooks edited abroad.

We sincerely believe that the topics treated in the two volumes, each containing two distinct parts, are self contained and compact. Each part and each chapter is provided with numerous exercises, including the answers corresponding to the even numbered ones.

We express our gratitude to Prof. Tunç Geveci, who kindly read the maniscript for his constructive criticism, and to our colleagues for their constant encouragements and to Miss Zehra Öner for her careful typing.

1979

Ankara

C O N T E N T S

CHAPTER I

FUNCTION, LIMIT, CONTINUITY

1.1 NUMBERS

A. Integers

Following historical development, the earliest numbers were the counting numbers $1, 2, 3, \ldots n, \ldots$. Introducing the number zero, one obtains the numbers $0, 1, 2, \ldots n, \ldots$ called the natural numbers. The natural numbers, except 0, that is, the counting numbers are all positive and are referred to as positive integers. Assigning "–" sign to these numbers one gets the negative integers, namely, $-1, -2, -3, \ldots$. A positive integer, a negative integer or zero is called an integer.

B. Rational numbers

Any number in the form of a ratio p/q of two integers $(q \neq 0)$ is called a rational number or a fraction. Any integer p is a rational number $(p = p/1)$. Thus $3/4$, $17/5$, $-11/7$, 6, -9 are rational numbers.

The decimal expansion of any rational number p/q obtained by ordinary division is either finite or else infinite. It is known from Arithmetic that an infinite expansion of a rational number contains a repeating block as given in the following examples:

$$0{,}19771977 \ldots 1977 \ldots \quad (= 0{,}\overline{1977})$$

$$-5{,}112323 \ldots 23 \ldots \quad (= -5{,}11\overline{23})$$

A finite expansion can be considered as an infinite expansion with "0" as repeating block:

$$12{,}75 \quad (= 12{,}75\overline{0})$$

Example. Find the (repeating) decimal expansion of the rational number 152/55.

Dividing 152 by 55 one gets

$$\begin{array}{r|l} 152 & 55 \\ \underline{110} & 2,76363\ldots63\ldots = 2,7\overline{63} \\ 420 & \\ \underline{385} & \\ 350 & \end{array}$$

Conversely, any decimal expansion with repeating block (cyclic expansion) represents a rational number.

Example. Express the repeating decimal expansion $3,71\overline{05}$ as a ratio of two integers.

Solution. Set

$$r = 3,71\overline{05}$$

Multiply each side by 10000 to bring "," just after the repeating block, and also multiply each side by 100 to bring "," just before the repeating block:

$$10000\, r = 37105,\overline{05}$$
$$100\, r = 371,\overline{05}$$

Subtraction gives

$$9900\, r = 37105 - 371 = 36734$$
$$r = \frac{36734}{9900}$$

Properties. If $r_1(=p_1/q_1)$, $r_2(=p_2/q_2)$ are two rational numbers, then the numbers

$$1)\quad r_1+r_2 \qquad \left(\frac{p_1}{q_1}+\frac{p_2}{q_2}=\frac{p_1q_2+p_2q_1}{q_1q_2}\right),$$

$$2)\quad r_1-r_2 \qquad \left(\frac{p_1}{q_1}-\frac{p_2}{q_2}=\frac{p_1q_2-p_2q_1}{q_1q_2}\right),$$

$$3)\quad r_1 \cdot r_2 \qquad \left(\frac{p_1}{q_1}\cdot\frac{p_2}{q_2}=\frac{p_1p_2}{q_1q_2}\right),$$

$$4)\quad r_1 : r_2 \qquad \left(\frac{p_1}{q_1}:\frac{p_2}{q_2}=\frac{p_1q_2}{q_1p_2}\right)$$

are all rational.

Corollary. Between any two distinct rational numbers there exists at least one rational number, hence infinitely many.

Proof. Let the given rational numbers be r_1 and r_2 : r_1+r_2 rational $\Rightarrow \frac{1}{2}(r_1+r_2)$ rational (why this arithmetic mean is between r_1 and r_2 ?)

This process can be continued indefinitely.

C. Irrational numbers

A number which is not rational is called an irrational number. Since any cyclic decimal expansion is a rational number, then non cyclic ones represent irrational numbers:

0,818818881888881 ... (Number of 8's increases by 1 in each step)

4,303003000300003 ...

The existence of irrational numbers may also be shown by the following theorem:

Theorem. If n is not the square of a positive integer, then $\sqrt{n}$ is irrational.

Proof. Suppose $\sqrt{n} = p/q$ where the integers p, q have no common factor (divisor) other than 1. Any fraction can be reduced into this form by simplification.

$$\sqrt{n} = p/q \quad \Rightarrow \quad q^2 n = p^2$$

Since $n \mid q^2 n$ (n divides $q^2 n$), then $n \mid pp$ implying $n \mid p$. Therefore for some integer k we have $p = kn$.

$$q^2 n = k^2 n^2 \quad \Rightarrow \quad k^2 n = q^2 \quad \Rightarrow \quad n \mid q .$$

The results $n \mid p$, $n \mid q$ show that p, q have a common factor $n (>1)$, contradicting the assumption that p, q had no common factor.

Some irrational numbers of this form are

$$\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6} \quad \text{(Why } \sqrt{4} \text{ is not irrational?)}$$

Properties. Let r be a rational and α be an irrational number.

1) $r + \alpha$ 2) $r - \alpha$ 3) $r\alpha$ 4) r/α

are all irrational.

Proof. 1) Suppose that $r + \alpha$ is equal to a rational number s: Then, $r + \alpha = s \Rightarrow \alpha = s - r \Rightarrow \alpha$ is a rational number, since $s-r$ is rational. This contradicts the hypothesis. Hence $r + \alpha$ is is irrational.

The proofs of other cases can be done similarly.

Remark. The sum, difference, product and the ratio of two irrational numbers may not be an irrational number:

$$(3+\sqrt{2})+(5-\sqrt{2}) = 8, \qquad (3+\sqrt{2})-(5+\sqrt{2}) = -2$$

$$(\tfrac{2}{3}+\sqrt{5})(\tfrac{2}{3}-\sqrt{5}) = -\tfrac{41}{9} \qquad \sqrt{18/2} = 3$$

Corollary. Between any two distinct rational numbers, there exists at least one irrational number, and hence infinitely many.

Proof. Let the given rational numbers be r_1 and r_2 ($r_1 < r_2$). $\sqrt{2}$ being irrational, for a sufficiently large positive integer m, the irrational number $\sqrt{2}/m$ is less than the difference r_2-r_1. Then $r_1 + (\sqrt{2}/m)$ is irrational and lies between r_1 and r_2.

For all integers $n > m$ the irrational numbers $r_1+(\sqrt{2}/n)$ lie between r_1 and r_2.

D. Real numbers

A rational or an irrational number is called a real number.

The four arithmetic operations (rational operations) for any two real numbers will always yield real numbers (excluding the case a/b where $b=0$)

The above definition provides a classification of real numbers as rational and irrational. Real numbers can also be classified as algebraic and non algebraic (transcentental) numbers: The roots of a polynomial equation

$$a_0x^n + \ldots + a_{n-1}x + a_n = 0$$

with rational coefficients are called algebraic numbers, and non algebraic real numbers are called transcendental numbers.

According to this definition all rational numbers are algebraic $(x - \frac{p}{q} = 0)$. Some irrational numbers which are algebraic are $\sqrt{2}$, $5 - \sqrt{3}$; for $x = \sqrt{2} \Rightarrow x^2 - 2 = 0$, and

$x = 5 - \sqrt{3} \Rightarrow (x - 5)^2 = 3 \Rightarrow x^2 - 10x + 22 = 0$. Some irrational numbers which are transcendental are the well known number π and the base e of natural logarithm.

Real number axis

A line (straight line) on which real numbers are represented in some manner is called a real number axis or shortly a number axis. In general a representation is done by choosing on the axis a fixed point O as origin corresponding to zero, a positive sense, and a unit length to locate first, integers in succession.

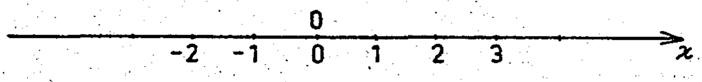
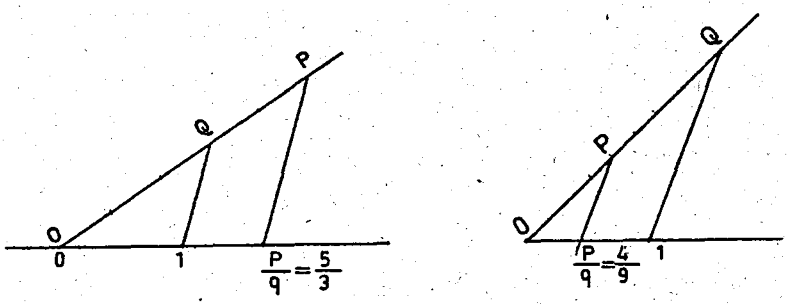


Fig. 1 Number axis

By the use of Thales Theorem, a rational number p/q can be constructed on the number axis. To find the point on the number axis corresponding to the number p/q, a ray OT (non parallel to Ox) is drawn on which line segments [OP], [OQ] of lengths p, q units are taken (Fig.2). Then Q is joined to the point represented by 1. The line passing through P and parallel to [Q1] intersects the number axis at the required point.

When $p<q<0$, the point Q is joined to the point representing -1 instead of 1.

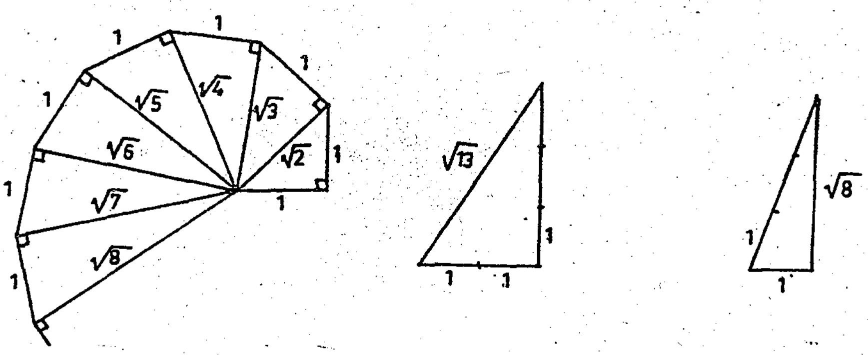


a) p/q>1 b) p/q<1

Fig. 2.
Construction of a rational number

By the use of Pythagorean Theorem, an irrational number in the form $\sqrt{n}$ where n is not the square of a positive integer, can be constructed on the number axis.

Construction of successive right triangles as shown in Fig. 3a may be used to find a line segment of length $\sqrt{n}$.



(a) By successive right triangles

(b) By a right triangle when $n=a^2+b^2$

(c) By a right triangle when $n=a^2 - b^2$

Fig. 3
Construction of $\sqrt{n}$

<u>Corollary</u>. Between two distinct irrational numbers there exists at least one irrational number, and hence infinitely many.

<u>Proof</u>. Let the given irrational numbers be $\alpha, \beta\, (\alpha < \beta)$. Then $\frac{1}{2}(\alpha+\beta)$ is either a rational number r or an irrational number γ. For the second case the corollary is established. For the first case $\frac{1}{2}(\alpha+r)$, $\frac{1}{2}(r+\beta)$ are two irrational numbers between α and β.

The existence of rational numbers between two distinct irrational numbers can be assumed.

<u>Axiom</u>. To any real number corresponds one and only one point on the number axis, and conversely to a any point on the number axis corresponds one and only one real number.

This axiom establishes a one-to-one correspondance between the points on the number axis and the real numbers.

The number x associated with a point P on the number axis is called the <u>coordinate</u> of P, and the point associated with a real number x is called the <u>graph</u> of x. The correspondance between P and x is represented by writing $P(x)$ or $P = (x)$. From now on we make no distinction between "number" and "point" on an axis so that we may say, for instance, "point 3" instead of "number 3".

<u>Square root</u>

A real number "a" distinct from zero is either greater or else smaller than zero. In the former case one writes "$a>0$" and "a" is said to be <u>positive</u>, in the other case one writes "$a<0$" and "a" is said to be <u>negative</u>. If $a \geq 0$, that is, if "a" is positive or zero, "a" is said to be <u>non-negative</u>.

A positive real number "a" has two square roots (as roots of the polynomial equation $x^2 = a$), one positive the other negative. For $a=4$, for instance, the square roots are clearly 2 and -2.

The positive square root of $a(>0)$ is denoted by $\sqrt{a}$ and the negative one by $-\sqrt{a}$. Thus,

$$\sqrt{4} = 2, \quad -\sqrt{4} = -2, \quad \sqrt{(-3)^2} = \sqrt{9} = 3$$

The number 0 which is neither positive nor negative has only one square root, namely 0, as a double root of $x^2 = 0$.

Absolute value

The absolute value of a real number "a" is a non negative real number, denoted by $|a|$ and defined by

$$|a| = \sqrt{a^2} \quad (\geqslant 0)$$

or equivalently, by

$$|a| = \begin{cases} a & \text{if } a>0 \\ 0 & \text{if } a=0 \\ -a & \text{if } a<0 \end{cases}$$

The equivalency of two definitions can be seen by considering three cases $a>0$, $a=0$, $a<0$ separately.

$$|5| = \sqrt{5^2} = 5, \qquad |-3| = \sqrt{(-3)^2} = \sqrt{9} = 3$$
$$|-2| = -(-2) = 2, \qquad |2| = 2$$

As an immediate corollary we have

Corollary

1. $|a|^2 = a^2$ 2. $-|a| \leqslant a \leqslant |a|$

Some other properties are stated in the next theorem.

Theorem. If a, b are real numbers, then

1. $|ab| = |a|\,|b|$ 2. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

3. $|a + b| \leqslant |a| + |b|$

Proof.

1. $|ab| = \sqrt{(ab)^2} = \sqrt{a^2}\sqrt{b^2} = |a|\,|b|$

2. Proved similarly.

3. $$\begin{aligned} |a+b|^2 &= (a+b)^2 \\ &= a^2 + 2ab + b^2 \\ &= |a|^2 + 2ab + |b|^2 \\ &\leqslant |a|^2 + 2|a|\,|b| + |b|^2 \\ &= (|a| + |b|)^2 \end{aligned}$$

$$|a+b|^2 \leqslant (|a| + |b|)^2$$

where $|a+b|$, $|a| + |b|$ being non negative, taking positive square roots of each side,

$$|a + b| \leqslant |a| + |b|$$

follows.

Changing b to $-b$ in the last inequality the latter is seen to include the inequality:

$$|a - b| \leqslant |a| + |b|$$

Distance

The distance between two points A and B with coordinates a, b on the number axis, denoted by

$$d(A, B) = d(a, b) = |AB|,$$

is defined as the non negative real number $|b-a|$.

Example.

1. $d(3, 5) = |5 - 3| = 2$ $\quad$ $d(5, 3) = |3 - 5| = 2$

2. $d(3, -5) = 3 + 5 = 8$ $\quad$ $d(-2, 7) = |7 + 2| = 9$

E. Complex numbers

The roots of the quadratic equation

$$ax^2 + bx + c = 0 \qquad (a \neq 0)$$

are given by

$$x_{1,\,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

They are real if and only if (iff) the discriminant $\Delta = b^2 - 4ac$ is non negative. Then for a real k if $\Delta = -k^2 < 0$, the roots become non real (imaginary) and have the form

$$x_{1,\,2} = \frac{-b \pm ki}{2a} = u + iv$$

where u and v are real numbers and $i = \sqrt{-1}$, unit imaginary number, with $i^2 = -1$.

Hence in the general case for any Δ the roots of a quadratic equation are numbers of the form

$$u + iv$$

which is called a complex number.

A complex number

$$z = a + ib$$

is real or imaginary according as $b = 0$ or $b \neq 0$. The real numbers a and b are called, respectively, the real part and imaginary part of z, written

$$a = \operatorname{Re} z, \qquad b = \operatorname{Im} z.$$

Equality. Two complex numbers are equal iff their real parts are equal and imaginary parts are equal:

$$a + ib = c + id \iff a = c, \quad b = d.$$

Hence $\quad a + ib = 0 \iff a = 0, \quad b = 0$

Conjugation. If $z = a + ib$, then the number $a - ib$ is called the complex conjugate or simply conjugate of z, written $\bar{z} = a - ib$.

From $a + ib = a - ib \iff b = 0$, it follows that a complex number is real iff it is equal to its conjugate: $z = \bar{z} \iff$ real z.

Addition and subtraction: If $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$, then their sum and difference are defined as follows:

1. $z_1 + z_2 = a_1 + a_2 + i(b_1 + b_2)$,
2. $z_1 - z_2 = a_1 - a_2 + i(b_1 - b_2)$.

One concludes that

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2, \qquad \overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$$

In words: The conjugate of a sum (difference) is the sum (difference) of conjugates.

A complex number is multiplied by a real scalar k by multiplying its real and imaginary parts by k:

$$k(a + ib) = ka + ikb$$

Example. Simplify

a) $u = (2 - 3i) - 2(4 + 2i)$

b) $v = \overline{2(3 - 2i) + 3i}$

Solution.

a) $u = 2 - 3i - 8 - 4i = 2 - 8 - (3i + 4i) = -6 - 7i$

b) $v = \overline{6 - 4i} + \overline{3i} = \overline{6 - i} = 6 + i$

Multiplication and division: The product of two complex numbers is obtained as follows:

3. $(a + ib)(c + id) = ac + iad + ibc + i^2bd$

$= ac + i(ad + bc) - bd$ (Note that $i^2 = -1$)

$= (ac - bd) + i(ad + bc)$

Corollary. $z = a + ib \Rightarrow z\bar{z} = a^2 + b^2$

Example. Perform multiplications:

a) $u = (2 - 3i)(5 + i)$

b) $v = (2 - 3i)(2 + 3i)$

Solution.

a) $u = 10 + 2i - 15i - 3i^2 = 10 - 13i + 3 = 13 - 13i$

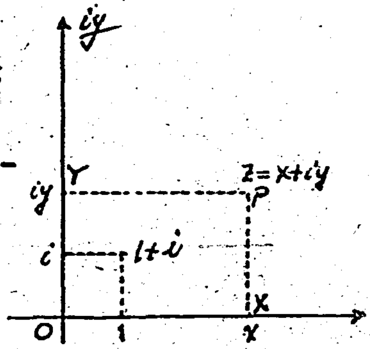
b) $v = (2 - 3i)(2 + 3i) = 2^2 + 3^2 = 4 + 9 = 13$

4. In view of above Corollary, division u/v is carried out by multiplying the numerator and denominator by the conjugate $\bar{v}$ of the denominator:

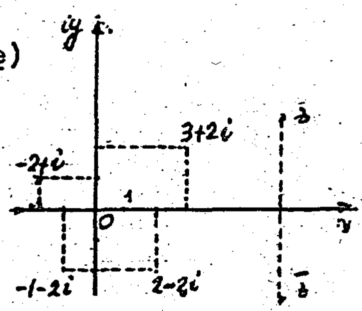
$$\frac{u}{v} = \frac{u\,\bar{v}}{v\,\bar{v}} = \frac{1}{v\,\bar{v}}\,u\bar{v}$$

Geometric Representation.

By taking two perpendicular axes with a common origin .O, and considering the horizontal axis as the real axis containing pure imaginary numbers (See fig.) any complex numbers $z = x + iy$ will be represented by a point P as the vertex of the rectangle OXPY where X is on the real axis with abscissa x, and Y is on the imaginary axis iy. The plane in which complex numbers represented this way is called complex, plane (z-plane or ARGAND plane)



On the accompanying figure, the numbers 1, i, 3 + 2i, -, 2 + i, ,1 , 2i, 2-2i are plotted.



The conjugate numbers $z = x + iy$ and $z = x - iy$ will symmetrically placed with respect to real axis.

<u>Example.</u> $\frac{2+3i}{1-i} = \frac{2+3i}{1-i}\,\frac{1+i}{1+i} = \frac{-1+5i}{2} = -\frac{1}{2} + \frac{5}{2}i$

One may show that

$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2, \qquad \overline{z_1/z_2} = \bar{z}_1/\bar{z}_2$$

In words: The conjugate of a product (ratio) is the product (ratio) of conjugates.

<u>Theorem.</u> (The Fundamental Theorem of Algebra)
A polynomial equation with real coefficient of degree n has at least one root, real or imaginary, and hence n roots, real or imaginary, simple or repeated.

<u>Proof.</u> Omitted.

<u>Corollary.</u> If a polynomial equation with real coefficients has an imaginary root it admits its conjugate as another root.

<u>Proof.</u> The proof is an applications of conjugation: Let the equation $P(x) = a_0 + a_1 x + \dots + a_n x^n = 0$ which can be represented as

$$P(x) = \sum_{k=0}^{n} a_k x^k = 0$$

admit the imaginary number z as root. Then

$$\begin{aligned} & 0 = P(z) = \sum a_k z^k \\ \Rightarrow \quad & 0 = \overline{\sum a_k z^k} = \sum \overline{a_k z^k} = \sum \bar{a}_k \overline{z^k} \\ & \quad = \sum a_k (\bar{z})^k = P(\bar{z}) \Rightarrow P(\bar{z}) = 0. \ \blacksquare \end{aligned}$$

<u>Corollary.</u> A polynomial equation with real coefficients, of odd degree has at least one real root.

Polar form of complex numbers and related properties will be treated in Chapter.

We conclude this section by two classification of numbers:

I. Complex numbers $(a + ib)$
- Real numbers $(a+ib)$ $(b=0)$
  - Rational
    - Integers
    - Fractions
  - Irrational
- Imaginary numbers $(a+ib)$ $(b\neq 0)$
  - Pure imaginary numbers ib $(b\neq 0)$

II. Complex numbers
- Algebraic numbers
- Transcendental numbers

EXERCISES (1, 1)

1. Construct the following numbers on the number axis:

   a) $3/5$ b) $-7/3$ (use Thales Theorem)

   c) $\sqrt{8}$ d) $\sqrt{12}$ (use Pythagoreas Theorem)

2. Give examples of two irrational numbers such that their

a) sum b) difference c) product d) ratio
is a rational number

3. Let e_1, e_2 be two even and o_1, o_2 be two odd numbers. Then prove the following:

a) e_1+e_2, e_1e_2, o_1+o_2 are even numbers

b) $e_1 + o_1$, o_1o_2 are odd numbers

4. If the product of two consective

a) even numbers is 624, b) odd numbers is 1155
find them. [a) ± 24, ± 26 , b) ± 33, ± 35]

5. If the sum of three consective

a) integers is 294 b) even integer is 288

c) odd integers is 327
find them. [*Hint: Take the middle number as a variable*]

6. Prove that the square

a) of an even number is an even number

b) of an odd number is an odd number

7. Prove the irrationality of the numbers

a) $\sqrt{7}$ b) $3 + \sqrt{2}$

8. Find the value of $|2x + 15|$ for

a) $x = -9$ b) $x = -7{,}8$ [a) 3 , b) 0,6]

9. Show the following properties of absolute value:

a) $|a|^2 = a^2$ b) $-|a| \leqslant a \leqslant |a|$

c) $|a - b| = |b - a|$ d) $|a| = 0 \Leftrightarrow a = 0$

e) $|ab| = |a|\ |b|$ f) $|a / b| = |a| / |b|$

g) $|a + b| \leqslant |a| + |b|$ h) $\big||a| - |b|\big| \leqslant |a - b|$

10. Find the distance between the given points. First express them as absolute value, and then compute.

a) 2,72 and 5,16 b) 3,86 and -7,28

c) -3,86 and 7,28 d) -1,23 and -12,35

11. $(3 + i)^3 = ?$ *Ans.* $18 + 26i$

12. $\dfrac{2 + i}{3 - 2i} = ?$ *Ans.* $(4+7i)/13$

13. Write a polynomial of least degree with real coefficients having the roots 3, $1 - 2i$. $[x^3 - 5x^2 + 11x - 15]$

14. Solve for real x and y :

$$\frac{2 - i}{3 + iy} = \frac{2x - 3iy}{2 + i}$$ *Ans.* $x = 5/6, \quad y = 0$

15. If $z = 5 + 4i$ find $z^2 - 2z + z\bar{z}$ *Ans.* $60 + 32i$

1.2 SETS

A. Definitions

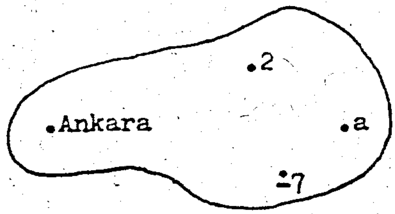
Any collection of objects (concrete or abstract) is called a set, and the objects in the set are its elements or members.

The sets are usually represented by capital letters A, B, ... Two sets formed by the same elements are said to be equal.

The set A consisting of elements, say, 2, a, Ankara, -7, is denoted either by listing the elements within two braces, or by a diagram (Venn diagram) in which the elements are marked arbitrarily in the plane and enclosed by a closed curve:

$$A = \{2, a, \text{Ankara}, -7\}$$

$$A = \{\text{Ankara}, 2, -7, a\}$$



The symbol $\in$ is used to mean "is an element of" or "belongs to", and $\notin$ is used otherwise. Then

$$2 \in A, \quad \text{Ankara} \in A, \quad 7 \notin A, \quad \text{Anka} \notin A$$

A set having finitely many elements is said to be a <u>finite set</u>, and one having infinity of distinct elements an <u>infinite set</u>. Thus $\{2, a, \text{Ankara}, -7\}$ is finite, while the set $\{1, 2, 3, \ldots, n, \ldots\}$ of natural numbers is infinite.

If S is a finite set, the number of its distinct elements is denoted by $n(S)$.

<u>Example 1.</u>

1. For the set $D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ of digits (<u>numerals</u>), $n(D) = 10$
2. For $E = \{\text{Venus, Earth, Izmir}, 3, \text{Earth}, 3, -5\}$, $n(E) = 5$

Another way of representing sets is by the use of a property common to all elements. If such a property is expressed by a true statement $p(x)$, then the symbol

$$\{x : p(x)\} \quad \text{or} \quad \{x \mid p(x)\}$$

represents the set of <u>all objects</u> having the property $p(x)$.

The meanings of the symbols $\{x: p(x)$ and $q(x)\}$ and $\{x: p(x)$ or $q(x)\}$ are clear.

Example 2. (for finite sets):

1. $D = \{n: n \text{ is a digit}\} = \{0, 1, 2, \ldots, 9\}$

2. $\{n: n \in D, \; n \text{ is prime}\} = \{2, 3, 5, 7\}$

3. $\{n: n \in D, \; 1 \leqslant n < 7\} = \{1, 2, 3, 4, 5, 6\}$

Example 3. The following infinite sets of numbers are used frequently in mathematics:

1. $\mathbb{N} = \{ n: \; n \text{ is a natural number}\}$
$= \{ 0, 1, 2, \ldots, n, \ldots \}$

2. $\mathbb{Z} = \{ n: \; n \text{ is an integer}\}$
$= \{ \ldots, -2, -1, 0, 1, 2, \ldots\}$

3. $\mathbb{Q} = \{ r: \; r \text{ is a rational number}\} = \{ \frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}$

4. $\mathbb{Q}' = \{r': r' \text{ is an irrational number}\}$

5. $\mathbb{R} = \{ x: \; x \text{ is a real number}\} = \{ x: \; x \in \mathbb{Q} \text{ or } x \in \mathbb{Q}'\}$

6. $\mathbb{C} = \{ z: \; z \text{ is a complex number}\} = \{a+ib: a, b \in \mathbb{R}, i^2 = -1\}$

A set worth of mentioning is the one having no element at all. It is called the <u>empty set</u> (<u>null set</u>) and denoted by $\emptyset$, so that $n(\emptyset) = 0$.

Example 4. Each one of the following is the null set $\emptyset$:

1. $\{x: x^2 + 1 = 0, \; x \in \mathbb{R}\}$, 2. $\{x: |x| < 0, \; x \in \mathbb{R}\}$

3. $\{x: x$ is a box, x is open and x is closed$\}$

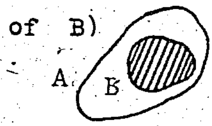
In any particular discussion, a set that contains all the objects that enter into that discussion is called the universal set. Clearly numerous universal sets exist corresponding to numerous particular discussions. A universal set is denoted by U.

If real numbers are taken into consideration, $\mathbb{R}$ is the universal set.

B. Subsets

A set A is said to be a subset of a set B, if every element of A is also an element of B, and one writes

$A \subseteq B$ (Read: A is a subset of B)



where B is said to be a superset of A

It follows that any set is a subset of itself, and we agree that the empty set is a subset of any set. Thus

$$\emptyset \subseteq \emptyset \subseteq \{1\} \subseteq \{1\} \subseteq \{1, 2, 3\} \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}.$$

If $A \subseteq B$, but $A \neq B$ one uses the notation

$A \subset B$ (Read: A is a proper subset of B)

where B contains at least one element not contained in A. With this notation the above relations can be written in the form

$$\emptyset \subseteq \emptyset \subset \{1\} \subseteq \{1\} \subset \{1, 2, 3\} \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

Example 5. Write all subsets of $\{1, 2, 3\}$.

Answer: $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1, 2, 3\}$.

If each of two sets is a subset of the other, then clearly they are equal, and vice versa:

$$A \subseteq B \quad \text{and} \quad B \subseteq A \Leftrightarrow A = B$$

This implication can be used to prove equality of sets.

Some subsets of R are in so frequent use that they bear special symbols, namely:

$$R^{+} = \{x: \ x > 0, \ x \in R\}, \qquad R^{-} = \{x: \ x < 0, \ x \in R\}$$
$$R^{*} = \{x: \ x \in R, \ x \neq 0\}$$

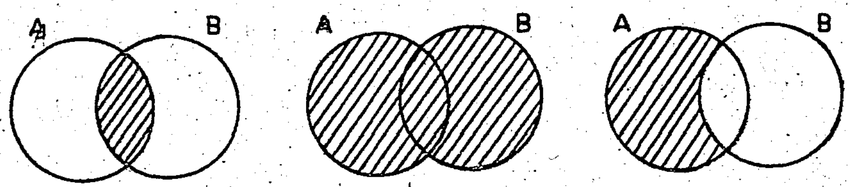
In the same way we may talk about the subsets of $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ (Why $\mathbb{N}^{-} = \emptyset$?) However some authors use $\mathbb{N}^{-}$ for $\mathbb{Z}^{-}$. In our notation, $\mathbb{N}^{-}$ is the set of all negative elements of $\mathbb{N}$, which is the empty set).

C. Operations with sets

Given two sets A and B, by means of three operations "$\cap$", "$\cup$" and "$\setminus$" we define the three sets, namely

1. $A \cap B = \{x: \ x \in A \ \text{ and } \ x \in B\}$ "A intersection B"
2. $A \cup B = \{x: \ x \in A \ \text{ or } \ x \in B\}$ "A union B"
3. $A \setminus B = \{x: \ x \in A, \ x \notin B\}$ "A minus B"

Venn diagrams of these sets are indicated by shaded sets given below:



$A \cap B$ $\qquad$ $A \cup B$ $\qquad$ $A \setminus B$

The intersection $A \cap B$ (which is also denoted by AB or $A.B$) is the set of all elements common to A and B, the <u>union</u> $A \cup B$ (which is also denoted by $A + B$) is the set of all elements belonging to A or B, or both A and B, and the <u>difference</u> $A \setminus B$ (which is also denoted by $A - B$) is the set of all elements of A that are not contained in B.

When $A \cap B = \emptyset$, then the sets A, B are said to be <u>disjoint</u>.

<u>Example 6.</u>

1. $\{1, 2, 3\} \cap \{2, 3, 4, 5\} = \{2, 3\}$
2. $\{1, 2, 3\} \cap \{4, 5\} = \emptyset$ ($\{1, 2, 3\}$, $\{4, 5\}$ are disjoint sets)
3. $\{1, 2, 3\} \cap \{1, 2, 3\} = \{1, 2, 3\}$
4. $\{1, 2, 3\} \cup \{2, 3, 4, 5\} = \{1, 2, 3, 4, 5\}$
5. $\{1, 2, 3\} \cup \{1, 2, 3\} = \{1, 2, 3\}$
6. $\{1, 2, 3\} \setminus \{1, 2, 3\} = \emptyset$
7. $\{1, 2, 3\} \setminus \{4, 5\} = \{1, 2. 3\}$
8. $\{1, 2, 3\} \setminus \{3, 4, 5\} = \{1, 2\}$

<u>Example.</u> 1. $\mathbb{R} \setminus \mathbb{Q} = \mathbb{Q}'$ 2. $\mathbb{R} \setminus \mathbb{Q}' = \mathbb{Q}$ 3. $\mathbb{Q} \cap \mathbb{Q}' = \emptyset$

<u>Corollaries.</u>

1. $A \cap B = B \cap A$ 1'. $A \cup B = B \cup A$

2. $(A \cap B) \cap C = A \cap (B \cap C)$ 2'. $(A \cup B) \cup C = A \cup (B \cup C)$

<u>Corollaries.</u>

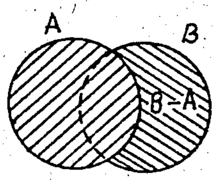
1. $A \cap A = A$ 1'. $A \cup A = A$

2. $A \cap \emptyset = \emptyset$ 2'. $A \cup \emptyset = A$

3. $A \cap U = A$ 3'. $A \cup U = U$

Examining the accompanying Venn diagram one immediately gets the relationships

$$n(A \cup B) = n(A) + n(B - A)$$
$$n(B - A) = n(B) - n(A \cap B)$$



which when added member to member give

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

D. Complement

When $A \subseteq S$, the difference $S - A$ is called the complement of A with respect to (w.r.to) the set S, and denoted by

$$C_S A \quad \text{(Read: The complement of A w.r. to S)}$$

If S is taken as a universal set U the notation for the complement of A is simply A'. The immediate corollaries are clear:

$$(A')' = A, \quad U' = \emptyset, \quad \emptyset' = U$$

Example 7. For $S = \{2, 4, 5, 6, 9\}$ and $A = \{2, 6, 9\} \subseteq S$ find the complement of A w.r. to S.

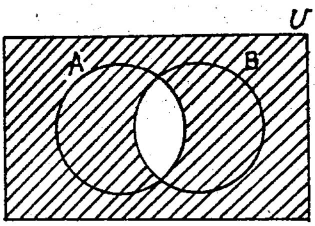
$$C_S A = S - A = \{4, 5\}$$

Example 8. $C_R Q = Q'$

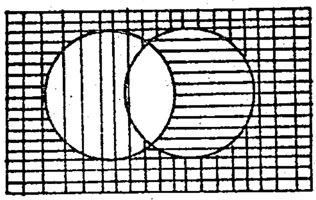
Example 9. Verify the following relations by the use of Venn diagrams

a) $(A \cap B)' = A' \cup B'$, b) $(A \cup B)' = A' \cap B'$

<u>Solution</u>. a) Taking a (rectangular) region as the universal set U and representing A and B by (circular) regions in U one obtains a Venn diagram of U, A, B on which the left hand side $(A \cap B)'$ is the shaded region, since the unshaded region represents $A \cap B$.



$(A \cap B)'$



$A' \cup B'$

In the second diagram for $U, A, B,$ the set A' is shaded by horizontal line segments, and B' by vertical ones, and their union is seen to be the shaded set in the first diagram.

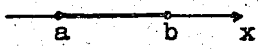
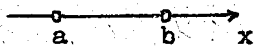
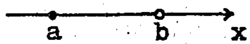
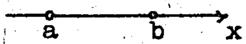
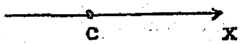
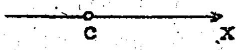
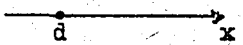
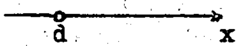
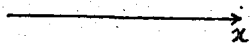
b) It can be shown in the same manner. It can also be derived from (a) setting $A = S'$, $B = T'$, The proof runs as follows:

$$\begin{aligned}(A \cup B)' &= (S' \cup T')' \\ &= ((S \cap T)')' \quad \text{(from (a))} \\ &= S \cap T \\ &= A' \cap B'. \ \blacksquare\end{aligned}$$

The two properties (a) and (b) given in Example 3 above are known as <u>De Morgan laws</u>.

E. Intervals

Some subsets of $\mathbb{R}$, denoted by $[a, b]$, (a, b), $[a, b)$, $(a, b]$ and called intervals, are of particular importance. Their definitions as well as their graphs on the number axis are given below:

1. $\{x: a \leq x \leq b\} = [a, b]$ 
2. $\{x: a < x < b\} = (a, b)$ 
3. $\{x: a \leq x < b\} = [a, b)$ 
4. $\{x: a < x \leq b\} = (a, b]$ 
5. $\{x: x \leq c\} = (-\infty, c]$ 
6. $\{x: x < c\} = (-\infty, c)$ 
7. $\{x: d \leq x\} = [d, \infty)$ 
8. $\{x: d < x\} = (d, \infty)$ 
9. $\{x: \text{all } x\} = (-\infty, \infty) = \mathbb{R}$ 

The interval $[a, b]$, representing a line segment as its graph on the x-axis and including the end points a, b is called a closed interval which reduces to a single point in case the end points coincide

The interval (a, b) is distinguished from $[a, b]$ by not containing the end points a, b and accordingly is called an open interval and reduces therefore to the empty set $\emptyset$ when $a = b$: $(a, a) = \emptyset$.

The intervals $[a, b)$ and $(a, b]$ being closed at one end open at the other are referred to as semi open or semi closed intervals.

The remaing intervals in the list are expressed by the use of symbols $-\infty$ and ∞ called respectively minus infinity and plus infinity. It is important to note that they are not numbers, but are convenient symbols for denoting points at infinity of a numbers axis.

In a closed interval $\{x: a \leqslant x \leqslant b, \quad x \in R\} = [a, b]$ the end points a, b are extreme values of the variable x. This is the reason for calling the interval closed.

In an open interval $\{x: a < x < b, \quad x \in R\} = (a, b)$, for the variable x there is no extreme value.

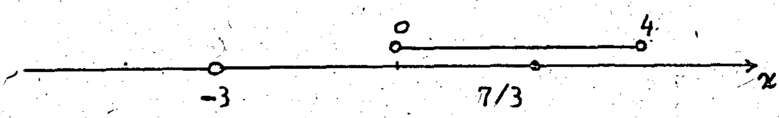
Example 10. Show that in the interval $[-2, 5)$ there is no largest number.

Suppose there is a largest number M in $[-2, 5)$. Then, since $M < \frac{1}{2}(M + 5) < 5$, the midpoint $\frac{1}{2}(M + 5)$ lies in the interval which is larger than M, contradicting that M was the largest number.

Since the intervals are number sets, operations with interval, can be performed.

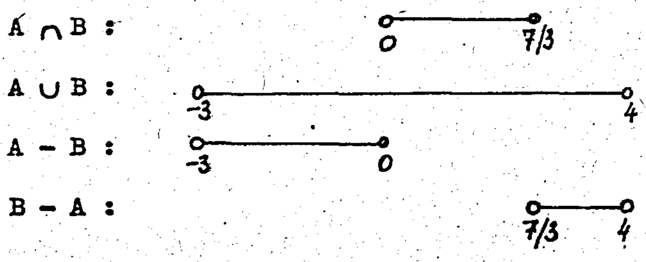
Example 11. For the intervals $A = (-3, 7/3)$, $B = (0, 4)$ sketch the graphs of the sets (a) $A \cap B$, (b) $A \cup B$, (c) $A - B$ (d) $B - A$

Solution. First, one sketches the graphs of A and B on the same number axis. If the graphs overlap (which is the case for A and B), one sketches (in practice) one of the intervals on the axis, and the other not on the axis, but just above the axis (these graphs should of course be thought as drawn on the same axis).



Then the graphs of $A \cap B$, $A \cup B$, $A - B$ and $B - A$

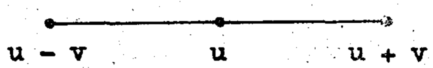
follow immediately:



Closed or open **bounded** intervals $[a, b]$, (a, b) can also be defined by a single inequality involving absolute value such as

$$\{x: |x - u| \leqslant v\} \text{ or } \{x: |x - u| < v\} \quad (v > 0)$$

of which the first one, consisting of all numbers x whose distance from the point u is less than or equal to v,



represents the closed interval with midpoint u and end points $u - v$, $u + v$, while the second represents an open interval with the same elements.

Example.

1. Find the interval defined by $|x + 5| \leqslant 3$
2. Express $(-7, 3)$ as an inequality involving an absolute value.

Solution.

1. $|x + 5| \leqslant 3 \Rightarrow -3 \leqslant x + 5 \leqslant 3 \Rightarrow -8 \leqslant x \leqslant -2 \Rightarrow [-8, -2]$.

2. $u = \frac{1}{2}(-7 + 3) = -2$, $v = \frac{3 - (-7)}{2} = 5 \Rightarrow |x + 2| < 5$.

EXERCISES (1. 2)

16. Write the following each interval as inequality with absolute value.

a) $(3, 12)$ b) $[-15, 3]$

c) $[-3 - \sqrt{2}, -3 + \sqrt{2}]$ d) (a, b)

17. Write the following as a single interval:

a) $[-2, 3] \cup (2, 5]$ b) $[-2, 3] \cap (2, 5]$

c) $[-7, 5] \cup (-8, 3)$ d) $[-7, 5] \cap (-8, 3)$

e) $|x - 3| < 5$ f) $|x + 6| \leqslant 2$

18. Find the set of solution of the following equations for $x \in \mathbb{R}$

a) $|x| - x = 0$ b) $|x + 4| - 5 = 0$

c) $2|x - 1| - |x| + 2 = 0$ d) $|x^2 - 3x - 10| - 3x + 15 = 0$

19. Write the following intervals as inequalities in absolute value:

a) $(-7, 4)$ Ans. $|x+\frac{3}{2}| < \frac{11}{2}$ b) $[-11, -2]$ Ans. $|x+\frac{13}{2}| \leqslant \frac{9}{2}$

20. Write the interval defined by

a) $|x + 3| < 8$ Ans. $(-11, 5)$ b) $|x - 2| \geqslant 0$ Ans. $\mathbb{R}$

Answers:

16. a) $|x-\frac{15}{2}| < \frac{9}{2}$, b) $|x+6| \leqslant 9$, c) $|x+3| \leqslant \sqrt{2}$

17. a) $[-2, 5]$, b) $(2, 3)$, c) $(-8, 5]$, d) $[-7, 3)$, e) $(-2, 8)$, f) $[-8, -4]$

18. a) $[0, \infty)$; b) $\{-9, 1\}$, c) ϕ; d) $\{5\}$

1.3 INDUCTION

Some theorems $p(n)$ in mathematics which involve the integer n as a variable are usually proved by a method called induction. These theorems are very often expressed by the use of some notations which we define below.

Let $a_m, \ldots, a_i, \ldots, a_n$ be any numbers with a_i as the general term where the integer " i " is called the index variable or simply the index. $(m \leq i \leq n)$

The sum $a_m + \ldots + a_i + \ldots + a_n$ where i runs from m up to n is denoted by the use of capital Greec letter Σ (sigma) as

$$\sum_{i=m}^{n} a_i = a_m + \ldots + a_n$$ (summation of a_i from m to n), Σ being

called the summation notation and the product $a_1 \ldots a_i \ldots a_n$ is represented by the use capital letter Π (pi) as

$$\prod_{i=m}^{n} a_i = a_m \ldots a_n$$ (product of a_i from m to n), Π

being called the product notation.

Example.

1. $$\sum_{i=3}^{i=6} (2i^2 + 5) = (2.3^2 + 5) + (2.4^2 + 5) + (2.5^2 + 5) + (2.6^2 + 5)$$
$$= 2(3^2 + 4^2 + 5^2 + 6^2) + 4.5 =$$
$$= 2.86 + 20 = 182$$

2. $$\prod_{i=2}^{4} (2i^2 + 5) = (2.2^2 + 5)(2.3^2 + 5)(2.4^2 + 5)$$
$$= 13.23.37$$

3. $$\prod_{i=1}^{n} i = 1 \ldots n$$

The last example gives the product of all positive integers from 1 up to n. This particular product is abreviated by the use of notation " ! " called the <u>factorial notation:</u>

$$1\ldots m = m! \quad \text{(read: } m \text{ factorial, or factorial } m\text{)}$$

Defining in addition $0!$ as 1 we have

$$0! = 1, \quad 1! = 1, \quad 2! = 1.2, \quad 3! = 1.2.3$$

$$4! = 1.2.3.4, \quad 5! = 1.2.4.5 = 4!5$$

$$(n+1)! = 1\ldots n(n+1) = n!\,(n+1)$$

Another symbol is "|" which is put between two integers or between two polynomials to mean that the left quantity divides the right one :

$$5 \mid 25, \quad 9 \mid 72, \quad -7 \mid 91, \quad x-4 \mid x^2 - 4^2$$

Some statements to be proved by induction are the following:

$$p(n):\ \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \text{ for all } n \in \mathbb{N}$$

$$q(n):\ n! > 2^n, \quad \text{for all } n \in \mathbb{N}_4$$

$$r(n):\ x - y \mid x^n - y^n, \quad \text{for all } n \in \mathbb{N}_1$$

where the sets $\mathbb{N}_1$, $\mathbb{N}_4$ or in general $\mathbb{N}_m$ means

$$\mathbb{N}_m = \{ m, m+1, m+2, \ldots \}$$

which consists of all successive integers, smallest of which is the integer $m \in \mathbb{N}$

The proof of a theorem

$$"p(n), \quad \text{for all} \quad n \in \mathbb{Z}_m = \{m, m+1, m+2, \ldots\}"; \; m \in \mathbb{Z}$$

by induction is done in four steps:

1. Verifying the truth of $p(m)$, or verifying $p(n)$ for the first integer m in $\mathbb{Z}_m$,
2. Assuming the truth of $p(k)$ for a number $k \in \mathbb{Z}_m$
3. Proving $p(k + 1)$ using (2),
4. Arguring as follows:
   $p(m)$ is true by (1). Since $p(m)$ is true, then $p(m + 1)$ must be true by (3). Since $p(m + 1)$ is true, then $p(m + 2)$ must be true again by (3). Continuing this way $p(n)$ must be true for all $n \in \mathbb{Z}_m$.

__Example__. Prove by induction:

$$p(n): \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \text{for} \quad n \in \mathbb{Z}_1$$

__Proof__. Here $\mathbb{Z}_m$ is $\mathbb{Z}_1$, since 1 is the least value taken by n.

1) $p(1)$: $\displaystyle\sum_{i=1}^{1} i^2 = \frac{1(1+1)(2+1)}{6} \Leftrightarrow 1 = 1$ (true)

(In case $p(m)$ is false the statement is disproved and hence there is no need to go further.)

2) Suppose $p(k)$ is true for some $k \in \mathbb{Z}_1$, that is, suppose

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

holds.

3) We need to prove

$$p(k+1): \quad \sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

under the hypothesis (2). Indeed,

$$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad (\text{by}(2))$$

$$= (k+1)\left[\frac{k(2k+1)}{6} + k + 1\right]$$

$$= (k+1)\,\frac{k(2k+1) + 6(k+1)}{6} =$$

$$= (k+1)\,\frac{2k^2 + 7k + 6}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

which is $p(k+1)$.

4) Then $p(n)$ is true for all $n \in Z_1$. ▪

Example 2. Prove $n! > 2^n$, for all $n \in Z_4$

Proof.

1) For $m = 4$, $4! > 2^4$ (true).

2) Suppose $k! > 2^k$ is true for k Z_4.

3) To prove $(k+1)! > 2^{k+1}$, having
$(k+1)! = k!(k+1) > 2^k(k+1)$ (by (2))

it will suffice to show

$$2^k(k + 1) > 2^{k + 1}$$

or $k + 1 > 2$ which is true since $k \in Z_4$.

4) $n! > 2^n$ is true for all $n \in Z_4$.

<u>Example 3</u>. Prove $x - y \mid x^n - y^n$, for all $n \in Z_1$.

<u>Proof</u>.

1) For $n = 1$, $x - y \mid x - y$ (true)

2) Suppose $x - y \mid x^k - y^k$ for some $k \in Z_1$.

We have supposed divisibility of $x^k - y^k$ by $x - y$, that is, the existence of a polynomial $B(x, y)$ such that

$$x^k - y^k = B(x, y).(x - y)$$

3) We prove $x - y \mid x^{k + 1} - y^{k + 1}$ using (2).

To use (2) we express $x^{k + 1} - y^{k + 1}$ in terms of $x^k - y^k$:

$$\begin{aligned} x^{k+1} - y^{k+1} &= x^{k+1} - x^k y + x^k y - y^{k+1} \\ &= x^k(x - y) + y(x^k - y^k) \\ &= x^k(x - y) + y.B(x, y)(x - y) \quad (\text{by}(2)) \\ &= [x^k + yB(x,y)]\ (x - y). \\ &= C(x, y).(x - y) \end{aligned}$$

meaning that

$$x - y \mid x^{k + 1} - y^{k + 1}$$

4) Then disivibility holds for all $n \in \mathbb{Z}_1$.

EXERCISES (1.3)

21. Evaluate

a) $\sum_{i=2}^{6} i^2$ Ans. 90 b) $\prod_{i=2}^{4} i^2$ Ans. 576

c) $\prod_{j=1}^{7} \frac{j}{i}$ Ans. $\frac{7!}{i^7}$ d) $\sum_{i=2}^{7} \frac{j^2}{i}$ Ans. $j^2 \sum_{i=2}^{7} \frac{1}{i}$

22. Write the following by the use of Σ, Π or ! .

a) $2^2 + 3^2 + 4^2 + 5^2 + 6^2$ b) $2^2 \cdot 3^2 \cdot 4^2 \cdot 5^2 \cdot 6^2$

c) $3 + 6 + 9 + 12 + 15$ d) $3 \cdot 6 \cdot 9 \cdot 12 \cdot 15$

e) $5 \cdot 10 \cdot 15 \cdot 20 \cdot 25 \cdot 30$

f) $5 + 10 + 15 + 20 + 25 + 30$

23. Write the following in the forms $(n-1)!n$ and $(n-2)!\ (n-1)n$

a) 2! b) 10! c) 32! d) 50! Ans. 49! 50, 48! 49.50

e) 12! f) 100! g) 3! h) 5!

24. The symbol $\overline{a_n \ldots a_0}$ represents a positive number with $n+1$ digits (for instance $\overline{1977} = 1977$). A mathematicians proved that the equality

$$\sum_{k=0}^{n} a_k! = \overline{a_n \ldots a_0}$$

holds only for numbers 1, 145 and 40585. Verify the equality for these numbers.

25. Simplify the following

a) $\frac{9!}{8!}$ b) $\frac{10!}{11!}$ c) $\frac{12!}{14!}$ d) $\frac{27!}{25!}$

a) [9] b) $[\frac{1}{11}]$ c) $[\frac{1}{13.14}]$ d) [26. 27],

e) $\frac{12!}{7!}\ \frac{5!}{13!}$ *Ans.* $\frac{1}{6\cdot 7\cdot 13}$ f) $\frac{12!}{72!}\ \frac{70!}{9!}$ *Ans.* $\frac{5\cdot 11}{3\cdot 71}$

Through 26 to 45 prove by induction

26. $\sum_{k=0}^{n} r^k = \frac{r^{n+1}-1}{r-1}\ (r \neq 1)$

27. $a + 1 > 0,\ a \neq 0$ $\begin{cases} \text{a)}\ (1+a)^n > 1 + na, & n \in Z_2 \\ \text{b)}\ (1+a)^n > 1 + na + \frac{1}{2}n(n-1)a^2, & n \in Z_1 \end{cases}$

28. $2^n > n^2,\ n \in Z_5$

29. a) $3 \mid 4^n - 1$ b) $8 \mid 9^n - 1,\ n \in Z_1$

30. $n! > 2^n,\ n \in Z_4$

31. a) $2n^3 - 9n^2 + 13n + 25 > 0,\ n \in Z_{-1}$

b) $2n^3 + 9n^2 + 13n + 7 > 0,\ n \in Z_{-2}$

32. $|a_1 + \dots + a_n| \leq |a_1| + \dots + |a_n|,\ n \in Z_1$

33. If a_i are all positive or all negative with $a_i \geqslant -1$, then

$$(1 + a_1) \dots (1 + a_n) \geqslant 1 + a_1 + \dots + a_n,\ n \in Z_1$$

34. $\frac{1}{1\cdot 2} + \dots + \frac{1}{(n-1)n} = 1 - \frac{1}{n},\ n \in Z_2$

35. $\sum_{k=1}^{n} (2k-1)^2 = \frac{n(4n^2-1)}{3}$

36. a) $\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$

b) $\sum_{k=1}^{n} k(k+1)(k+2) = \frac{n(n+1)(n+2)(n+3)}{4}$

37. $\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1$

38. $9 \mid n^3 + (n+1)^3 + (n+2)^3, \quad n \in Z_0$

39. $133 \mid 11^{n+2} + 12^{2n+1}, \quad n \quad Z_0$

40. $\sum_{i=1}^{n} (-1)^{i-1} \quad i^2 = (-1)^{n-1} \quad \frac{n(n+1)}{2}$

41. $a_0 = 2, \quad a_1 = 3, \quad a_{n+1} = 3a_n - 2a_{n-1} \Rightarrow a_n = 2^n + 1$

42. a) $\frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n} \quad n, \quad n \quad Z_2$

b) $\sum_{k=1}^{2n+1} (n^2 + k) = n^3 + (n+1)^3, \quad n \in Z_1$

43. a) $3 \mid n^3 - n$, b) $5 \mid n^5 - n$ c) $7 \mid n^7 - n \quad n \in Z_2$

44. $a_1 = \cos\theta, \qquad a_2 = \cos 2\theta,$

$a_n = 2a_{n-1}\cos\theta - a_{n-2} \Rightarrow a_n = \cos n\theta, \quad n \in Z_3$

45. a) $\sqrt[n]{n} < 1 + \sqrt{2/n}, \quad n \in Z_1$

b) $2^n \leqslant \frac{(2n)!}{n!\, n!} \leqslant 2^{2n}, \quad n \in Z_0$

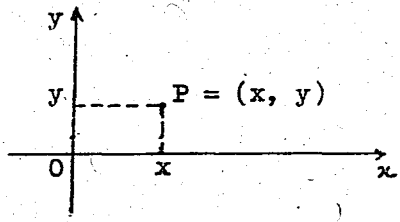
Answers:

22. a) $\sum_{2}^{6} i^2$, b) $\prod_{2}^{6} i^2$, c) $\sum_{1} 3k$, d) $\prod_{1}^{5} 3k$
e) $\prod_{1}^{6} 5n$, f) $\sum_{1}^{6} 5n$.

1.4 RELATIONS

A. Rectangular Coordinate System

Consider in the plane two perpendicular number axes Ox and Oy with common origin O, called a rectangular coordinate system or a cartesian coordinate system. A plane provided with such a system Oxy is called an analytic plane. Let P be any point in the analytic plane with coordinate x and y where x is the abscissa and y the ordinate of P. P is represented by the symbol (x, y), called an ordered pair of real numbers, and written $P = (x, y)$ or $P(x, y)$

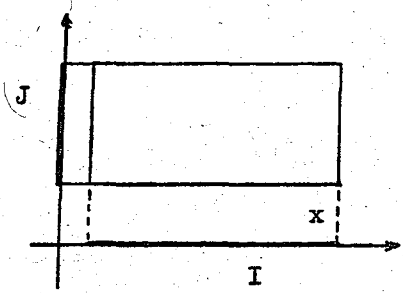


Rectangular coordinate system

If I is an interval taken on the horizontal axis Ox (x-axis), and J an interval on the vertical axis Oy (y-axis), we define the cartesian product IxJ as the set

$$\{(x, y): \; x \in I, \quad y \in J\}$$

of all ordered pairs (or points in the analytic plane) whose first coordinate x is in the first set I, and whose second coordinate y is in the second set J. The set I x J is then the shaded rectangular region in the figure.



Graph of I x J

If J = I, the cartesian product I x I is abbreviated as I^2 and accordingly the cartesian product $\mathbb{R}^2$ will mean the whole analytic plane.

B. Relations

A statement p(x, y) involving variables x, y with $x \in I$, $y \in J$ (or with $(x, y) \in I \times J$) is called an open statement which is true or false in $I \times J$, or in a subset of $I \times J$.

Examples.

1. $p(x, y)$: $x^2 + y^2 \leq 25$, $(x, y) \in \mathbb{R}^2$

   True for instance for $(-3, 4)$, $(1, -2)$ and false for instance for $(4, 6)$.

2. $q(x, y)$: $x = 5$, $(x, y) \in \mathbb{R}^2$

   True only for $x = 5$ for any y, and false in other cases.

3. $r(x, y)$: $2 = 2$, $(x, y) \in \mathbb{R}^2$

   Since $2 = 2$ is always true, $r(x, y)$ is true for all (x, y)

4. $s(x, y)$: $2 = 3$, $(x, y) \in \mathbb{R}^2$

   Since $2 = 3$ is false, $s(x, y)$ is false for all (x, y).

Let $I \subseteq \mathbb{R}$ be an interval (Read: Let I, which is a subset of $\mathbb{R}$, be an interval) on the horizontal axis, and $J \subseteq \mathbb{R}$ one taken on the vertical axis. Then by a <u>relation</u> ρ from I to J is meant the set

$$\rho = \{(x, y): \quad x \in I, \quad y \in J, \quad p(x, y)\} \qquad (1)$$

$$\rho : \quad I \to J, \quad p(x, y) \qquad (1')$$

consisting of all points in the analytic plane for which the statement $p(x, y)$ is true.

The relation (1') can be written more generally as

$$\rho : \quad \mathbb{R} \to \mathbb{R}, \quad p(x, y) \qquad (1'')$$

to mean that ρ is a relation from reals to reals.

ρ can be restated as a <u>rule</u> $p(x, y)$ by which to a number x in I there is assigned either no y or one or <u>more</u> y in J. Any y assigned with a given x is said to be an <u>image</u> of x under the relation ρ.

When (1) is plotted in the analytic plane, the resulting set of points is called the <u>graph</u> of the relation ρ. Depending upon the case, either ρ is the empty set (with no graph), or the graph is a curve or a part of the analytic plane (a region).

In connection with a relation $\rho : \mathbb{R} \to \mathbb{R}$, $p(x, y)$ from reals to reals we define the set

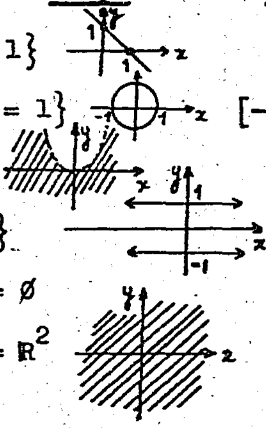
$$D_\rho = \{x: p(x, y)\},$$

consisting of all x of all pairs (x, y) in ρ as the <u>domain</u> of ρ, and the set

$$R_\rho = \{y: p(x, y)\}$$

of all images as the <u>range</u> of ρ. Certainly $D_\rho \subseteq I$, $R_\rho \subseteq J$.

<u>Examples</u>

	Graph	D_ρ	R_ρ
1. $\{(x, y): x + y = 1\}$		$\mathbb{R}$	$\mathbb{R}$
2. $\{(x, y): x^2 + y^2 = 1\}$		$[-1, 1]$	$[-1, 1]$
3. $\{(x, y): y < x^2\}$		$\mathbb{R}$	$\mathbb{R}$
4. $\{(x, y): \lvert y \rvert = 1\}$		$\mathbb{R}$	$[-1, 1]$
5. $\{(x, y): 2 = 3\} = \emptyset$		$\emptyset$	$\emptyset$
6. $\{(x, y): 2 = 2\} = \mathbb{R}^2$		$\mathbb{R}$	$\mathbb{R}$

C. Inverse of a Relation

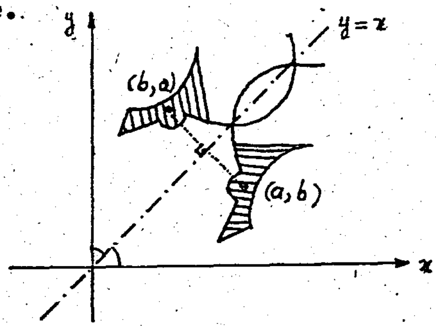
If $\rho = \{(x, y): (x, y) \in I \times J, \ p(x, y)\}$ is a relation by the rule $p(x, y)$, then the relation

$$\rho^{-1} = \{(x, y): (x, y) \in J \times I, \ p(y, x)\}$$

obtained from ρ by interchanging I and J, and also x, y in p(x, y) is called the inverse of ρ.

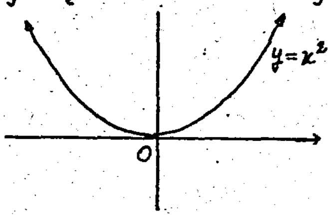
Clearly inverse of the inverse is the original relation.

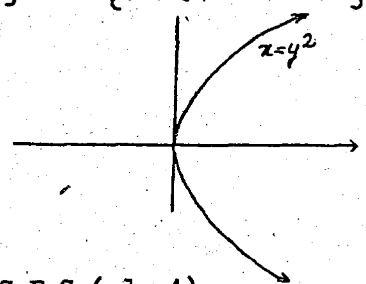
From the definition it follows that the domain (range) of ρ^{-1} is the range (domain) of ρ, and also the graphs of ρ and ρ^{-1} are symmetric with respect to the line $y = x$, since the point (a, b) in ρ is the point (b, a) in ρ^{-1} are symmetric in that line.



Example.

$\rho = \{(x, y): \ y = x^2\}$ $\qquad$ $\rho^{-1} = \{(x, y): \ x = y^2\}$





EXERCISES (1. 4)

In the following exercises if the graph of a relation is a region, represent it by shading, and in case its boundary does not belong to the graph draw it as a dotted curve.

Sketch each of the following relations from $\mathbb{R}$ to $\mathbb{R}$:

46. a) $\{(x, y): x \geqslant 3\}$ b) $\{(x, y): y = -1\}$

47. a) $\{(x, y): x > -1\}$ b) $\{(x, y): x > 2, y \leqslant -2\}$

48. a) $\{(x, y): x^2 + 4y^2 = 4\}$ b) $\{(x, y): x^2 - 4y^2 = 4\}$

49. a) $\{(x, y): x^2 + y^2 \geqslant 0\}$ b) $\{(x, y): x^2 + y^2 > 0\}$

50. a) $\{(x, y): |y| > 0\}$ b) $\{(x, y): \frac{x}{y} > 0\}$

51. a) $\{(x, y): x + y - 2 \leqslant 0\}$ b) $\{(x, y): x + y - 1 > 0\}$

52. a) $\{(x, y): y - x^3 < 0\}$ b) $\{(x, y): y - |x| + 1 < 0$

53. a) $\{(x, y): x - 2y < -1, 2x + y > 3\}$

b) $\{(x, y): 2x + y > 3, x - 2y < -1, y < 3\}$

c) $\{(x, y): 2x + y > 3, x - 2y < -1, y < 3, x + y \leqslant 5\}$

54. Find the inverse of each relation

a) $\{(x, y): x = -2\}$ b) $\{(x, y): x - 2y + 3 < 0\}$

c) $x \rightarrow y = \sqrt{9 - x^2}$ d) $x \rightarrow y = |9 - x^2|$

55. Sketch the graphs of the relations for $(x, y) \in \mathbb{R}^2$:

a) $\{(x, y): |x| = 1, |y| = 2\}$

b) $\{(x, y): |x - y| > x - y\}$

c) $\{(x, y): |x| + y \leqslant x + |y|$

d) $\{(x, y): |x - y|\,|x + y| = 0$

1.5 FUNCTIONS

A. Definitions

By a function from an interval I to an interval J is meant a relation $\varsigma : I \to J$ such that

1) I is the domain, J is the range, and

2) Every number x in the domain I have a single image in J.

If $f: I \to J$ is a function, the image of a number x of I, written $f(x)$, is said to be the value of the function f at x, or one says that x is mapped to $f(x)$ under f.

The notation for a function f will be

$$f: I \to J, \quad y = f(x); \qquad f: I \to J, \quad x \to f(x)$$

$$\{(x, y): x \in I, \quad y = f(x)\}$$

where the equality $y = f(x)$ is the defining rule for f.

Then $f: \mathbb{R} \to \mathbb{R}, \; y = f(x)$ is the general notation to indicate that f is from real numbers to real numbers and the domain D_f or dom f is the set on which $f(x)$ is defined and is determined by the rule $f(x)$, and the range R_f or ran f will be the subset $f(D_f)$ of $\mathbb{R}$.

For example for the function

$$f: \mathbb{R} \to \mathbb{R}, \qquad x \to \sqrt{1 - x^2}$$

the domain is

$$D_f = \{x: 1 - x^2 \geqslant 0\} = \{x: \; x^2 \leqslant 1\} = [-1, 1].$$

The graph of f is a curve, in general, lying entirely in the rectangular region $I \times J$, and since every x in the domain I has a single image, any vertical line (in the analytic plane) through the domain I intersects the graph at exactly one point.

Examples. Which ones of the following relations from $\mathbb{R} \to \mathbb{R}$ are functions and find their domains.

1. $\{(x, y): \ y = x^2\}$ 2. $\{(x, y): |y| = x\}$

3. $\{(x, y): \ x = -2\}$ 4. $\{(x, y): \ y = 5\}$

5. $\{(x, y): \ y = \frac{x}{x}\}$ 6. $\{(x, y): |y| = 1 - |x|\}$

Answers.

(1), (4) and (5) are functions, since every x in D_f has one image. For (1), $D_f = \mathbb{R}$; for (4), $D_f = \mathbb{R}$, and for (5), $D_f = R^*$.

Piecewisely defined functions:

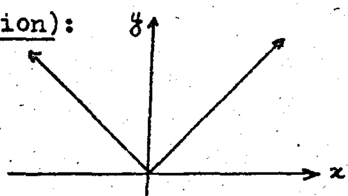
If f, g, ... are functions with pairwise disjoint domains I, J, ... respectively then the function h with domain $I \cup J \cup \ldots$ and defined by the rule

$$h(x) = \begin{cases} f(x) & \text{when } x \in I \\ g(x) & \text{when } x \in J \end{cases}$$

is called a piecewisely defined function.

Example 1. (Absolute value function):

$$x = \begin{cases} -x & \text{if } x \in R^- \\ 0 & \text{if } x = 0 \\ x & \text{if } x \in R^+ \end{cases}$$



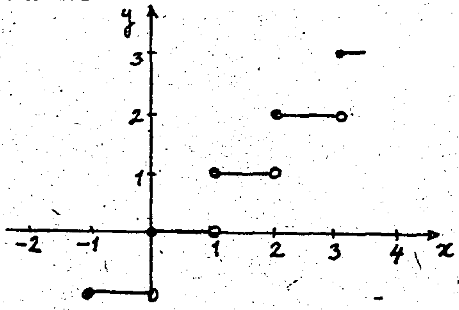
The graph of $|x|$

more generally

$$|f(x)| = \begin{cases} -f(x) & \text{if } f(x) < 0 \\ f(x) & \text{if } f(x) \geq 0 \end{cases}$$

Example 2. (Greatest integer function)

$$[\![x]\!] = \begin{cases} \vdots & \vdots & \vdots \\ -2 & \text{if} & x \in [-2, -1) \\ -1 & \text{if} & x \in [-1, 0) \\ 0 & \text{if} & x \in [0, 1) \\ 1 & \text{if} & x \in [1, 2) \\ 2 & \text{if} & x \in [2, 3) \\ \vdots & \vdots & \vdots \end{cases}$$



The graph of $[\![x]\!]$

more generally

$$[\![f(x)]\!] = n \quad \text{if} \quad n \leq f(x) < n + 1, \ \forall n \in \mathbb{Z} \ \text{(for all } n)$$

The reason for calling the function $[\![\ ]\!]$ the greatest integer function, is that the number $[\![x]\!]$ represents the greatest integer not exceeding x, in other words, if

$$u = a,\alpha_1\alpha_2 \cdots = a + 0,\alpha_1\alpha_2 \cdots$$
$$v = -b,\beta_1\beta_2 \cdots = -b - 0,\beta_1\beta_2 \cdots$$
$$= -b - 1 + 1 - 0,\beta_1\beta_2 \cdots$$
$$= -(b + 1) + 0,\gamma_1\gamma_2 \cdots$$

then

$$[\![u]\!] = a, \qquad [\![v]\!] = -(b + 1)$$

Example. Solve the equation $[\![7x + 3]\!] = 5$ for $x \in \mathbb{R}$:

Solution. $[\![7x+3]\!] = 5 \Rightarrow 5 \leqslant 7x+3 < 6$

$\Rightarrow 2 \leqslant 7x < 3 \Rightarrow 2/7 \leqslant x < 3/7$

B. TYPES OF FUNCTIONS

a. Polynomial functions:

A function

$$P: \mathbb{R} \quad \mathbb{R}, \quad P(x) = \sum_{k=0}^{n} a_k x^k = a_n x^n + \dots + a_o \quad (a_i \in \mathbb{R})$$

is called a polynomial function where the rule

$$a_n x^n + \dots + a_o$$

for P is a polynomial of degree n (if $a_n \neq 0$). The only polynomial without degree is the zero polynomial where all coefficients are zero.

The polynomials of degree 0 are constant, and $P: \mathbb{R} \to \mathbb{R}$, $P(x) = c$ is called a constant function whose graph is a horizontal line. A polynomial

$$P: \mathbb{R} \to \mathbb{R}, \quad P(x) = ax + b, \quad (a \neq 0)$$

of degree 1 is called a linear function of which the particular case

$$I: \mathbb{R} \to \mathbb{R}, \quad I(x) = x$$

is called the identity function whose graph is the line $y = x$.

b. Rational and irrational functions:

A function

$$R: \mathbb{R} \to \mathbb{R}, \quad R(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$, $Q(x)$ are polynomials with $Q(x)$ having a degree, is called a rational function.

The domain of the rational function R is the largest possible subset of $\mathbb{R}$ on which $Q(x) \neq 0$,

$$D_R = \mathbb{R} - \{x: Q(x) = 0\}$$

Clearly any polynomial function is a rational function (with $Q(x) = 1$). A function whose rule cannot be expressed as a rational function is said to be an irrational function. Simple examples of such functions are given by the rules $y = \sqrt{x}$, $y = \cos x$, $y = 2^x$.

c. Algebraic functions

A function

$$f: \mathbb{R} \rightarrow \mathbb{R}, \quad y = f(x)$$

is called an algebraic function, if y satisfies the relation

$$\sum_{k=0}^{n} A_k(x)y^k = 0 \quad \text{or} \quad A_n(x)y^n + \ldots + A_0(x) = 0$$

where the coefficients are polynomials in x. By rearranging, the rule can be written as

$$\sum_{k=0}^{m} B_k(y)x^k = 0 \quad \text{or} \quad B_m(y)x^m + \ldots + B_0(y) = 0$$

From $A(x)y - B(x) = 0$, it follows that any rational function (and hence any polynomial function) is algebraic. Among other algebraic functions we mention ones involving radicals:

$$x \to \sqrt{1 - x^2}, \quad x \to \frac{x + \sqrt{x}}{2 - \sqrt{x}}, \quad x \to \sqrt[3]{x}$$

Example. Show that the function defined by the rule

$$y = \frac{\sqrt{x} + x}{\sqrt{x} - 2} \qquad (1)$$

is an algebraic function.

Solution. We need to show that y satisfies a polynomial equation with polynomial coefficients.

$$\begin{aligned}
(1) &\Rightarrow y\sqrt{x} - 2y = \sqrt{x} + x \quad \Rightarrow \quad (y - 1)\sqrt{x} = 2y + x \\
&\Rightarrow (y - 1)^2 x = (2y + x)^2 \\
&\Rightarrow xy^2 - 2xy + x = 4y^2 + 4xy + x^2 \\
&\Rightarrow (x - 4)y^2 - 6xy + x - x^2 = 0 \qquad (2)
\end{aligned}$$

Remark. The relation (2) admits (1) as a root. Since (2) is of second degree in y, admits another root as a rule of another algebraic function.

Let the roots of (2) be y_1, y_2. Since $y_1 y_2 = \frac{x - x^2}{x - 4}$ and y_1 is given by (1), we get

$$y_2 = \frac{x - x^2}{x - 4} \, \frac{\sqrt{x} - 2}{\sqrt{x} + x} = \frac{x - x^2}{\sqrt{x} + 2} \, \frac{1}{\sqrt{x} + x} = \frac{\sqrt{x} - x}{\sqrt{x} + 2}$$

The relation (2) of degree 2 in y defines two functions (they are algebraic since coefficients are polynomials in x).

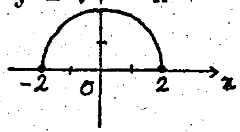
In general, a relation

$$\sum_{k=0}^{n} A_k(x) y^k = 0 \qquad (A_n(x) \neq 0)$$

of degree n in y, defines at most n algebraic functions which we call an <u>implicitely defined functions</u>.

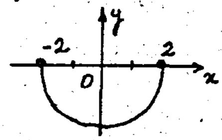
<u>Example</u>. The relation $\{(x, y): x \in R, x^2 + y^2 = 4\}$, where $x^2 + y^2 = 4$ is of second degree in y, defines two functions whose rules are obtained by solving $x^2 + y^2 = 4$ for y:

$y = \sqrt{4 - x^2}$



Graph of the function $y = \sqrt{4 - x^2}$

$y = -\sqrt{4 - x^2}$



Graph of the function $y = -\sqrt{4 - x^2}$

More generally a function defined by a relation $f(x, y) = 0$ is said to be an <u>implicitely defined function.</u> For instance $xy^2 - (x + 1)y + 1 = 0$, $y \cos y + x^3 + x = 0$ define some implicitely defined function.

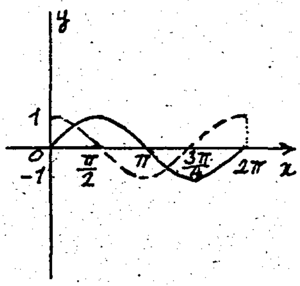
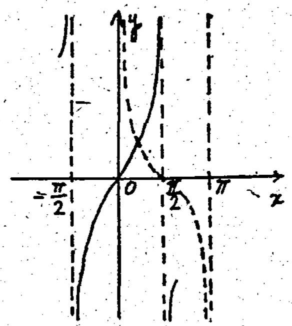
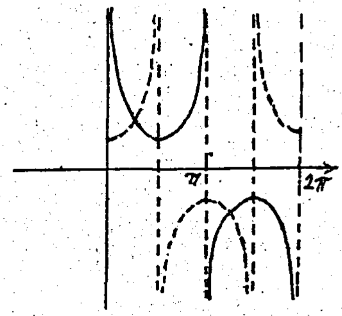
d. <u>Trigonometric Functions</u>

A function which is not algebraic is called a <u>transcendental function</u>. As some examples for transcendental functions we give trigonometric functions which we will represent simply by their rules:

<u>Rules for trig.fn.</u>	<u>Domain</u>	<u>Range</u>	<u>Period=T</u>
$y = \sin x$	$\mathbb{R}$	$[-1, 1]$	2π
$y = \cos x$	$\mathbb{R}$	$[-1, 1]$	2π
$y = \tan x$	$\mathbb{R} - \{x: x = (2k + 1)\frac{\pi}{2}, k \in Z\}$	$\mathbb{R}$	π
$y = \cot x$	$\mathbb{R} - \{x: x = k\pi, k \in Z\}$	$\mathbb{R}$	π
$y = \sec x$	$D_{\tan}$	$R-(-1, 1)$	2π

$y = \cos x$ $\qquad D_{\cot}$ $\qquad R - (-1, 1)$ $\qquad 2\pi$

their graphs are given in an interval of length T:

sin: ——— $\qquad$ tan: ——— $\qquad$ csc: ———

cos: ---- $\qquad$ cot: ---- $\qquad$ sec: -----

<u>Identities:</u>

$\cos^2 x + \sin^2 x = 1, \quad 1 + \tan^2 x = \sec^2 x, \quad 1 + \cot^2 x = \csc^2 x$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

$$\left.\begin{array}{l} \sin 2x = 2 \sin x \cos x \\ \cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x \\ \tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x} \end{array}\right\} \text{Double angle formulas}$$

$$\left.\begin{array}{l} \sin^2 \dfrac{x}{2} = \dfrac{1 - \cos x}{2} \\ \cos^2 \dfrac{x}{2} = \dfrac{1 + \cos x}{2} \end{array}\right\} \text{Half angle formulas}$$

$$\left.\begin{aligned}
\sin x + \sin y &= 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2} \\
\sin x - \sin y &= 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2} \\
\cos x + \cos y &= 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2} \\
\cos x - \cos y &= -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}
\end{aligned}\right\} \quad \text{(Factor form)}$$

C. <u>Monotonic increasing (decreasing) functions:</u>

A function $f: D \to \mathbb{R}$ is said to be an <u>increasing function</u> on an interval I which is a subset of the domain D, if

$$f(x_2) > f(x_1) \quad \text{or} \quad f(x_2) - f(x_1) > 0$$

for any two numbers x_1, $x_2 \in I$ for which $x_1 < x_2$.

The graph of such a function rises as x increases on I, and we say that f <u>increases</u> on I.

Under the same conditions for x_1, x_2 if

$$f(x_2) < f(x_1) \quad \text{or} \quad f(x_2) - f(x_1) < 0,$$

than f is called a <u>decreasing function</u> on I.

The graph of a decreasing function falls as x increases on I, and we say that f <u>decreases</u> on I.

<u>Example</u>. Show that $y = 4 - x^2$ increases on the interval $\mathbb{R}_o^-$, and decreases on $\mathbb{R}_o^+$.

<u>Solution</u>. For $x_1, x_2 \in D = \mathbb{R}$ with $x_1 < x_2$, we have

$$f(x_2) - f(x_1) = (4 - x_2^2) - (4 - x_1^2) = x_1^2 - x_2^2$$

$$= (x_1 - x_2)(x_1 + x_2)\begin{cases} > 0 & \text{when } x_1, x_2 \in \mathbb{R}_0^- \\ < 0 & \text{when } x_1, x_2 \in \mathbb{R}_0^+ \end{cases}$$

If f is an increasing (or decreasing) function on an interval $I \subseteq D$, then f is said to be a monotonic increasing (or monotonic decreasing) function in the interval I.

The function given in the above example, is monotonic increasing in R_0^- and monotonic decreasing in R_0^+.

A monotonic increasing (or decreasing) function f in an interval is expressed usually by saying that f is one-to-one (or simply $1 - 1$) in I to mean that to distinct numbers x_1, x_2 in I correspond distinct images $f(x_1), f(x_2)$.

D. Inverse of a function

A function

$$f: D \to \mathbb{R},\ y = f(x) \quad \text{or} \quad f = \{(x, y): x \in D,\ y = f(x)\} \quad (1)$$

with D as the domain and $\mathbb{R}$ as the range, being a relation from $D \to \mathbb{R}$, its inverse

$$f^{-1} = \{(x, y): \ x \in R, \ \ x = f(y)\} \qquad (2)$$

is a relation from $\mathbb{R}$ to D. If the relation f^{-1} is a function we call f^{-1} the inverse function of f, and f is said to be an invertible on the set D.

Since f is a function it maps an x in D into a image y in $\mathbb{R}$, and since f^{-1} is a function from $\mathbb{R}$ to D it maps y backward to the single image x in D. This means that f is an one-to-one function and consequently f^{-1} is one-to-one function.

The graphs of f and f^{-1} are symmetric with respect to the line $y = x$. (The pairs (x, y) of f and (y, x) of f^{-1} are symmetrical in $y = x$)

Example. Show that $f: \mathbb{R} \to \mathbb{R}$, $y = 2x - 1$ is invertible on $\mathbb{R}$. and find its inverse g.

$$f = \{(x, y):\ x \in R,\quad y = 2x - 1\}$$

$$f^{-1} = \{(x, y):\ x \in R,\quad x = 2y - 1\}$$

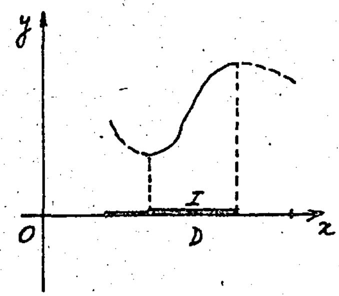
$$= \{(x, y):\ x \in R,\quad y = \frac{x+1}{2}\}$$

$$g: \mathbb{R} \to \mathbb{R},\quad g(x) = \frac{x+1}{2}$$

Corollary. If $f: D \to R$, $y = f(x)$ is monotone increasing (or decreasing) on a interval $I \subseteq D$, then f is invertible on that interval I.

Proof. It will suffice to give the proof for the case where f is monotone increasing on I.

Since f is monotone increasing it maps distinct numbers in I to distinct numbers in $\mathbb{R}$.



If the relation f^{-1} is not a function then some distinct numbers $y_1, y_2 \in \mathbb{R}$ are mapped to the same number x in D, contradicting that f is monotone on I. ∎

Let $f: \mathbb{R} \to \mathbb{R}$, be a function with a domain $D \subseteq R$. If I is a subset of D, then $f: I \to J$ is said to be a restricted function in the restricted domain I.

If there are some intervals on which a function f satisfies required conditions, then f is said to be restricted on each interval or a subset of it, and the interval itself is the largest.

Example. Find a restriction on the domain D of the function given by the rule $y = |x - 1| - 2|x| + x$ to be

a) a constant function,

b) an invertible function.

Solution. The given function is the piecewisely defined function:

$$y = \begin{cases} 1 + 2x & \text{if } x \in (-\infty, 0) \\ 1 - 2x & \text{if } x \in (0, 1] \\ -1 & \text{if } x \ (1, \infty). \end{cases}$$

a) A domain of restriction is $(1, \infty)$,

b) A domain of restriction is $(-\infty, 0]$ on which the function is increasing, or $(0, 1]$ on which it is decreasing.

E. Operation with functions:

Let

$$f\colon I \to \mathbb{R}, \quad y = f(x)$$

be a function with domain I. If $c \in \mathbb{R}$, then the function

$$cf\colon I \to \mathbb{R}, \quad y = (cf)(x) = cf(x) \qquad (0)$$

is called a scalar multiple of f.

Let now be given two functions

$$f\colon I \to \mathbb{R}, \quad y = f(x)$$
$$g\colon J \to \mathbb{R}, \quad y = g(x)$$

with non disjoint domain I and J, then $f + g$, $f - g$, fg,

f/g, called the <u>sum</u>, <u>difference</u>, <u>product</u> and <u>ratio</u> of f and g, are defined as follows:

<u>Domain</u>

$$f+g: \quad I \cap J, \quad y = (f+g)(x) = f(x) + g(x) \qquad (1)$$

$$f-g: \quad I \cap I, \quad y = (f-g)(x) = f(x) - g(x) \qquad (2)$$

$$f \quad g: \quad I \cap J, \quad y = (fg)(x) = f(x)\, g(x) \qquad (3)$$

$$f/g: \quad D, \quad y = (f/g)(x) = f(x)/g(x) \qquad (4)$$

where $D = (I \cap J) - \{x: g(x) = 0\}$.

Another function is gof, called <u>composite function</u> which is defined as

$$gof: D, \qquad y = (gof)(x) = g(f(x))$$

where the domain D is the largest possible subset of $\mathbb{R}$ on which $g(f(x))$, $f(x)$ and $g(x)$ are defined.

Because of the rule $g(f(x))$ we call also a <u>function of function</u> or a <u>chain function</u>.

<u>Example</u>. Let $f(x) = \frac{|x|}{x}$ and $g(x) = x\sqrt{1-x}$ be two functions. We have

$$D_f = R^* = R - \{0\}, \quad D_g = (-\infty, 1],$$

and

0) $(3f)(x) = 3f(x) = 3\frac{|x|}{x}$

1) $(f+g)(x) = f(x) + g(x) = \frac{|x|}{x} + x\sqrt{1-x}$

2) $(f-g)(x) = f(x) - g(x) = \frac{|x|}{x} - x\sqrt{1-x}$

3) $(fg)(x) = f(x)\, g(x) = \frac{|x|}{x}\, x\sqrt{1-x} = |x|\sqrt{1-x}$;$(x \neq 0)$

where cancellation by x is permissible under $x \neq 0$ and this condition is jointly written with the rule.

4) $(\frac{f}{g})(x) = \frac{f(x)}{g(x)} = \frac{|x|}{x^2\sqrt{1-x}}$

As to the compositions gof and fog we have

5) $(gof)(x) = g(f(x) = g(\frac{|x|}{x}) = \frac{|x|}{x}\sqrt{1-\frac{|x|}{x}}$

$$(fog)(x) = f(g(x)) = f(x\sqrt{1-x}) = \frac{|x\sqrt{1-x}|}{x\sqrt{1-x}} = \frac{|x|\sqrt{1-x}}{x\sqrt{1-x}}$$

$$= \frac{|x|}{x} \quad (x \neq 1)$$

and

$$D_{gof} = (-\infty, 1] - \{0\}, \quad D_{fog} = (-\infty, 1] - \{0, 1\} = (-\infty, 1) - \{0\}$$
$$= (-\infty, 1)^{*}$$

<u>Example</u>. Given the functions

$$f: \mathbb{R} \to \mathbb{R}, \quad f(x) = \frac{x}{x-2}; \quad g: \mathbb{R} \to \mathbb{R}, \quad g(x) = x^2 - x$$

find the rules for the composite functions gof and fog, and then determine their domains.

<u>Solution</u>.

1. $(gof)(x) = g(f(x)) = f^2(x) - f(x) = \frac{x^2}{(x-2)^2} - \frac{x}{x-2}$

$$= \frac{x^2 - x(x-2)}{(x-2)^2} = \frac{2x}{(x-2)^2}$$

2. $(fog)(x) = f(g(x)) = \frac{g(x)}{g(x)-2} = \frac{x(x-1)}{(x+1)(x-2)}$

$$D_{gof} = \mathbb{R} - \{2\}, \qquad D_{fog} = \mathbb{R} - \{-1, 2\}$$

<u>Corollary</u>. If f is an invertible function, then

$$f^{-1} of = f of^{-1} = I$$

where I is the identity function under a necessary restriction.

<u>Proof</u>. Let $f: D \to \mathbb{R}$, $y = f(x)$ with $x = f^{-1}(y)$; then

$$(f^{-1} of)(x) = f^{-1}(f(x)) = f^{-1}(y) = x = I(x),$$

$(f^{-1} of)(x) = I(x)$ for all x implying that $f^{-1} of = I$.

Also,

$$(f of^{-1})(y) = f(f^{-1}(y)) = f(x) = y = I(y) \Rightarrow fof^{-1} = I. \blacksquare$$

<u>Corollary</u>. $(hog)of = ho(gof)$.

For, $((hog)of)(x) = (hog)(f(x))$

$$= h(g(f(x))) = ho((gof)(x)) = (ho(gof))(x)$$

for all x. $\blacksquare$

<u>Corollary</u>. If f, g are invertible functions, then

$$(gof)^{-1} = f^{-1} og^{-1}$$

<u>Proof</u>. We need to show that $(gof)o(f^{-1} og^{-1}) = I$. Indeed,

$$(gof)o(f^{-1} og^{-1}) = go(fof^{-1})og^{-1}$$

$$= go\, I\, og^{-1} = go(Iog^{-1}) = gog^{-1} = I. \blacksquare$$

F. Even and odd functions

Let $f: D \to \mathbb{R}$ be a function with $x \in D \Rightarrow -x \in D$. Then f is called

1) an even function if $f(-x) = f(x)$ for all $x \in D$,

2) an odd function if $f(-x) = -f(x)$ for all $x \in D$.

Example. For $n \in N$

1) $f(x) = x^{2n}$ is an even function,

2) $g(x) = x^{2n+1}$ is an odd function.

Solution.

1) $f(-x) = (-x)^{2n} = x^{2n} = f(x)$ for all $x \in \mathbb{R}$

2) $g(-x) = (-x)^{2n+1} = -x^{2n+1} = -f(x)$ for all $x \in \mathbb{R}$

The reader can show that the function $f(x) = x^3 - x^2$ is neither even nor odd, and that the zero function $0(x) = 0$ is both even and odd.

Why the graph of an even (odd) function is sym. w.r.to y-axis (origin)?

G. Periodic Functions

A function $f: \mathbb{R} \to \mathbb{R}$ with domain $\mathbb{R}$ is said to be periodic if there exists a number $T (\neq 0)$ such that

$$f(x + T) = f(x) \quad \text{for all } x \in \mathbb{R}$$

where T is called a period of $f(x)$.

If T is a period, certainly, all integral multiples of T are also periods.

The smallest of all positive periods is called the fundamental period or the least period or the period of f, written T_f. As a period of a constant function may be taken any real number.

Examples.

1. $\sin x$, $\cos x$ $(T_f = 2\pi)$,
2. $\tan x$, $\cot x$ $(T_f = \pi)$
3. $x - [x]$ $(T_f = 1)$

The graph of a periodic function is obtained with the repetition of the graph of f in the interval of length T_f.

Corollaries.

1. $f(x + T) = f(x) \Rightarrow f(x + a + T) = f(x + a)$
2. $T_{cf} = T_f$ $(c \in \mathbb{R})$
3. If the period of $f(x)$ is T_f, then the period of $f(ax + b)$ is T_f/a: Suppose $f(ax + b)$ is periodic with period T'. Then

   $f(a(x + T') + b) = f(ax + b)$ holds implying

   $f(ax + b + aT') = f(ax + b) \Rightarrow aT' = T_f \Rightarrow T' = T_f/a$.

Example. Find the periods of $\cos(3x + 2)$ and $\tan \frac{x}{5}$.

Answer. $\frac{2\pi}{3}$, 5π

4. If the periods of f, g are T_f, T_g respectively, then $f + g$, $f - g$, fg, f/g are periodic and a positive period T is an interval of length T such that T/T_f, T/T_g are positive integers.

Example. Find a period of $\cos x + \cos 3x$

Solution. Let $f(x) = \cos x$, $g(x) = \cos 3x$. Then we have $T_f = 2\pi$, $T_g = 2\pi/3$ implying that $T = 2\pi$ since $T/T_f = 1$, $T/T_g = 3$.

Example. Find a period of $2 \sin x \cos x$.

Solution. Periods of $\sin x$, $\cos x$ being 2π, 2π, a period is $T = 2\pi$, but this not the least period, because $2 \sin x \cos x = \sin 2x$ has period $2\pi/2 = \pi$.

5. gof is periodic if f is periodic:

$$(g \circ f)(x + T_f) = g(f(x + T_f)) = g(f(x)) = (g \circ f)(x).$$

H. Inverse Trigonometric Functions

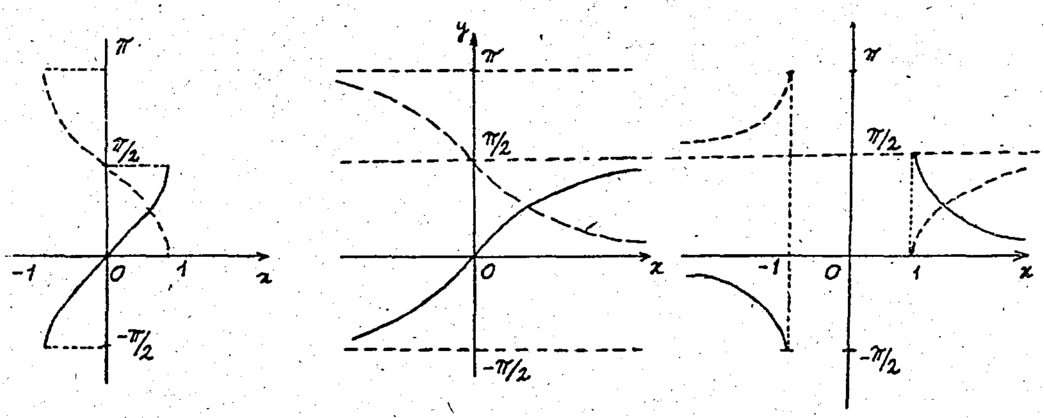
Each of the six trigonometric functions has an inverse in an interval in which it is increasing or decreasing. For each one, a fundamental restricted interval is selected. This interval for a particular function will be the fundamental range of the inverse of that function.

Trigonometric functions,
their intervals of increase or decrease,
and chosen fundamental intervals

f	Intervals of increase or decrease of f	Fundamental interval
$y = \sin x$	$\left[(2k-1)\frac{\pi}{2}, (2k+1)\frac{\pi}{2}\right]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos x$	$[k\pi, (k+1)\pi]$	$[0, \pi]$
$y = \tan x$	$\left((2k-1)\frac{\pi}{2}, (2k+1)\frac{\pi}{2}\right)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$y = \cot x$	$(k\pi, (k+1)\pi)$	$(0, \pi)$
$y = \csc x$	$\left((2k-1)\frac{\pi}{2}, (2k+1)\frac{\pi}{2}\right)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$y = \sec x$	$(k\pi, (k+1)\pi)$	$(0, \pi)$

Inverse trigonometric functions, their domains and fundamental ranges

f^{-1}	Domain	Fundamental range
$y = \arcsin x \; (=\sin^{-1}x) \Rightarrow x=\sin y$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$y = \arccos x \; (=\cos^{-1}x) \Rightarrow x=\cos y$	$[-1, 1]$	$[0, \pi]$
$y = \arctan x \; (=\tan^{-1}x) \Rightarrow x=\tan y$	$(-\infty, \infty)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$
$y = \operatorname{arccot} x \; (=\cot^{-1}x) \Rightarrow x=\cot y$	$(-\infty, \infty)$	$(0, \pi)$
$y = \operatorname{arccsc} x \; (=\csc^{-1}x) \Rightarrow x=\csc y$	$(-\infty, -1]\cup[1, \infty)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$
$y = \operatorname{arcsec} x \; (=\sec^{-1}x) \Rightarrow x=\sec y$	$(-\infty, -1]\cup[1, \infty)$	$(0, \pi)$



Arcsin: ——— Arctan: ——— Arccsc: ———

Arccos: ----- Arccot: ----- Arcsec: —·—·—

<u>Example</u>. Show that

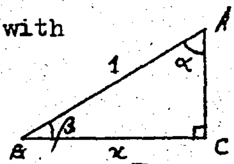
$$\arcsin x = \arccos \sqrt{1 - x^2}$$

in the common range $[0, \frac{\pi}{2}]$

<u>1 st Proof</u>. Let $u = \arcsin x$, $v = \arccos\sqrt{1-x^2}$. Then

$$x = \sin u,\ \sqrt{1-x^2} = \cos v \Rightarrow \sin^2 u + \cos^2 v =$$
$$= x^2 + (1-x^2) = 1$$
$$\Rightarrow \cos^2 u = \cos^2 v \Rightarrow u = v.$$

<u>2 nd Proof</u>. Let ABC be a right triangle with



$\alpha = \arcsin x \Rightarrow x = \sin\alpha$. Then

$$|AC| = \sqrt{1-x^2} \Rightarrow \cos\alpha = \sqrt{1-x^2} \Rightarrow \alpha = \arccos\sqrt{1-x^2}$$
$$\Rightarrow \arcsin x = \arccos\sqrt{1-x^2}$$

<u>Example</u>. Show that, for the common range,

$$\arcsin x + \arccos x = \frac{\pi}{2}$$

<u>Solution</u>. Let $\alpha = \arcsin x$, $\beta = \arccos x$

Then

$$x = \sin\alpha, \quad x = \cos\beta \Rightarrow \sin\alpha = \cos\beta$$
$$\Rightarrow \alpha + \beta = \frac{\pi}{2}$$

<u>Example</u>. Evaluate

$$\arcsin\frac{12}{13} - \operatorname{arccot}\frac{17}{7}$$

<u>Solution</u>. Let $\alpha = \arcsin\frac{12}{13}$, $\beta = \operatorname{arccot}\frac{17}{7}$ or

$$\sin\alpha = \frac{12}{13}, \quad \cot\beta = 17/7. \quad \text{Then}$$

$$\tan\alpha = \frac{12}{5}, \quad \tan\beta = \frac{7}{17}$$

$$\Rightarrow \quad \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \frac{\frac{12}{5} - \frac{7}{17}}{1 + \frac{84}{85}} = 1$$
$$\Rightarrow \quad \alpha - \beta = \frac{\pi}{4}$$

EXERCISES (1.5)

56. Which ones of the following relations are functions?

a) $\{(x, y): y = [\![x^2]\!]\}$ b) $\{(x, y): [\![y]\!] = x\}$

c) $\{(x, y): x = y^2\}$ d) $(x, y): x^2 + y^2 - 8 = 0\}$

57. Given the functions

$$f(x) = \begin{cases} |x| & \text{if } x < 1 \\ -3 & \text{if } 1 \leq x < 2\frac{1}{2} \\ [\![x]\!] & \text{if } 2\frac{1}{2} < x \end{cases}, \quad g(x) = \begin{cases} [\![2x - 3]\!] & \text{if } x < 3 \\ -\sqrt{(-x)^2} & \text{if } x = 3 \\ 2 & \text{if } x > 3 \end{cases}$$

find

a) $f(1)$, $g(1)$ b) $f(3)$, $g(3)$ c) $f(-1)$, $g(-1)$

58. Which ones of the following functions are polynomial, rational, irrational, algebraic or transcendental functions?

$f(x) = \sqrt{x}$, $g(x) = -\sqrt{3}\, x^{-1}$, $h(x) = \sqrt{1-x^2}$, $i(x) = \tan x$,

$F(x) = -3$ $G(x) = |x|$, $H(x) = 3 - x^3$

59. Show that the following are algebraic functions

a) $y = \sqrt{x + 1} + \sqrt{x - 1}$ b) $y = \sqrt[3]{x + 1} + \sqrt[3]{x - 1}$

60. Find the domain and range of

a) $y = \dfrac{x - 1}{x^2 - 1}$ b) $y = \sqrt{\dfrac{x + 1}{2 - x}}$

c) $y = \begin{cases} x & \text{if } -\infty < x < 2 \\ x^2 & \text{if } 4 \leq x < \infty \end{cases}$

61. Evaluate the value $f(x) = [\![x]\!]$ at

a) 2 b) $2.\overline{713}$ c) π d) 355/113

e) -3 f) $-3 + 0,17$ g) $-3 - 0,27$ h) $-\pi$

i) α if $4<\alpha<5$ j) β if $-5<\beta<-4$

k) $\sqrt{17}$ l) $-\sqrt{17}$

62. Find two examples of function whose graph is symetric with respect to the line $y = x$.

63. Find the intervals in which the given function is increasing or decreasing

a) $y = \frac{1}{x}$ b) $y = x^2$ c) $y = x^3+1$ d) $y = \frac{1}{x^2+1}$

64. Using their graphs find the intervals in which the given function is increasing or decreasing

a) $y = \sin x$ b) $y = \tan x$

c) $y = \sec x$ d) $y = \cos^2 x$

65. Defining $f^+ = \begin{cases} f & \text{if } f(x) \geqslant 0 \\ 0 & \text{if } f(x) < 0 \end{cases}$ and

$f^- = \begin{cases} f & \text{if } f(x) \leqslant 0 \\ 0 & \text{if } f(x) > 0 \end{cases}$ sketch the graphs of f^+

and f^- if $f(x)$ is

a) x b) $|x|$ c) $x^2 - 1$ d) $4 - x^2$

e) $\cos x$ f) $\sin x$ (for (e), (f), $x \in [0, 2\pi]$)

66. Sketch the graph of each function on the indicated interval:

a) $f(x) = |x| - x$, $\mathbb{R}$ b) $y = |x| - [\![x]\!]$, $[-2, 2]$

67. Let $f: \mathbb{R} \to \mathbb{R}$, $x \to x^3 + 3$ and

$g: \mathbb{R} \to \mathbb{R}$, $x \to -3x + 7$

Then find

a) $-2f$ b) $f + g$ c) fg d) f/g

e) fog f) gof g) fof h) gog

68. Two functions which are both even or both odd are said to be of the same parity. Show that the product of two functions

a) of the same parity is an even function,

b) of different parities is an odd function.

69. Let e_i, o_i denote even, odd functions respectively. What can be said about eveness or oddness of:

a) $e_1 + e_2$ b) $e_1 + o_1$ c) $o_1 + o_2$

d) $e_1 - e_2$ e) $e_1 - o_1$ f) $o_1 - o_2$

70. Write the inverse of each function. Which ones of them are function?

a) $\{(x, y): y = x - 5\}$ b) $\{(x, y): x = -1\}$

c) $\{(x, y): y = 3\}$ d) $\{(x, y): y = x^2 - 1\}$

e) $\{(x, y): y = \arccos x\}$ f) $\{(x, y): y = \sin x\}$

71. Write the inverse functions of

a) $y = \dfrac{x - 3}{2x + 1}$ b) $y = \dfrac{2x + 1}{x - 3}$

c) $y = \dfrac{3x - 1}{x}$ d) $y = \dfrac{x}{3x - 1}$

72. Defining

$$\operatorname{sgn} f = \begin{cases} 1 & \text{when } f(x) > 0 \\ 0 & \text{when } f(x) = 0 \\ -1 & \text{when } f(x) < 0 \end{cases} \quad \text{(read: \underline{signum} } f)$$

Sketch the graphs of sgn f if $f(x)$ is

a) x b) $|x|$ c) $x^2 - 1$ d) $4 - x^2$

e) $\cos x$ f) $\sin x$ $\left(\text{for (e), (f), } x \in [0, 2\pi]\right)$

73. Prove

a) $(f + g)\circ h = f\circ h + g\circ h$

b) $(f - g)\circ h = f\circ h - g\circ h$

c) $(fg)\circ h = (f\circ h)(g\circ h)$

d) $(f/g)\circ h = (f\circ h)/(g\circ h)$

74. Write the intervals in which the following functions are monotone (you may use graph):

a) $y = \frac{1}{x + 3}$ b) $y = \sin x + \cos x$ c) $y = |x^2 - 4| + 4$

75. Find the inverse of the function given in Exercise 74 choosing one proper interval.

76. Find the inverse of the function

$$y = \begin{cases} 3x - 1 & \text{when } x \leqslant -1 \\ \frac{3x}{x + 2} & \text{when } x > -1 \end{cases}$$

77. Find the points of intersection, if any, of the given pairs of functions:

a) $y = \frac{x + 2}{x - 1}$, $y = \frac{x - 2}{x + 1}$

b) $y = \frac{2x - 1}{x + 3}$, $y = \frac{3x + 1}{2 - x}$

78. If $f(x) = \sin x$ and $g(x) = x^2 + 2$, then find

a) $f(\frac{x}{2} + \pi)g(2x - 1)$ b) $f(3a)\, g(\sin a)$

79. Find the ranges of the following functions (Hint: Solve for x)

a) $y = \dfrac{x^2 - 3x}{x + 1}$ b) $y = \dfrac{x^2}{x^2 - 2x - 3}$

80. Find the periods of

a) $\cos(2x + 3)$ b) $\sin(\frac{x}{3} - 2)$ c) $\tan(\frac{x}{2} + \pi)$

d) $\cot(3x - \pi)$ e) $\cos(\pi x - \pi)$ f) $\sin(2\pi x - \pi^2)$

g) $\sin x \cos x$ h) $\tan^2 x$.

81. Examine the following functions for evenness and oddness:

a) $|x|$ b) $3 - x$ c) $x + 2x^3$ d) $x|x|$

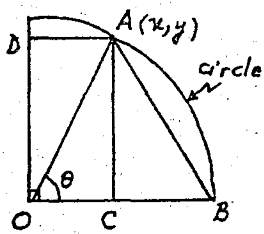
e) $|x| - x^2$ f) -3 g) $\sin^3 2x$ h) $\dfrac{\sin 2x}{\sin 3x}$

82. Find fog and gof if

$$f(x) = \sqrt{x + 1}, \quad g(x) = \frac{x}{x^2 - 4x + 3}$$

and determine the domain of each of these composite functions.

83. Express the area of



a) the triangle AOB in terms of θ

b) the triangle AOB in terms of x

c) the rectangle ACOD in terms of θ

d) the rectangle ACOD in terms of x.

84. Find a domain of restriction in which the relation $|x + y| - y + 2 = 0$ is a function.

85. Given the relation $9x^2 - 36x + 16y^2 + 96y + 36 = 0$ Write two functions equivalent to this relation.

Answers:

56. a) function, b), c), d) relations
57. a) $f(1)=-3$, $g(1)=-1$ b) $f(3)=3$, $g(3)=-3$, c) $f(-1)=1$, $g(-1)=-5$
58. f= algebraic, irrational, g= rational, h: algebraic, irrat., i: transcendental, F: polynonial, G: algebraic, H: polynonlal

1.6 LIMIT

A. Definitions

Let f be a function defined at every point x of an open interval I, expect perhaps at a point $x_0 \in I$, that is, it is not required that f be defined at x_0.

We first introduce the notations:

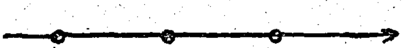
$x \to x_0$ (Read: x approaches x_0 or x tends to x_0) to mean that $|x - x_0|$ can be made arbitrarily small by taking x sufficiently close to x_0 (without being equal to x_0), or to mean that for any positive δ (which can be taken as small as we please)

$$0 < |x - x_0| < \delta$$

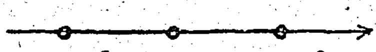
holds for all x in the set

$$(x_0 - \delta\ ;\ x_0 + \delta) - \{x_0\}$$

where the open interval $N[x_0] = (x_0 - \delta, x_0 + \delta)$ is called a neighborhood of x_0 while the set $N(x_0) = N[x_0] - \{x_0\}$ a deleted neighborhood of x_0.

$N[x_0]$: 
$x_0 - \delta$ x_0 $x_0 + \delta$

A δ-neighborhood of x_0

$N(x_0)$: 
$x_0 - \delta$ x_0 $x_0 + \delta$

A deleted δ-neighborhood of x_0

Now we are defining the limit of the function f at x_o:

The function f is said to have a <u>limit</u> ℓ at the point x_o, if $|f(x) - \ell|$ can be made arbitrarily small when $x \to x_o$, or more precisely, a function f has a limit ℓ at x_o, if corresponding to any given positive number ε, one can find a positive number $\delta(\varepsilon)$ such that

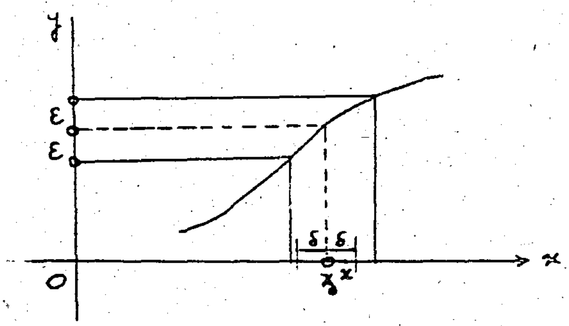
$$|f(x) - \ell| < \varepsilon$$

for all values of x for which

$$0 < |x - x_o| < \delta, \quad \text{or} \quad x \in N(x_o)$$

If this is the case, one writes

$$\lim_{x \to x_o} f(x) = \ell \quad \text{or} \quad f(x) \to \ell \quad \text{as} \quad x \to x_o.$$



It should be noted that the statement $0 < |x - x_o| < \delta$ excludes the point x_o from considerations. Moreover no restriction is imposed on to the way in which x approaches x_o from left or right.

As a first result we have the

<u>Corollary.</u> <u>A function cannot have more than one limit at a given point.</u>

<u>Proof</u>. Suppose f has the limits ℓ, ℓ' at a point x_o. We prove $\ell = \ell'$.

Since ℓ, ℓ' are limits at x_o, given $\varepsilon > 0$ there exist deleted neighborhoods N, N' of x_o such that

$$x \in N(x) \Rightarrow |f(x) - \ell| < \varepsilon,$$
$$x \in N'(x_o) \Rightarrow |f(x) - \ell'| < \varepsilon$$

Forming $|\ell - \ell'|$, we have

$$|\ell' - \ell| = |(f(x)) - \ell) - (f(x) - \ell')|$$
$$\leqslant |f(x) - \ell| + |f(x) - \ell'| < \varepsilon + \varepsilon = 2\varepsilon$$

Since ε can be taken arbitrarily small, it follows that $\ell = \ell'$.

<u>Remark</u>. To test a number ℓ to be the limit of a function f at a point x_o, it is sometimes useful to apply the equivalent definition of limit, namely, <u>a function f has the limit ℓ at x_o, when corresponding to a given $\delta > 0$ one can find a $\varepsilon(\delta) > 0$ such that</u>

$$x \in N(x_o) \Rightarrow |f(x) - \ell| < \varepsilon$$

where $\varepsilon \to 0$ as $x \to x_o$.

<u>Example 1</u>. Let I be the identity function which is defined by the rule $I(x) = x$. Then show that the limit of $I(x)$ at the point x_o is x_o.

<u>Proof</u>. We are using the above equivalent definition of limit.

For a given $\delta > 0$ with

$$0 < |x - x_o| < \delta \qquad (1)$$

we must find an $\varepsilon > 0$ corresponding to this δ with $\varepsilon \to 0$ as $\delta \to 0$, such that

$$|I(x) - x_o| < \varepsilon \qquad (2)$$

Since $I(x)$ is equal to x, in view of (1) we have

$$|I(x) - x_o| = |x - x_o| < \varepsilon$$

and taking δ as equal to ε we have (2), and $\varepsilon \to 0$ as $\delta \to 0$. ∎

Example 2. Let $f(x) = x^2 + 5$. Show that

$$\lim_{x \to 2} f(x) = 9$$

Proof. By above Remark for a given $\delta > 0$ an $\varepsilon(\delta) > 0$ is to be determined such that

$$|x - 2| < \delta \ldots (1) \Rightarrow |(x^2 + 5) - 9| < \varepsilon \ldots (2)$$

with $\varepsilon \to 0$ as $\delta \to 0$.

In view of (1), for the left hand side of (2) we have

$$|x^2 - 4| = |x - 2|\,|x + 2| < \delta\,|(x - 2) + 4| < \delta(\delta + 2).$$

Taking $\varepsilon = \delta(\delta + 2)$ we have (2), and $\varepsilon \to 0$ as $\delta \to 0$.

The verification of a number ℓ to be the limit of a given function at a given point can be done in the same way, but the limit of a function at a given point is determined in general by the use of theorems on limits (practically).

Left and right limits:

The limit of a function f at a point x_o under the conditions

$$x < x_o, \qquad 0 < |x - x_o| < \delta$$

is called the left limit of f at x_o, and the limit of f at x_o under the conditions

$$x > x_o, \qquad 0 < |x - x_o| < \delta$$

is called the right limit of f at x_o.

The notations for left limit are

$$\lim_{\substack{x \to x_o \\ x < x_o}} f(x), \quad \lim_{x \uparrow x_o} f(x), \quad \lim_{x \nearrow x_o} f(x), \quad \lim_{x \to x_o^-} f(x), \quad \lim_{x \to x_o -} f(x)$$

and those for right one are:

$$\lim_{\substack{x \to x_o \\ x > x_o}} f(x), \quad \lim_{x \downarrow x_o} f(x), \quad \lim_{x \searrow x_o} f(x), \quad \lim_{x \to x_o^+} f(x), \quad \lim_{x \to x_o} f(x)$$

At a given point x_o some functions have both the left and right limit, some others have only one, and still others have none.

If both the left and right limit exist at x_o for a function f, and are equal to each other $(= \ell)$, then we say that $f(x)$ has the limit ℓ, and one writes

$$\lim_{x \to x_o} f(x) = \ell$$

If $f: I \to \mathbb{R}$, where I is an interval with end points a and b $(a<b)$, then the limit of f at "a" is defined to be the right limit at "a", and the limit of f at "b" is defined to be the left limit of f at "b".

Example. Evaluate the following limits.

1. $\lim_{x \to -1} x|x|$ 2. $\lim_{x \to 3} [\![x^2 - 1]\!]$

Solution.

1. Since $x|x| = \begin{cases} -x^2 & \text{when } x < 0 \\ 0 & \text{when } x = 0 \\ x^2 & \text{when } x > 0 \end{cases}$

then $x|x| = -x^2$ in a deleted neighborhood N(-1) of -1, and

$$\lim_{x \to -1} x|x| = \lim_{x \to -1} (-x^2)$$

$$\left.\begin{aligned} \lim_{x \nearrow -1} (-x^2) &= \lim_{\substack{h \to 0 \\ h > 0}} -(-1-h)^2 = \lim_{h \to 0} (-1-2h-h^2) = -1 \\ \lim_{x \searrow -1} (-x^2) &= \lim_{\substack{h \to 0 \\ h > 0}} -(-1+h)^2 = \lim_{h \quad 0} (-1+2h-h^2) = -1 \end{aligned}\right\}$$

$\Rightarrow \lim_{x \to -1} x|x| = -1$ since the left and right limits exist and equal to each other.

2. $$\lim_{x \to 3} [\![x^2-1]\!] = \lim_{\substack{h \to 0 \\ h > 0}} [\![(3-h)^2 - 1]\!] = \lim_{\substack{h \to 0 \\ h > 0}} [\![8-6h + h^2]\!]$$

$$= \lim_{\substack{h \to 0 \\ h > 0}} [\![8 - h(6 - h)]\!]$$

where for sufficiently small h one has $0 < h(6-h) < 1$ implying $7 < 8 - h(6-h) < 8$ and $\lim_{x \to 3^-} [\![x^2 - 1]\!] = 7$

$$\lim_{x \to 3^+} [\![x^2 - 1]\!] = \lim_{\substack{h \to 0 \\ h > 0}} [\![(3+h)^2 - 1]\!] = \lim_{\substack{h \to 0 \\ h > 0}} [\![8 + 6h + h^2]\!] = 8$$

Since the two limits are distinct, there is no limit at 3.

<u>Remark</u>: In case $f(x)$ increases or decreases indefinitely to ∞ or to $-\infty$ when $x \to x_0^+$ or $x \to x_0^-$ or $x \to x_0$ then we say that f has no limit.

Though we say that there is no limit at such cases, it is however <u>customary</u> to write

$$\lim_{x \to x_0^-} f(x) = -\infty \qquad \text{or} \qquad \lim_{x \to x_0^-} f(x) = \infty \qquad \text{or}$$

$$\lim_{x \to x_0^+} f(x) = -\infty \qquad \text{or} \qquad \lim_{x \to x_0^+} f(x) = \infty$$

<u>Example 3</u>.

1. $\lim_{x \uparrow \pi/2} \tan x = \infty$ (no limit)
2. $\lim_{x \uparrow 1} \frac{2}{(x-1)^3} = -\infty$ (no limit)

If $f: I \to \mathbb{R}$ is a function where I is an interval involving ∞ or $-\infty$ we can talk about the limits

$$\lim_{x \to \infty} f(x) \qquad \lim_{x \to -\infty} f(x) \qquad (1)$$

Setting $x = 1/t$ and noting that

$$\lim_{x\to\infty} x = \lim_{t\to 0^+} \frac{1}{t}, \qquad \lim_{x\to -\infty} x = \lim_{t\to 0^-} \frac{1}{t}$$

the limits (1) are defined by the equalities

$$\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f(\tfrac{1}{t}), \qquad \lim_{x\to -\infty} f(x) = \lim_{t\to 0^-} f(\tfrac{1}{t}).$$

If f is a periodic function with period T, one sets $x = a + kT$ implying $\lim_{x\to\infty} f(x) = \lim_{k\to\infty} f(a + kT)$ for any $a \in R$.

Example 4.

$$\lim_{t\to\infty} \frac{\text{Sin } t}{t} = ?$$

Solution 1. Since $|\text{Sin } t| \leqslant 1$, then $\left|\frac{\text{Sin } t}{t}\right| \leqslant \frac{1}{|t|} \to 0$.

2. Setting $t = 1/x$, we have

$$\lim_{t\to\infty} \frac{\text{Sin } t}{t} = \lim_{x\to 0^+} \frac{\text{Sin } \frac{1}{x}}{\frac{1}{x}} = \lim_{x\to 0^+} x \sin \frac{1}{x} = 0$$

since $\sin(1/x)$ is bounded.

Example.

$$\lim_{x\to\infty} \frac{x^7 + 8x}{5x^7 + 3} = \lim_{x} \frac{x^7(1 + \frac{8}{x^6})}{x^7(5 + \frac{3}{x^7})} = \lim_{x} \frac{1 + \frac{8}{x^6}}{5 + \frac{3}{x^7}} = \frac{1}{5}$$

or

$$\lim_{x\to\infty} \frac{x^7 + 8x}{5x^7 + 3} = \lim_{t\quad 0^+} \frac{\frac{1}{t^7} + \frac{8}{t}}{\frac{5}{t^7} + 3} = \lim_{t\to 0^+} \frac{1 + 8t^6}{5 + 3t^7} = \frac{1}{5}$$

B. Theorems on Limits

Theorem 1.

a) If $f(x) = c$ (c is a constant), then $\lim_{x \to x_0} f(x) = c$

b) If $\lim_{x \to x_0} f(x) = \ell$, and c is a constant, then

$$\lim_{x \to x_0} c\, f(x) = c \lim_{x \to x_0} f(x)$$

c) If $\lim_{x \to x_0} f(x) = \ell$ and $\ell \neq 0$, then

$$\lim_{x \to x_0} \frac{1}{f(x)} = \frac{1}{\ell}$$

d) If $f: I \to J$ is invertible on I and if

$$\lim_{x \to x_0} f(x) = y_0, \quad \text{then} \quad \lim_{x \to y_0} f^{-1}(x) = x_0$$

Proof.

a) Since $|f(x) - c| = |c - c| = 0$, then $|f(x) - c| < \varepsilon$ for any $\varepsilon > 0$ and for any $N(x_0)$

b) Since for a given $\varepsilon > 0$, there exists $N(x_0)$ such that

$$x \in N(x_0) \Rightarrow |f(x) - \ell| < \varepsilon$$

we have for $x \in N(x_0)$

$$|c\, f(x) - c\ell| = |c|\, |f(x) - \ell| < |c|\, \varepsilon = \varepsilon'$$

c) First we obtain the inequality (a) below for f:

$$|f(x) - \ell| < \varepsilon \ (\varepsilon < |\ell| \text{ is taken})$$

$$\Rightarrow \ \big||f(x)| - |\ell|\big| \leqslant |f(x) - \ell| < \varepsilon \quad (\text{From } \big||a| - |b|\big| \leqslant |a - b|)$$

$$\Rightarrow \ \big||f(x)| - |\ell|\big| < \varepsilon$$

$$\Rightarrow \quad -\varepsilon < |f(x)| - |\ell| < \varepsilon$$

$$\Rightarrow \ 0 < |\ell| - \varepsilon < |f(x)| < |\ell| + \varepsilon \qquad \ldots \text{(a)}$$

Now for $x \in N(x_o)$

$$\left|\frac{1}{f(x)} - \frac{1}{\ell}\right| = \frac{|f(x) - \ell|}{|f(x)|\,|\ell|} < \frac{\varepsilon}{|f(x)|\,|\ell|} < \frac{\varepsilon}{(|\ell| - \varepsilon)\,|\ell|}$$

d) Since f is invertible we have $y = f(x) \Leftrightarrow x = f^{-1}(y)$ so that

$$\lim_{x \to x_o} f(x) = y_o \Leftrightarrow \lim_{y \to y_o} f^{-1}(y) = x_o \Leftrightarrow \lim_{x \to y_o} f^{-1}(x) = x_o$$

<u>Theorem 2</u>. If the functions f, g have limits at a point x_o, then

a) $\lim\limits_{x \to x_o} \left[f(x) + g(x)\right] = \lim\limits_{x \to x_o} f(x) + \lim\limits_{x \to x_o} g(x)$

b) $\lim\limits_{x \to x_o} \left[f(x) - g(x)\right] = \lim\limits_{x \to x_o} f(x) - \lim\limits_{x \to x_o} g(x)$

c) $\lim\limits_{x \to x_o} \left[f(x) \quad g(x)\right] = \lim\limits_{x \to x_o} f(x) \cdot \lim\limits_{x \to x_o} g(x)$

d) $\lim\limits_{x \to x_o} \left[f(x) : g(x)\right] = \lim\limits_{x \to x_o} f(x) : \lim\limits_{x \to x_o} g(x)$

(if $\lim\limits_{x \to x_o} g(x) \neq 0$)

Proof. Let

$$\lim_{x \to x_0} f(x) = \alpha , \qquad \lim_{x \to x_0} g(x) = \beta .$$

Then given $\varepsilon > 0$, there exist deleted neighborhoods N_1, N_2 of x_0 such that

$$x \in N_1 \Rightarrow |f(x) - \alpha| < \varepsilon, \quad x \in N_2 \Rightarrow |g(x) - \beta| < \varepsilon.$$

Taking $N = N_1 \cap N_2$, we have

$$x \in N \Rightarrow |f(x) - \alpha| < \varepsilon, \quad |g(x) - \beta| < \varepsilon$$

a) $x \in N \Rightarrow |f(x) + g(x) - (\alpha+\beta)| = |f(x) - \alpha + g(x) - \beta|$
$\leqslant |f(x) - \alpha| + |g(x) - \beta| < \varepsilon + \varepsilon = 2\varepsilon.$

Since $\varepsilon\,(>0)$ is arbitrary, then $f(x) + g(x) \to \alpha + \beta$ as $x \to x_0$.

b) Similarly proved.

c) $x \in N \Rightarrow | f(x)\, g(x) - \alpha \beta |$

$$= |f(x)\, g(x) - \alpha g(x) + \alpha g(x) - \alpha\beta|$$
$$= |(f(x) - \alpha)\, g(x) + \alpha (g(x) - \beta)|$$
$$\leqslant |f(x) - \alpha|\, |g(x)| + |\alpha|\, |g(x) - \beta|$$
$$< \varepsilon |g(x)| + |\alpha|\, \varepsilon$$
$$< \varepsilon (|\beta| + \varepsilon) + |\alpha|\, \varepsilon$$

$x \in N \Rightarrow |f(x)\, g(x) - \alpha\beta| < (|\alpha| + |\beta| + \varepsilon)\, \varepsilon \to 0.$

d) $$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \left[f(x) \cdot \frac{1}{g(x)}\right] = \lim_{x \to x_0} f(x) \cdot \lim_{x \to x_0} \frac{1}{g(x)} \qquad (3)$$

$$= \alpha \cdot \frac{1}{\beta} = \frac{\alpha}{\beta} \quad \text{(Theorem 1 c)}$$

Corollary. Let a composite function gof be given. Then

$$\lim_{x \to x_0} f(x) = \alpha \quad \text{and} \quad \lim_{x \to \alpha} g(x) = g(\alpha) \Rightarrow \lim_{x \to x_0} (gof)(x) = g(\alpha).$$

Theorem 3.

1) If $f(x) < g(x)$ holds for all x in a deleted neighborhood $N(x_o)$ and if f, g have limits α, β at x_o; then $\alpha \leqslant \beta$.

2) If $f(x) < u(x) < g(x)$ holds for all $x \in N(x_o)$ and if f, g have the same limit ℓ at x_o, then

$$\lim_{x \to x_o} u(x) = \ell.$$

Proof.

1)

$$g(x) - f(x) > 0 \Rightarrow \lim_{x \to x_o} [g(x) - f(x)] \geqslant 0 \Rightarrow \lim_{x \to x_o} g(x) -$$

$$\lim_{x \to x_o} f(x) \geqslant 0 \Rightarrow \beta - \alpha \geqslant 0 \Rightarrow \alpha \leqslant \beta$$

2) Since f, g have limits ℓ at $x_o \in D_f \cap D_g$ then there exist $N_1(x_o)$, $N_2(x_o)$ such that

$$x \in N_1(x_o) \Rightarrow |f(x) - \ell| < \varepsilon, \quad x \in N_2(x_o) \Rightarrow |g(x) - \ell| < \varepsilon$$

implying $\ell - \varepsilon < f(x) < \ell + \varepsilon$ and $\ell - \varepsilon < g(x) < \ell + \varepsilon$. Since $f(x) < u(x) < g(x)$ we have $\ell - \varepsilon < u(x) < \ell + \varepsilon$ which implies $|u(x) - \ell| < \varepsilon$ or that $\lim_{x \to x_o} u(x) = \ell$. ∎

Corollary 1.

$$P(x) = \sum_{k=0}^{n} a_k x^k \Rightarrow \lim_{x \to x_o} P(x) = P(x_o)$$

Proof.

$$\lim_{x \to x_o} P(x) = \lim_{x \to x_o} \sum_{k=0}^{n} a_k x^k$$

$$= \sum_{k=0}^{n} \lim_{x \to x_o} (a_k x^k), \ldots \text{ (Theorem 2a)}$$

$$= \sum_{k=0}^{n} a_k \left(\lim_{x \to x_o} x^k \right) \ldots \text{ (Theorem 1b)}$$

$$= \sum a_k \left(\lim_{x \to x_o} x \right)^k \ldots \text{ (Theorem 2c)}$$

$$= \sum a_k x_o^k \qquad (x \to x_o)$$

$$= P(x_o).$$

Corollary 2. If $P(x)/Q(x)$ is a rational function with $Q(x_o) \neq 0$, then

$$\lim_{x \to x_o} \frac{P(x)}{Q(x)} = \frac{P(x_o)}{Q(x_o)}$$

Proof. $$\lim_{x \to x_o} \frac{P(x)}{Q(x)} = \frac{\lim_{x \to x_o} P(x)}{\lim_{x \to x_o} Q(x)} \qquad \text{(Theorem 2d)}$$

$$= \frac{P(x_o)}{Q(x_o)} \qquad \text{(Coroll. 1)}$$

C. Indeterminate forms

If $\lim f(x) = 0$, $\lim g(x) = 0$ when $x \to x_o$ or $x \to \infty$, the use of property

$$\lim \frac{f(x)}{g(x)} = \frac{\lim f(x)}{\lim g(x)}$$

does not help in getting the limit of $f(x) / g(x)$, since the form $0/0$ is not defined and may be taken as equal to any number k. Indeed, the equality $0/0 = k$ is equivalent to $0 = 0.k$ and the latter holds true for any $k \in R$. For this reason $0/0$ is called an indeterminate form. The indeterminate forms that we encounter in this chapter are

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty \cdot 0, \quad \infty - \infty$$

There are also three other which arise in considering limit of a function of the form $f(x)^{g(x)}$, and are 0^0, 1^∞ and ∞^0. These indeterminate forms will be taken up in a later chapter where, by the use logarithms, they will be reduced to above mentioned indeterminate forms.

a. The indeterminate form 0/0:

A remarkable example is the following

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} \quad \left[\frac{0}{0}\right]$$

which we state as a theorem:

Theorem: If θ is measured in radian, then

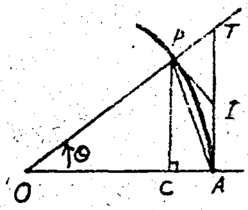
$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \quad \text{or} \quad \lim_{\theta \to 0} \frac{\theta}{\sin \theta} = 1$$

Proof.

a) Right limit: $0 < \theta < \frac{\pi}{2}$, $\theta = |\widehat{AP}|$ (See Fig.)

$\sin\theta = |CP| < |AP| = |\widehat{AP}| = \theta < |AI| + |IP|$

$< |AI| + |IT| = |AT| = \tan\theta$



$\Rightarrow \quad \sin\theta < \theta < \tan\theta$

$$\Rightarrow \quad 1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

$$\Rightarrow \quad 1 > \frac{\sin\theta}{\theta} > \cos\theta$$

Since $\lim\limits_{\theta\to 0} \cos\theta = 1$ we have, from Theorem 3,

$$\lim_{\theta\to 0+} \frac{\sin\theta}{\theta} = 1$$

b) Left limit: Set $\theta = -\theta'$ $(\theta' > 0)$.

$$\lim_{\substack{\theta\to 0\\ \theta<0}} \frac{\sin\theta}{\theta} = \lim_{\substack{\theta'\to 0\\ \theta'>0}} \frac{\sin(-\theta')}{-\theta'} = \lim_{\substack{\theta'\to 0\\ \theta'>0}} \frac{\sin\theta'}{\theta'} = 1 \text{ (from a)}$$

Since left and right limits exist and equal to each other the proof is completed. ■

Examples. Use the previous theorem to evaluats:

1. $\lim\limits_{t\to 0} \dfrac{\sin 3t}{2t}$ 2. $\lim\limits_{t\to 0} \dfrac{\sin 2t}{\sin 3t}$ (t is in radion)

Solution.

1. $$\lim_{t\to 0} \frac{\sin 3t}{2t} = \lim_{t\to 0} \frac{3}{2}\,\frac{\sin 3t}{3t} = \frac{3}{2} \lim_{h\to 0} \frac{\sin h}{h} = \frac{3}{2}$$

2. $$\lim_{t\to 0} \frac{\sin 2t}{\sin 3t} = \lim_{t\to 0} \left(\frac{\frac{\sin 2t}{2t}}{\frac{\sin 3t}{3t}} \cdot \frac{2}{3}\right) = \frac{2}{3} \lim_{t\to 0} \frac{\sin 2t}{2t} \Big/ \lim_{t\to 0} \frac{\sin 3t}{3t} = \frac{2}{3}$$

b. Indeterminate forms ∞/∞, $\infty-\infty$, $0.\infty$:

We note that $-\infty/-\infty$, $-\infty/\infty$, $\infty/-\infty$, $-\infty+\infty$. $0.(-\infty)$ are also indeterminate forms. All are reducible to the indeterminate form $0/0$.

Let $f(x)\to\infty$, $g(x)\to\infty$ when $x\to x_0$ or $x\to\infty$.

Then

1. $\lim \frac{f(x)}{g(x)} = \left[\frac{\infty}{\infty}\right] = \lim \frac{1/g(x)}{1/g(x)} = \left[\frac{0}{0}\right]$

2. $\lim \left[f(x) - g(x)\right] = \left[\infty - \infty\right] = \lim \left[\frac{1}{1/f(x)} - \frac{1}{1/g(x)}\right]$

$$= \lim \frac{1/g(x) - 1/f(x)}{1/(f(x)\ g(x)} = \left[\frac{0}{0}\right]$$

Let now, $f(x)\to 0$, $g(x)\to\infty$ when $x\to x_0$ or $x\to\infty$. Then

3. $\lim \left[f(x)\ g(x)\right] = \left[0.\infty\right] = \lim \frac{f(x)}{1/g(x)} = \left[\frac{0}{0}\right]$

Algebraic methods of evaluation:

a) For a rational function $P(x)/Q(x)$: (in cases $\frac{0}{0}$ $\frac{\infty}{\infty}$ $0.\infty$) let

$$P(x) = \sum_{i=0}^{n} p_i x^i, \quad Q(x) = \sum_{i=0}^{m} q_i x^i.$$

1) $$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \left[\frac{\infty}{\infty}\right] = \begin{cases} 0 & \text{when } n < m \\ \frac{p_n}{q_n} & \text{when } n = m \\ \infty \text{ (or } -\infty) & \text{when } n > m \end{cases}$$

2) $$\lim_{x\to x_0} \frac{P(x)}{Q(x)} = \left[\frac{0}{0}\right] = \lim_{x\to x_0} \frac{p(x)(x - x_0)}{q(x)(x - x_0)}$$ (by remainder Theorem)

$$= \lim_{x \to x_0} \frac{p(x)}{q(x)} = \frac{p(x_0)}{q(x_0)} \quad \text{(if not indeterminate)}$$

If indeterminate, repeat the process, or setting $x = x_0 + h$ write

$$\lim_{x \to x_0} \frac{p(x)}{Q(x)} = \lim_{h \to 0} \frac{P(x_0 + h)}{Q(x_0 + h)}$$

This method of incremant is applicable more generally to the ratio of any two functions.

Example. Evaluate the following limits:

1) $\displaystyle\lim_{x \to -\infty} \frac{x^3 + 5x^2 + 6}{-3x^2 + 4x}$ 2) $\displaystyle\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{\frac{\pi}{2} - x}$

3) $\displaystyle\lim_{x \to 2} \frac{x^2 - 4}{x^3 - 8}$ 4) $\displaystyle\lim_{x \to -2} \frac{x^3 + 2x^2 - 4x - 8}{x^4 + 2x^3 + 8x + 16}$

Solution.

1. $$\lim_{x \to -\infty} \frac{x^3 + 5x^2 + 6}{-3x^2 + 4x} = \lim_{x \to -\infty} \frac{x^2(x + 5 + 6/x)}{x^2(-3 + \frac{4}{x})} =$$

$$= \lim_{x \to -\infty} \frac{x + 5 + \frac{6}{x}}{-3 + \frac{4}{x}} = \infty \quad \text{(no limit)}$$

2. Setting $x = \frac{\pi}{2} + h$, we have

$$\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{\frac{\pi}{2} - x} = \left[\frac{0}{0}\right] = \lim_{h \to 0} \frac{\cos(\frac{\pi}{2} + h)}{-h} = \lim_{h \to 0} \frac{-\sin h}{-h} = 1$$

3.

$$\lim_{x \to 2} \frac{x^2 - 4}{x^3 - 8} = \left[\frac{0}{0}\right] = \lim_{x \to 2} \frac{(x - 2)(x + 2)}{(x - 2)(x^2 + 2x + 4)}$$

$$= \lim_{x \to 2} \frac{x + 2}{x^2 + 2x + 4} = \frac{1}{3}$$

or, setting $x = 2 + h$, we have

$$\lim_{x \to 2} \frac{x^2 - 4}{x^3 - 8} = \lim_{h \to 0} \frac{(2 + h)^2 - 4}{(2 + h)^3 - 8} = \lim_{h \to 0} \frac{4h + h^3}{12h + 6h^2 + h^3}$$

$$= \lim_{h \to 0} \frac{4 + h^2}{12 + 6h + h^2} = \frac{1}{3}$$

4. $$\lim_{x \to -2} \frac{x^3 + 2x^2 - 4x - 8}{x^4 + 2x^3 + 8x + 16} = \left[\frac{0}{0}\right] = \lim_{x \to -2} \frac{(x + 2)(x^2 - 4)}{(x + 2)(x^3 + 8)}$$

$$= \lim_{x \to -2} \frac{x^2 - 4}{x^3 + 8} = \left[\frac{0}{0}\right] = \lim_{x \to -2} \frac{(x + 2)(x - 2)}{(x + 2)(x^2 - 2x + 4)}$$

$$= \lim_{x \to -2} \frac{x - 2}{x^2 - 2x + 4} = \frac{-4}{4 + 4 + 4} = -\frac{1}{3}$$

or, setting $x = -2 + h$, we have

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 4x - 8}{x^4 + 2x^3 + 8x + 16} = \lim_{h \to 0} \frac{(h - 2)^3 + 2(h - 2)^2 - 4(h - 2) - 8}{(h - 2)^4 + 2(h - 2)^3 + 8(h - 2) + 16}$$

$$= \lim_{h \to 0} \frac{h^3 - 4h^2}{h^4 - 6h^3 + 12h^2} = \lim_{h \to 0} \frac{h - 4}{h^2 - 6h + 12} = -\frac{1}{3},$$

b) For an algebraic function: (in case $\infty - \infty$)

As illustrated in the following examples, indeterminacy can be removed in general by multiplying numerator and denominator by the conjuguates of the expressions.

<u>Example 1.</u> $\lim_{x \to \infty} \left[\sqrt{x^2 + 3x} - \sqrt{x^2 + 2}\right] = ?$

Solution.

$$\lim_{x\to\infty} (\sqrt{x^2+3x} - \sqrt{x^2+2}) \frac{\sqrt{x^2+3x}+\sqrt{x^2+2}}{\sqrt{x^2+3x}+\sqrt{x^2+2}} =$$

$$= \lim_{x\to\infty} \frac{3x-2}{\sqrt{x^2+3x}+\sqrt{x^2+2}} = \frac{3}{2}$$

Example 2. $\lim\limits_{x\to 2} \dfrac{\sqrt{x+2}-2}{\sqrt{2x+5}-3} = ?$

Solution.

$$\lim_{x\to 2} \frac{\sqrt{x+2}-2}{\sqrt{2x+5}-3} \cdot \frac{\sqrt{x+2}+2}{\sqrt{x+2}+2} \cdot \frac{\sqrt{2x+5}+3}{\sqrt{2x+5}+3}$$

$$= \lim_{x\to 2} \frac{(x-2)(\sqrt{2x+5}+3)}{(2x-4)(\sqrt{x+2}+2)} = \frac{1}{2} \lim_{x\to 2} \frac{\sqrt{2x+5}+3}{\sqrt{x+2}+2} = \frac{3}{4}$$

Example 3. $\lim\limits_{x\to -8} \dfrac{\sqrt[3]{x}+2}{x+8} = ?$

Solution. In the identity

$$(\sqrt[3]{a} \pm \sqrt[3]{b})(\sqrt[3]{a^2} \mp \sqrt[3]{ab} + \sqrt[3]{b^2}) = a \pm b$$

each factor at the left hand side is the conjugate of the other, and we have

$$\lim_{x\to -8} \frac{\sqrt[3]{x}+2}{x+8} \cdot \frac{\sqrt[3]{x^2}-2\sqrt[3]{x}+4}{\sqrt[3]{x^2}-2\sqrt[3]{x}+4} = \lim_{x\to -8} \frac{x+8}{(x+8)(\sqrt[3]{x^2}-2\sqrt[3]{x}+4)} = \frac{1}{12}$$

EXERCISES (1.6)

86. By ε, δ technique show that limit of

a) $x^2 - 2x$ is 3 when $x \to 3$.

b) $\frac{|x-1|}{x-1}$ is 1 when $x \to 2$.

87. If $f(x) = (x+3)/(x-2)$, find a $\delta > 0$ corresponding to $\varepsilon = 1/10$ for all x in a deleted neighborhood of 3.

88. Find the left hand side and right hand side limits of

a) $\frac{|x-4|}{x-4}$ when $x \to 4$.   b) $\frac{x^2-4}{|x-2|}$ when $x \to 2$

c) $x[\![x]\!]$ when $x \to -2$ d) $\lim_{x \to -1} [\![1+x]\!][\![2+x]\!]$ when

89. Evaluate the limits, if any:

a) $\lim_{x \to \sqrt{5}} [\![x^2 - 5]\!]$ b) $\lim_{x \to 1} (x[\![x]\!] - [\![-x]\!])$

c) $\lim_{x \to 0} [\![\cos x + \sin x]\!]$ d) $\lim_{x \to \pi/4} [\![\cos x + \sin x]\!]$

90. Evaluate the following limits, if any:

a) $\lim_{x \to \infty} \sin x$ b) $\lim_{x \to \infty} \tan x$

c) $\lim_{x \to \infty} \frac{\cos x}{x}$ d) $\lim_{x \to \infty} \frac{x}{\sin x}$

e) $\lim_{x \to \infty} y$, where $x^3y^2 - 3xy^2 + 5y - 4x + 8 = 0$

91. Evaluate the following limits, if any:

a) $\lim_{x \to 3/2} [\![2 + x]\!]$ b) $\lim_{x \to -2} \frac{x}{|x|}(x^2 + 3)$

c) $\lim_{x \to 3} [\![x + \frac{1}{2}]\!] \sin \frac{\pi}{6} x$ d) $\lim_{x \to 4} \sin(\frac{\pi}{2} + 2k\pi)[\![x^2+1]\!]$

e) $\lim_{x \to \pi/3} \text{Sec } x$ f) $\lim_{x \to 1/3} \frac{1}{[\![2x^2 + 1]\!]}$

92. For each of the following cases, find examples of functions f, g and x_0 at which they have no limits but

a) $f + g$ b) $f - g$ c) fg d) f/g

has a limit at x_0.

93. Find limits, if any:

a) $\lim\limits_{x \to \pi/3} (\sin x + \cos x)$ b) $\lim\limits_{x \to -\pi/4} (|x| - \tan x)$

c) $\lim\limits_{x \to 0} x \cot x$ d) $\lim\limits_{x \to -\sqrt{5}} \dfrac{3x}{[\![x]\!]}$

94. Given $f(x) = \sqrt{x + 4}$ and $g(x) = x^2 - 5$, find

a) $\lim\limits_{x \to 3} (f \circ g)(x)$ b) $\lim\limits_{x \to 3} (g \circ f)(x)$

c) $\lim\limits_{x \to 5} (f \circ f)(x)$ d) $\lim\limits_{x \to 2} (g \circ g)(x)$

95. Evaluate

a) $\lim\limits_{x \to \sqrt{3}/2} \arcsin x$ b) $\lim\limits_{x \to 1/2} [\![\arccos x]\!]$

c) $\lim\limits_{x \to 2} y$, y being defined by $x^3y^2 - 5x^2y + 8x + 4y - 8=0$

d) $\lim\limits_{x \to 0} y$, y being defined by $x^3y^2 - 5x^2y - 8x + 4xy - 8=0$

e) $\lim\limits_{x \to 2} f^{-1}(x)$ if $f(x) = \dfrac{x + 1}{x - 2}$

96. Determine the constants a and b such that

a) $\lim\limits_{x \to -2} \dfrac{x^3 - ax + 2}{x + 2} = 9$ b) $\lim\limits_{x \to -2} \dfrac{x^3 - 6x + 2b}{x + 2} = 0$

97. Find the limits, if any:

a) $\lim\limits_{x \searrow 1} \dfrac{|x| + |x - 1| - 1}{x^2 - 1}$ b) $\lim\limits_{x \nearrow 1} \dfrac{|x - 1| - |x| + 1}{x^2 + 1}$

98. Evaluate the following limits, if any:

a) $\lim\limits_{x \to -1} \dfrac{x^3 + x^2 - 2x - 2}{x^3 + x^2 + 2x + 2}$ [-1/3]

b) $\lim\limits_{x \to \infty} \dfrac{x^3 - 5x + 7}{x^2 + 8x}$ [No limit, ∞]

c) $\lim\limits_{x \to \infty} \dfrac{x^2 + 8x}{x^3 - 5x + 7}$ [0]

d) $\lim\limits_{x \to \infty} \dfrac{3x^8 + 100x^3 + 12}{8x^8 + 1000x^3 - 9}$ [3/8]

e) $\lim\limits_{x \to 0} \dfrac{x^3 - 6x^2 - 8}{x + 5}$ [-8/5]

99. Find the limits, if any:

a) $\lim\limits_{x \to 2} \dfrac{x^3 - 3x^2 + 4}{x^3 - 5x^2 + 8x - 4}$,

b) $\lim\limits_{y \to x} \dfrac{x^2 - y^2}{y - x}$

c) $\lim\limits_{x \to \pi/2} \dfrac{\cos x}{1 - \sin x}$

d) $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{\sin x}$

100. Evaluate

a) $\lim\limits_{\theta \to \pi} \dfrac{1 + \cos\theta}{\sin^2\theta}$ [1/2] b) $\lim\limits_{x \to 0} \dfrac{\sin 2x \tan x}{1 - \cos x}$ [4]

101. Evaluate

a) $\lim\limits_{x \to \pi/2} \dfrac{\tan x}{\sec x}$ b) $\lim\limits_{x \to 0} \dfrac{\csc x}{\cot x}$

c) $\lim\limits_{x \to \pi/3} \dfrac{\cot(x - \pi/3)}{\tan(x + \frac{\pi}{6})}$ d) $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x^2}$

102. Evaluate $\lim\limits_{x \to 9} \dfrac{x^{3/2} - 9x^{1/2}}{x - 9}$ [3]

103. Evaluate the following limits, if any:

a) $\lim\limits_{x \to 3} \dfrac{3 - x}{\sin \pi x}$ b) $\lim\limits_{x \to 0} \dfrac{x \sin(\sin x)}{1-\cos(\sin x)}$

c) $\lim\limits_{x \to 0^+} \dfrac{\arctan x}{1 - \cos 2x}$ d) $\lim\limits_{x \to 0} \dfrac{\sin^2 x + \sin x^2}{x^2}$

104. Find the limits:

a) $\lim\limits_{x \to \pi/2} (x - \frac{\pi}{2}) \tan x$ b) $\lim\limits_{x \to 0} x \csc x$

c) $\lim\limits_{x \to \infty} x \sin(k\pi + x)$ d) $\lim\limits_{x \to \pi/2} \dfrac{1}{x - \frac{\pi}{2}} \cos x$

105. a) $\lim\limits_{t \to 0} \dfrac{\sqrt{1 + 2t} - (1 + t)}{t^2} = ?$ $[-1/2]$

b) $\lim\limits_{t \to 0} \dfrac{\sqrt[3]{1 + 3t} - (1 + t)}{t^2} = ?$ $[-3/2]$

106. Evaluate

a) $\lim\limits_{x \to 0} \dfrac{\sqrt{x + 4} - 2}{\sqrt{x + 9} - 3}$ b) $\lim\limits_{x \to 1} \dfrac{\sqrt[3]{2x + 6} - 2}{\sqrt{x + 3} - 2}$

107. Evaluate the limits, if any:

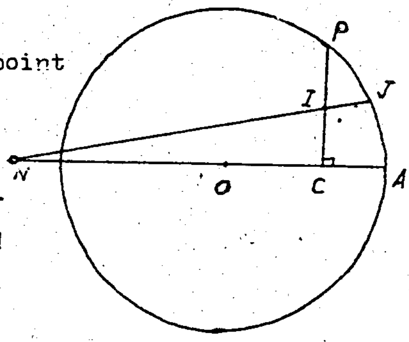
a) $\lim\limits_{x \to \pi/2} (\tan x - \sec x)$ b) $\lim\limits_{x \to 0} (\cot x - \csc x)$

c) $\lim\limits_{x \to \infty} (\sqrt{x^2 + 3x} - \sqrt{x^2 - 3x})$

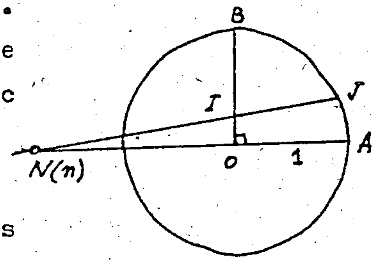
d) $\lim\limits_{x \to \infty} (\sqrt[3]{x + 3} - \sqrt[3]{x})$

108. Find an interval for the $\lim f(x)$ when $x \to 4$ if $2x + 3 > f(x) > x + 5$. $[[9, 11]]$

109. In the Figure, P is on the unit circle in the first quadrant. If I is the mid point of the segment [CP] and J is that of arc $\widehat{AP}$ and N is the point where IJ intersects the line OA, find the limting position of N on OA when $P \rightarrow A$. (<u>Hint</u>: Take $\theta = \measuredangle AOP$).



110. Let OA, OB be two perpendicular lines passing through the center O of a circle. Let I, J be points on the segment [OB] and on the arc $\widehat{AB}$ such that $|OI| = |OB|/n$, $|\widehat{AJ}| = |\widehat{AB}|/n$. Let N be the point where IJ intersects OA. Find the limting position of N(n) as $n \rightarrow \infty$.



$\left[\,|ON| = \frac{2}{\pi - 2}\,\right]$

1.7 CONTINUITY

A. Definitions

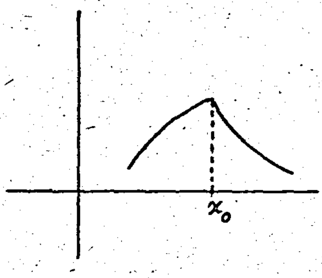
Continuity is among important concepts about functions.

A function continuous on a closed interval enjoys many properties, and the graph of such a function consists of a single piece that can be traced without lifting the pencil off the paper.

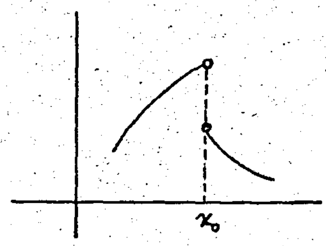
A function f, defined on a closed interval $[a, b]$, is said to be <u>continuous</u> at an interior point $x_0 \in (a, b)$ if

$$\lim_{x \to x_0} f(x) = f(x_0) \qquad (1)$$

Continuity of f at the end points a and b is defined by (1) by the use of right and left limits at a and b respectively. A function which is not continuous at a point $x_0 \in [a, b]$ is said to be <u>discontinuous</u> at x_0.



Graph of a function continuous at x_0



Graph of a function discontinuous at x_0.

A function $f: \mathbb{R} \to \mathbb{R}$, given by a rule, is discontinuous at a point x_0 at least in one of the following cases:

1) f is not defined at x_0 (<u>Missing point discontinuity</u>)

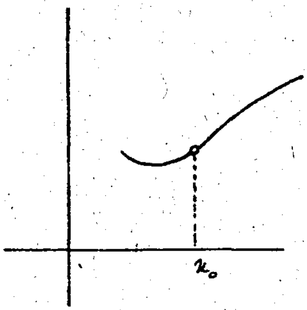
2) f is defined at x_0 but not in a deleted neightborhood[1] $N(x_0)$ of x_0. (<u>Isolated point discontinuity</u>)

1) Some authors regard f continuôus at an isolated point, in a deleted neighborhood of which f is undefined.

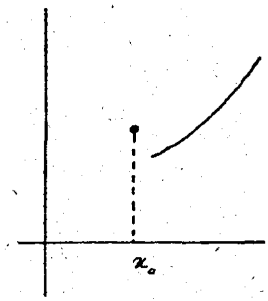
3) lim f(x) does not exist at x_0 <u>(jump discontinuity)</u>

4) lim f(x) exists but this limit is not equal to $f(x_0)$ <u>(Removable discontinuity)</u>

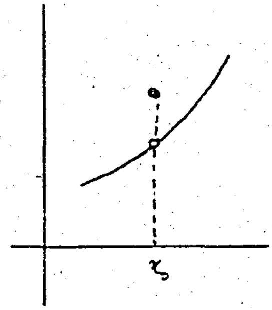
Then we have the following types of discontinuities as illustrated by graphs:



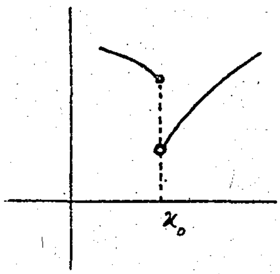
Missing point discontinuity or removable discontinuity



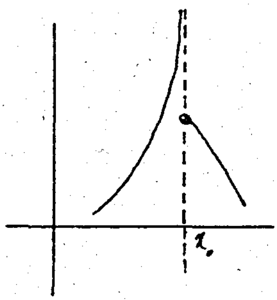
Isolated point discontinuity



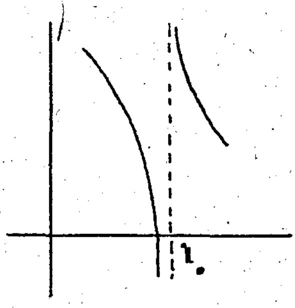
Finite jump discontinuity



Finite jump discontinuity



Infinite jump discontinuity



Infinite jump discontinuity

<u>Example 1</u>. Why the following functions are not continuous at indicated points, and what is the type of discontinuity?

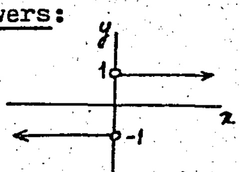
1. $f(x) = \frac{|x|}{x}$, $x_0 = 0$

2) $g(x) = \begin{cases} 1 & \text{when} \quad x = -1 \\ x & \text{when} \quad x \geqslant 0 \end{cases}$, $x_0 = -1$

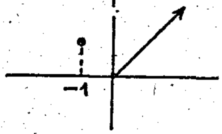
3) $h(x) = \dfrac{x-2}{x-2}$, $x_0 = 2$

4) $k(x) = \dfrac{1}{x-1}$, $x_0 = 1$

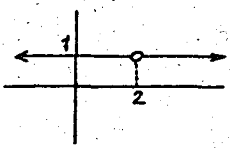
Answers:

1) 

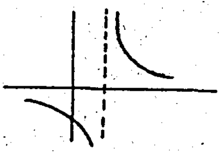
f is not defined at $x_0 = 0$ (finite jump)

2) 

$x = -1$ is an isolated point of g. (isolated point)

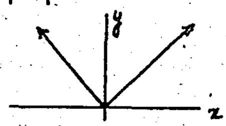
3) 

h is undefined at $x_0 = 2$. h having limit (=1) at $x_0 = 2$ the discontinuity is removable.

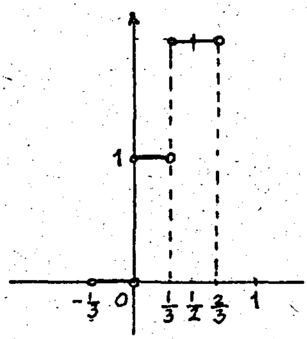
4) 

k is undefined at $x_0 = 1$ (infinite jump)

Example 2. Test the function $f(x) = |x|$ for continuity at the origin.



Solution. Since $\lim\limits_{x \to 0} |x| = 0$ and this limit is equal to $f(0)$, f is continuous at 0.



Example 3. Test the function $f(x) = [\![3x + 1]\!]$ at $x_0 = \frac{1}{2}$.

Solution. $\lim\limits_{x \to \frac{1}{2}} f(x) = 2 = f(\frac{1}{2})$.

It is continuous.

Example 4. Find the set of x's on which $f(x) = [x^2]$ is continuous.

Solution. Since f has (finite jump) discontinuities at, $x = \pm\sqrt{a}$ $(a \in N)$ only, then the set is $\mathbb{R} - \{x: x = \pm\sqrt{a}, a \in N\}$.

If $f: [a, b] \to \mathbb{R}$ is continuous at every point of the closed interval $[a, b]$, then f is said to be continuous on $[a, b]$. Similar definition is given for a function defined in an open interval (a, b).

Notation. $C[a, b] \doteq \{f: \ f \text{ is continuous on } [a, b]\}$
$C(a, b) = \{f: \ f \text{ is continuous in } (a, b)\}$

B. Properties of continuous functions:

Theorem 1.

$f, g \in C[a, b] \Rightarrow cf, f \pm g, fg, f/g \in C[a, b]$ where the latter holds when $g(x) \neq 0$, and c is constant.

Proof. We only prove the continuity of cf on $[a, b]$. Since $f \in C[a, b]$, then for $x_0 \in [a, b]$ we have $\lim_{x \to x_0} f(x) = f(x_0)$. Now,

$$\lim_{x \to x_0} (cf)(x) = \lim_{x \to x_0} cf(x) = c \lim_{x \to x_0} f(x) = c\, f(x_0) = (cf)(x_0)$$

The other cases are similarly proved using theorem on limits. ∎

Corollary. If $P(x)$, $Q(x)$ are two polynomials, then

1) $P \in C(-\infty, \infty)$

2) $P/Q \in C(-\infty, \infty)$ except at the roots of Q.

Theorem 2. If $f \in C[a, b]$, then

1) There exist $x_1, x_2 \in [a, b]$ such that

$$f(x_1) = m \leqslant f(x), \quad f(x_2) = M \geqslant f(x)$$

for all $x \in [a, b]$, where m, M are the smallest and largest values of $f(x)$ on $[a, b]$.

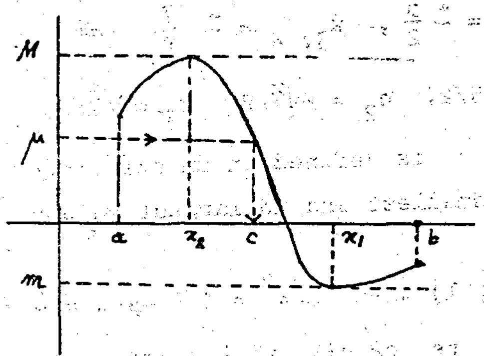
2. For any μ with $m < \mu < M$, there exists at least one $c \in (a, b)$ such that $f(c) = \mu$. (Intermediate value theorem)

Sketchy proof.

Since f is continuous on the closed interval $[a, b]$, $f(x_0)$ is definite for all x_0 on the same interval, and the set of all $f(x_0)$ has a smallest value m and largest value M.

f being continuous on $[a, b]$, it is an onto function from $[a, b]$ to $[m, M]$.

The numbers m, M related to a function f continuous on a closed interval are called global (or absolute) minimum, maximum of $f(x)$, respectively.



Example. Find m, M and c, if any, with $\mu = f(c)$ for the following functions in the indicated intervals and μ

a) $f(x) = \dfrac{x}{x+3}$, $[1, 4]$, $\mu = 1/2$

b) $g(x) = x^2 - 2$, $[2, 5]$, $\mu = 14$

c) $h(x) = |x^2 - 4|$, $[-5/2, 3]$, $\mu = 9/4$

d) $k(x) = 1/x^2$, $(1, 5)$, $\mu = 4/9$

Solution.

a) Since f is increasing in $[1, 4]$, we have $m = f(1) = 1/4$, $M = f(4) = 4/7$. $\mu = \frac{1}{2} \in [1/4, 4/7]$ Then $\frac{x}{x+3} = \frac{1}{2} \Rightarrow c = 3 \in [1, 4]$.

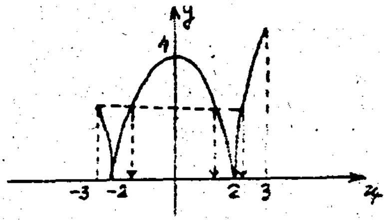
b) Since g is increasing in $[2, 5]$, we have $m = f(2) = 2$, $M = f(5) = 23$, $\mu = 14 \in [2, 23]$. Then $x^2 - 2 = 14 \Rightarrow x = \pm 4$, and $c = 4 \in [2, 5]$.

c) From the graph

$m = f(-2) = f(2) = 0$,

$M = f(3) = 5$.

$\mu = \frac{9}{4} \in [0, 5]$. Then



$$|x^2 - 4| = \frac{9}{4} \Rightarrow x^2 - 4 = \pm\frac{9}{4} \Rightarrow x^2 = \frac{16 \pm 9}{4} \Rightarrow$$

$$x_{1,2} = \pm\frac{5}{2}, \quad x_{3,4} = \pm\frac{7}{2} \Rightarrow$$

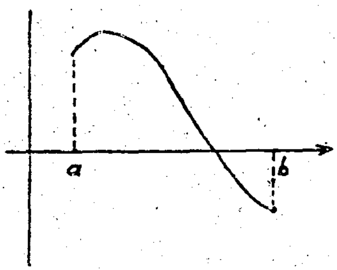
$c_1 = -5/2$, $c_2 = -\sqrt{7}/2$, $c_3 = \sqrt{7}/2$, $c_4 = 5/2 \in [-5/2, 3]$.

d) Since k is defined on an open interval there will be no smallest and no largest values, but $\frac{1}{25} < k(x) < 1$.

$\mu \in (1/25, 1)$ then $1/x^2 = 4/9 \Rightarrow x = \pm 3/2 \Rightarrow c = 3/2 \in (1, 5)$.

Corollary. If $f \in C[a, b]$ and $f(a)\,f(b) < 0$, then there exists at least one $c \in (a, b)$ such that $f(c) = 0$, in other words the equation $f(x) = 0$ has at least one root between a and b.

To find an approximate root of an equation $f(x) = 0$, in the first step, one determines an interval $[a, b]$ on which f is continuous and $f(a)\,f(b) < 0$ and the



corollary guarantees then the existence of a root in (a, b).

There are some methods for such a determination, two of which are mentioned here. Each consists of finding intervals $[a_n, b_n]$ with $f(a_n)\, f(b_n) < 0$ such that

$$[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \supset \cdots \supset [a_n, b_n] \supset \cdots$$

where the length of $[a_n, b_n]$ tends to 0 as $n \to \infty$ and each interval containing the same root c, the numbers $a_1, a_2, a_3 \ldots$ (or $b_1, b_2, b_3, \ldots$) can be taken as the first, second, third ... approximations to the root c. (If in the process $f(a_n)=0$, or $f(b_n) = 0$, for some value k of n, then the root c is at hand).

First way: Take $[a, b]$ as $[a_1, b_1]$. Let α_1 be the midpoint of $[a_1, b_1]$. If $f(\alpha_1) = 0$, then α_1 is a root, otherwise $f(\alpha_1) > 0$ or < 0. If $f(\alpha_1)\, f(a_1) < 0$, say, take $[a_1, \alpha_1]$ as $[a_2, b_2]$. Let α_2 be the midpoint of $[a_2, b_2]$. If $f(\alpha_2) = 0$, then α_2 is a root, otherwise $f(\alpha_2) > 0$ or < 0. If $f(\alpha_2)\, f(b_2) < 0$, say, take $[\alpha_2, b_2]$ as $[a_3, b_3]$, and so on. The process can be continued to any desired approximation.

Second way: Referring to the first way above α_1 is taken here as the point where the chord joining the end points of the curve intersects $[a_1, b_1] = [a, b]$. The process continues as in the first way taking α_i's as intersection of chords with x-axis.

From the equation

$$y - f(a) = \frac{f(b) - f(a)}{b - a}\,(x - a)$$

of the chord passing through the points $(a, f(a))$, $(b, f(b))$ we obtain

$$\alpha_1 = a - (b - a)\ \frac{f(a)}{f(b) - f(a)}$$

Example. Find interval(s) on each of which

$$f(x) = 2x^3 - 7x + 3 = 0$$

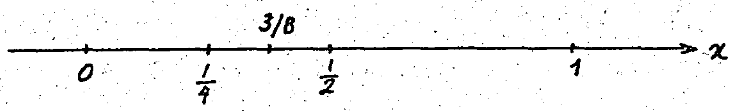
has one root, and then find one of them approximately applying the first and second way.

Solution 1. $f(-2) = -6 < 0$, $f(-1) = 8 > 0$, $f(0) = 3 > 0$. $f(1) = -2 < 0$, $f(2) = 5 > 0 \Rightarrow c_1 \in [-2, -1]$, $c_2 \in [0, 1]$. $c_3 \in [1, 2]$. Let us find the second root c_2.

$$\alpha_1 = \frac{0 + 1}{2} = \frac{1}{2}, \quad f(\tfrac{1}{2}) = -1/4 < 0, \quad \alpha_2 = 1/4$$

$$\Rightarrow \quad f(1/4) = 41/32 > 0. \quad \alpha_3 = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{4}\right) = 3/8.$$

Stopping at this step we get 3/8 as the third approximation of the root with largest error $\frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = 1/8$.



Solution 2. $\alpha_1 = a - (b - a)\dfrac{f(a)}{f(b) - f(a)} = 0 - (1-0)\dfrac{3}{-2-3}$ $= 3/5 \Rightarrow f(3/5) = -96/125 < 0 \Rightarrow \alpha_2 \in [0, 3/5]$.

$$\alpha_2 = 0 - (\tfrac{3}{5} - 0)\,\frac{3}{-\frac{96}{125} - 3} = 75/157$$

$$\Rightarrow \quad f(\alpha_2) = -59/628 < 0 \Rightarrow \alpha_3 \in [0,\ 75/157].$$

$$\alpha_3 = 0 - \frac{75}{157} \cdot \frac{3}{-\frac{59}{628} - 3} \cong \frac{25}{59}$$ with an error less than

$$\frac{1}{2}\left(\frac{75}{157} - 0\right) = 75/314.$$

E X E R C I S E S (1. 7)

111. Test the following functions for continuity at indicated points:

a) $f(x) = |x|, \quad x_1 = 3/2, \quad x_2 = 0$

b) $g(x) = [\![x]\!], \quad x_1 = 3/2, \quad x_2 = 0$

112. Same question for:

a) $F(x) = \begin{cases} x + 1 & \text{when} \quad x > 2 \\ 3 & \text{when} \quad x < 2 \end{cases}, \quad x_0 = 2$

b) $G(x) = \begin{cases} x^2 + 1 & \text{if} \quad x > 1 \\ 2 & \text{if} \quad x \leq 1 \end{cases} \quad x = 1$

c) $H(x) = \begin{cases} x^2 + 1 & \text{if} \quad x \geq 1 \\ 3 & \text{if} \quad x < 1 \end{cases} \quad x_1 = 2, \; x_2 = 1$

113. Show that the following functions are continuous for all $x \in R$:

a) $f(x) = \begin{cases} x^2, & x < -1 \\ 1, & x = -1 \\ x + 2, & x > -1 \end{cases}$ b) $g(x) = \sqrt{\dfrac{x^2 + 2x + 3}{x^2 + x + 1}}$

114. Show that the following functions are continuous at all x in their domain of definition:

a) $f(x) = |x^2 - 2| \dfrac{x}{x}$ b) $g(x) = \sqrt{x^2 - 5x - 4}$

c) $h(x) = \sqrt[3]{x + 5}$ d) $k(x) = \sqrt[4]{x^2 + 2}$

115. Find the points of discontinuity and identify their types of the following functions, if any:

a) $f(x) = \dfrac{x^2 + 3x - 10}{x - 2}$ b) $g(x) = \begin{cases} x^2 + 3, & x < 2 \\ 5 - x, & x > -2 \end{cases}$

c) $F(x) = \begin{cases} x + 4, & x < 2 \\ 7, & x = 2 \\ 2x + 2, & x > 2 \end{cases}$ d) $G(x) = [\![x]\!] - x$

116. Same question for:

a) $f(x) = \begin{cases} x, & x < 0 \\ 1, & x = 0 \\ \dfrac{1}{1-x} & x > 0 \end{cases}$ b) $g(x) = \dfrac{x}{\sin x}$

c) $F(x) = x \cot x$ d) $G(x) = \dfrac{\tan x}{\arctan x}$

117. Find the points and type if discontinuity of the following functions in the indicated intervals, if any:

a) $f(x) = \dfrac{x^2 + 3}{|x - 2| - 1}$, $[0, 2]$

b) $g(x) = \dfrac{x}{[\![2x]\!] - x}$, $[0, 5]$

c) $h(x) = \dfrac{\sin x + \cos x}{\sin x - \cos x}$, $[0, \pi/2]$

d) $k(x) = \dfrac{\sin x}{\arcsin x}$, $[0, \pi/2]$

118. Find the points of discontinuity and identify their types of the following functions on their domain, if any:

a) $F(x) = [\![\sin x]\!]$

b) A function defined by

$$x^3y^2 - 2x^2y - xy^2 + 8xy + 5x - y + 3 = 0$$

c) GoG if $G(x) = [\![x^2 + 1]\!]$

d) H^{-1} if $H(x) = \dfrac{1}{x + 1}$

119. Find the points of discontinuity of $f + g$, fg, f/g if

$$f(x) = 2x - \frac{1}{x^2}, \quad g(x) = x^2 + \frac{1}{x^2}$$

120. Find the points of discontinuity of fog and gof and determine their types, if any, where

a) $f(x) = x^2 - 1$, $g(x) = \sin x$

b) $f(x) = \cos x$, $g(x) = \dfrac{1}{x^2 + 1}$

121. Find m, M of the following functions in the given

interval if they are continuous; then find x for the given value of $f(x)$:

a) $y = x^3 - 8$, $[1, 2]$, $f(x) = -2$

b) $y = |x + 3| + x$, $[-2, 3]$, $f(x) = 0$

c) $y = |x^2 + 4x| - 5$, $[-1, 3]$, $f(x) = 2$

d) $y = [\![\frac{x}{5} + 1]\!]$, $[1, 4]$, $f(x) = 1$

122. Same question for

a) $f(x) = x^2 - [\![x^3]\!]$, $[-1, 0]$, $f(x) = 1$

b) $f(x) = \dfrac{x}{x + 1}$, $[3, 7]$, $f(x) = 5/6$

c) $f(x) = -\dfrac{2 - x}{x^2 + 3x}$, $[1, 5]$, $f(x) = 0$

d) $f(x) = \sec x + \csc x$, $[\pi/6, \pi/3]$, $f(x) = 1$

123. Find approximately a root of $f(x) = 0$ in $[0, 1]$ and determine the maximum error.

a) $f(x) = 2x^2 + x - 1 = 0$

b) $f(x) = 3x^2 + 2x - 1 = 0$

124. Find the root of $x^3 - 5x + 2 = 0$ in $(0, 1)$ approximately with an error less than 10%.
(Use both ways)

125. Find approximate root of $\sin x + \cos x$ in $(0, \pi)$ and give the maximum error.

A SUMMARY

(Chapter 1)

1.3 $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$, $\sum_{k=1}^{n} ar^k = a\,\frac{r^{n+1}-1}{r-1}$, $(r\neq 1)$

1.4 The <u>relation</u> from I to J with defining rule $p(x, y)$ is

$$\rho = \{(x, y):\ x\in I,\ y\in J,\ p(x, y)\}$$

and the inverse ρ^{-1} of ρ is

$$\rho^{-1} = \{(x, y):\ x\in J,\ y\in I,\ p(y, x)\}$$

1.5 The <u>function</u> from I to J is a relation with the rule $y = f(x)$ (assigning one only one $y\in J$ for each $x\in I$) where $I = D_f$ and $R_f = f(D_f)\subseteq J$ are the domain and range of f. f is written $f: D_f \to R_f$, $y = f(x)$. Types: Polynomial, rational, irrational; algebraic, transcendental, explicit, implicit; even, odd; periodic, piecewisely defined functions.

1.6 $\lim_{x\to x_0^-} f(x)$, $\lim_{x\to x_0^+} f(x)$, and $\lim_{x\to x_0} f(x)$ are left limit, right limit, and limit of $f(x)$ at x_0. Indeterminate forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty-\infty$, $\infty\cdot 0$; 0^0, 1^∞, ∞^0.

1.7 If $\lim_{x\to x_0} f(x) = f(x_0)$, $f(x)$ is said to be continuous at x_0. The notation $f\in C[a, b]$ denotes function f <u>continuous</u> on $[a, b]$. $f\in C[a, b]$ attains its minimum value m, its maximum value M, and all intermediate values in between m, M.

MISCELLANEOUS EXERCISES

(Chapter 1)

126. A natural number n is called composite if $n = a.b$ where a, b are natural numbers greater than 1, and called prime if $n > 1$ and not composite. Show that each of the natural numbers between 90 and 100 which is composite has a prime factor less than 10.

127. What is the value of $\left| |x - 6| - |-x| + |2 - x| \right|$ for $x = \sqrt{5}$?

128. Prove: $(ac + bd)^2 \leq (a^2 + b^2)(c^2 + d^2)$

129. Prove:

a) n^2 even $\Rightarrow$ n even

b) n^2 odd $\Rightarrow$ n odd

130. Solve the following for $x, y \in R$:

a) $x + 2i = 3 - 5yi$ b) $(2x - i)(y + 3i) = 7 + 4i$

c) $\dfrac{4x - iy}{2 + i} = 3 + yi$ d) $2x + i = (5 - i)\,\overline{x + yi}$

131. Evaluate

a) $(\sqrt{2} - \sqrt{2}i)^4$ b) $(\sqrt{3} + i)^3$

132. For each case find a relation between real x and y if $z^2 - 3z\bar{z} + 2\bar{z}$ $(z = x + iy)$ is

a) real b) pure imaginary

133. Write a polynomial of least degree with real coefficients having the roots $2-i$ and $1 + 2i$.

134. If $n(A) = 8$ and $n(B) = 5$, what may be the largest and smallest values of

a) $n(A \cup B)$ b) $n(A \cap B)$

135. If $A = \{n: 3|n, \ n \leqslant 100, \ n \in N^+\}$ and
$B = n: 4|n, \ n \leqslant 100, \ n \in N^+\}$,
compute

a) $n(A)$ b) $n(B)$ c) $n(A \cap B)$

d) $n(A \cup B)$, e) $n(A - B)$ f) $n(B - A)$

136. In a group of students there are $n(F) = 12$ football players, $n(V) = 8$ voleyball players and $n(B) = 9$ basketball players. If $n(F \cap V) = 4$, $n(F \cap B) = 3$, $n(V \cap B) = 5$ and $n(F \cup B \cup V) = 18$, find the number of students playing

a) football only [6] b) basketball only [2]

c) Voleyball only [0] d) all. [1]

137. Which of the following are the subsets of the set R of all rectangles?

a) {rectangles with area A} b) {squares}

c) {parallelograms} d) {rhombuses}

e) {rectangles with dimension $a \times 2a$, $a \in R^+$}

138. Which of the following are the subset of polygons?

a) {triangles} b) {squares}

c) {rectangles} d) {circles}

e) {hexagons} f) {pyramids}

g) {spheres}

139. Given the sets $A = \{(x, y) \in \mathbb{R}^2 : |x| - |y| \leqslant 1\}$,
$B = \{(x, y) \in \mathbb{R}^2 : |x^2 - y^2| < 4$ and
$C = \{(x, y) \in \mathbb{R}^2 : |y| > x^2\}$, graph them in the cartesian plane and then obtain the graphs of

a) $A \cup B$ b) $A - C$ c) $B \cap C$

140. Find the interval defined by

a) $|x - 3| \leqslant 2$ b) $|x + 2| < 3$

c) $|x + 7| < 9$ d) $|x - 9| \leqslant 7$

141. Express the given interval as an inequality involving an absolute value:

a) $(3, 8)$ b) $[5, -7]$

c) $[-4, 7]$ d) $(-2, 5)$

142. Find the set of solution of the following equation:

a) $|x^2 - 2x| - x - 1 = 0$ $[\{\frac{3-\sqrt{13}}{2}, \frac{3+\sqrt{13}}{2}\}]$

b) $|x + 3| - |2x - 1| - x = 0$ $[\{-1, 2\}]$

143. Same question for

a) $2|x + 4| - |x - 2| + x = 0$

b) $|x - 3| - |x + 1| + 4 = 0$

144. Prove by induction:

a) $x^n + y^n$ is divisible by $x + y$ for $n \in \mathbb{Z}_1$

b) $x^n - y^n$ is divisible by $x - y$ for $n \in \mathbb{Z}_1$

145. Prove by induction:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \dots + \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n},$$

$n \in \mathbb{Z}_1$.

146. Prove by induction:

$\left|\frac{\operatorname{Sin} nx}{\operatorname{Sin} x}\right| \leqslant n$ for $n \in \mathbb{Z}_0$, $x \neq k\pi$

147. Given the relation $x - |y| = 1$

a) Sketch it b) write its inverse

148. Same question for $|x| + |y| = 1$

149. Find the inverse of the relation

$$(x - y)(3y + x) + 1 = 0.$$

150. Which ones of the following relations are symmetric?

a) $\{(x, y): x^2 + y^2 > 4\}$ b) $\{(x, y): x + y < 2\}$

c) $\{(x, y): x - y > 1\}$ d) $\{(x, y): xy + 4 = 0\}$

e) $\{(x, y): |x + y| < 2\}$ f) $\{(x, y): xy^2 - 1 = 0\}$

151. Sketch the graph of the relations:

a) $\rho = \{(x, y): |y| - x + 1 > 0\}$

b) $\rho = \{(x, y): ||x| - y| - 3 < 0\}$

152. Same question for:

a) $\{(x, y): [\![x - 2]\!] = 3, [\![y - 3]\!] = 2\}$

b) $\{(x, y): [\![x]\!] + [\![y]\!] = 1\}$

153. Sketch:

a) $\{(x, y): |x| + |x - 1| = 3\}$

b) $\{(x, y): |y| - |y - 1| > 3\}$

c) $\{(x, y): |x| + |y - 1| < 3\}$

d) $\{(x, y): |y| - |x - 1| > 3\}$

154. Sketch:

a) $\{(x, y): |x| = 2\}$ b) $\{(x, y): [\![x]\!] = 2\}$

c) $\{(x, y): |x - 3| = 2, y=1\}$ d) $\{(x, y): [\![x - 3]\!] = 2, y=1\}$

155. Sketch

a) $\{(x, y): |2x + 3| = 5, [\![y]\!] = 2\}$

b) $\{(x, y): [\![2x + 3]\!] = 5, |y| = 2\}$

156. Sketch the graphs of the relations:

a) $\{(x, y): [\![x]\!][\![y]\!] = 1\}$

b) $\{(x, y): [\![x]\!][\![y]\!] = -1\}$

c) $\{(x, y): [\![x]\!][\![y]\!] = 0\}$

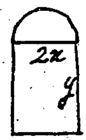
d) $\{(x, y): [\![x]\!][\![y]\!] = 4\}$

157. Prove

a) $[\![x]\!] \leqslant x < [\![x]\!] + 1$ b) (a) $\Leftrightarrow 0 \leqslant x - [\![x]\!] < 1$

c) $[\![x]\!] + [\![-x]\!] \leqslant 0$ d) $0 \leqslant [\![x]\!] - 2[\![\frac{x}{2}]\!] \leqslant 1\}$

158. Given in the Figure a window with constant area S. The glass in rectangular form permits the light half of that of semicircular form. Find the amount of light $\ell(x)$ passing through the window. (Glass in rectangular form permits amount of light ℓ_0 per unit area).



159. Find the area A of an isocales triangle with equal sides a and angle between them is x; then discuss the continuity of A as a function of x. Find m and M.

160. Find the distance function d(m) of the foot of the perpendicular from (4, 0) to the line y=mx. Find the domain D and range of this functions.

161. A variable point P on $(x - 2)^2 + y^2 = 4$ is given. Find the sum of the coordinates of P with respect to the line $y = x$, and $y = -x$.

162. If $f(\sqrt{2}x + 3) = x^2 + x$, find $f(x)$.

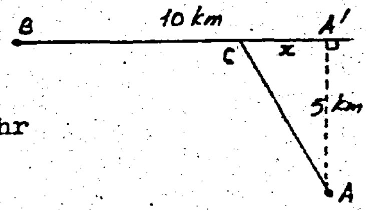
163. If $f(x) = \sqrt{x^2 + 1}$, $g(x) = x/(x^2 + 1)$, find

a) $(f\circ g)(x)$ b) $(g\circ f)(x)$

c) $f^{-1}(x)$ d) $g^{-1}(x)$

164. If $1/p + 1/q = 1$ show that $f(x) = x^{p-1}$ and $g(x) = x^{q-1}$ are inverse functions.

165. Using the data given in the Figure compute the time $t(x)$ for a man walking from A to B via C if the speed from A to C is 2 km/hr and C to B is 3 km/hr.



166. Let $e_1(x)$, $e_2(x)$ be two even and $O_1(x)$, $O_2(x)$ be two odd functions. What can be said about evenness or oddness of

a) $e_2 \circ e_1$ b) $e_1 \circ O_1$

c) $O_1 \circ e_1$ d) $O_2 \circ O_1$?

167. If F, G, H are three given invertible functions and f, g, h are unknown functions defined by $f \circ F = G$, $F \circ g = G$ and $F \circ h \circ G = H$ show that

a) $f = G \circ F^{-1}$ b) $g = F^{-1} \circ G$

c) $h = F^{-1} \circ H \circ G^{-1}$

168. Given $f(x) = \sqrt{x+1}$, $g(x) = \tan^2 x$ and $h(x) = 4x^2$ find the following:

a) $(f \circ g \circ h)(\sqrt{\pi}/4)$ b) $(f \circ h \circ g)(\pi/3)$

c) $(g \circ h \circ f)(3)$ d) $(h \circ f \circ g)(\pi/6)$

169. Prove:

$$\csc \frac{\pi}{7} - \csc \frac{2\pi}{7} - \csc \frac{3\pi}{7} = 0$$

170. Prove

$$\arctan \frac{1}{2} + \arctan \frac{1}{5} + \arctan \frac{1}{8} = \frac{\pi}{4}$$

171. Evaluate the following

a) $\arctan \frac{1}{2} + \arctan \frac{2}{\sqrt{5}}$ b) $\arcsin \frac{3}{5} + \arctan \frac{4}{3}$

c) $\arctan \frac{1}{4} + \arctan \frac{8-\sqrt{15}}{2}$ d) $\arcsin x + \arctan x$

172. Find the period of the following functions, if any:

a) $y = \sin^2 x \cos^2 x$ b) $y = \sin 3x \sin 7x$

c) $y = \frac{x}{3} - [\![\frac{x}{3}]\!]$ d) $y = \tan 3x + \cos 6x$

173. Find the interval in which the given function is monotone increasing:

a) $f(x) = \dfrac{1}{1+x^2}$ b) $f(x) = \sqrt{x}$

c) $f(x) = (2x+3)^3$ d) $f(x) = \dfrac{1}{1-x^2}$

174. Evaluate the limits, if any:

a) $\lim\limits_{x \to 4} \dfrac{\sqrt{x} - 2}{x - 2}$ b) $\lim\limits_{x \to \pi/2} \dfrac{\operatorname{Sin} x - 1}{\cos x}$

175. Evaluate:

a) $\lim\limits_{a \to t} \dfrac{a \operatorname{Sin} t - t \operatorname{Sin} a}{t^2 - a^2}$

b) $\lim\limits_{b \to u} \dfrac{b \cos au - u \cos ab}{b - u}$

c) $\lim\limits_{t \to 0} \dfrac{1}{t^2} \sin t \tan 2t$

d) $\lim\limits_{x \to 0} (1 + ax)^{a/x}$ if $\lim\limits_{x \to 0} (1 + x)^{1/x} = e$

176. Find the limit of f^{-1} when $x \to 2$ if $f(x) = (x+1)/(x-2)$

177. $\lim\limits_{x \to a} \dfrac{\sin(x-a)}{\sin x - \sin a} = ?$

178. Evaluate the following limits, if any:

a) $\lim\limits_{x \to 2^-} \left\{ x[\![x^2 - 1]\!] + [\![x]\!](x^2 - 1) \right\}$

b) $\lim\limits_{x \to 1/2} \dfrac{[\![x]\!]}{2x - 1}$ c) $\lim\limits_{x \to 2/3} \dfrac{3x - 2}{[\![x]\!]}$

d) $\lim\limits_{x \to 0} \left(\dfrac{1}{x} - \dfrac{1}{[\![x]\!]} \right)$

179. Given

$$f(x) = \begin{cases} \sin x & \text{when } -\infty < x < -\pi/2 \\ -1 & \text{when } x = -\pi/2 \\ \cos x - 1 & \text{when } x > -\pi/2 \end{cases}$$

a) find $\lim f(x)$ when $x \to -\pi/2$, if any

b) Is the function continuous at the same point?

180. Given $f(x) = [\![2x - 1]\!]$

a) find D_f b) find the set of x on which $f(x)$ is continuous,

c) Is it monotone? d) find the inverse relation.

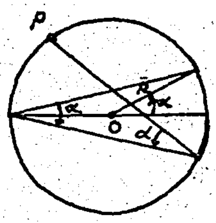
181. Find the set in which $f(x)$ is continuous:

a) $f(x) = \arcsin x + \arcsin x$

b) $f(x) = \arctan x + \text{arccot}\, x$

c) $f(x) = \dfrac{x^2}{|x^2 + 4x| + 3}$ d) $f(x) = \sqrt{\dfrac{[\![x]\!]}{x + 1}}$

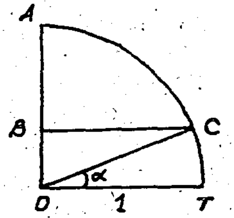
182. From the Figure, find the coordinates of the point P as a function of α, and find the acute angle α for which $x = y$. $(0 < \alpha < \pi/2)$



183. Write the area of an isoceles trapezoid as the

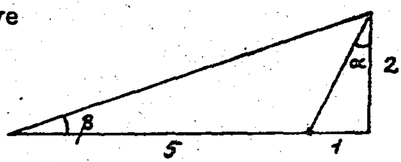
function of base x if upper base a is equal to lateral sides.

184. Find the area $A(\alpha)$ of $\triangle ABC$ as a function of α where C lies on a quarter of circle of radius R, and find m, M if $A(\alpha)$ is continuous.



185. In the given Figure, prove

$$\alpha + \beta = \frac{\pi}{4}.$$



ANSWERS TO EVEN NUMBERED
SELECTED EXERCISES OF CHAPTER 1

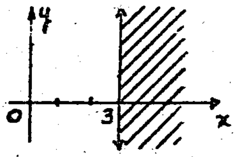
1. 1 (p. 16)

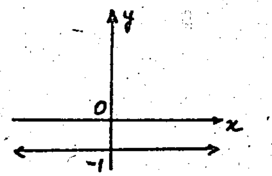
2. a) $(3+\sqrt{2})+(5-\sqrt{2})$, b) $(4+\sqrt{3})(4-\sqrt{3})$

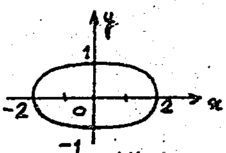
6. Hint: $k \in \mathbb{Z} \Rightarrow 2k$ is even, $2k+1$ is odd.

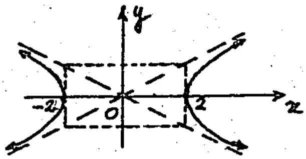
10. Use $d(a, b) = |b - a|$

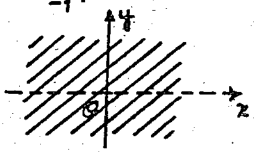
1. 4 (p. 41)

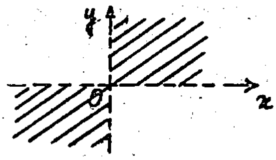
46. a)


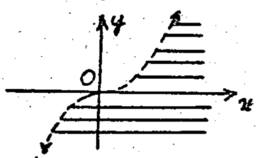
b)


48. a)


b)


50. a)


b)


52. a)


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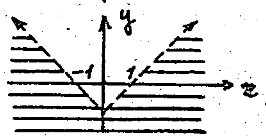


54. a) $y = -2$, b) $2x - y - 3 > 0$

c) $x = \pm\sqrt{9 - y^2}$ d) $x = |9 - y^2|$

1. 5 (p. 62)

60. a) $\mathbb{R} - \{-1, 1\}$, $\mathbb{R} - \{0, 1/2\}$

b) $[-1, 2]$, $[0, \infty)$, c) $\mathbb{R} - [2, 4]$, $\mathbb{R} - [2, 16]$

62. $y = 1/x$, $y = x$.

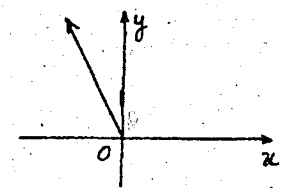
64. a) $[2k\pi - \pi/2, \ 2k\pi + \pi/2]$, increasing;
$[2k\pi + \pi/2, \ 2k\pi + 3\pi/2]$, decreasing.

b) $[k\pi - \pi/2, \ k\pi + \pi/2]$, increasing

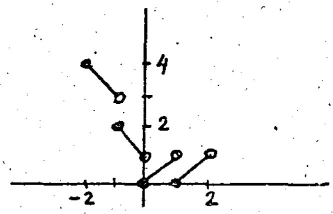
c) $[2k\pi, \ (2k + 1)\pi]$ incr., $[(2k + 1)\pi, \ (2k + 2)\pi]$ decr.

d) $[k\pi, \ k\pi + \pi/2]$ decr., $[k\pi + \pi/2, \ k\pi + \pi]$ incr.

66. a)



b)

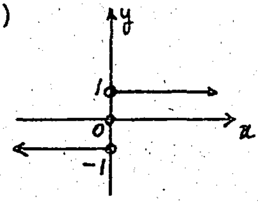


70. a) $y = x + 5$, b) $y = -1$, c) $x = 3$

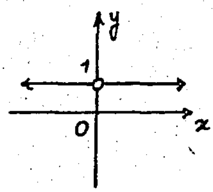
d) $x = y^2 - 1$, not a function e) $y = \cos x$

f) $y = \arcsin x$

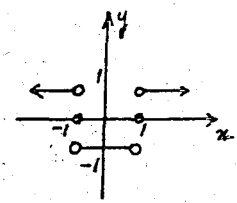
72. a)



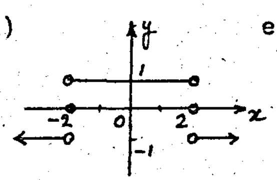
b)



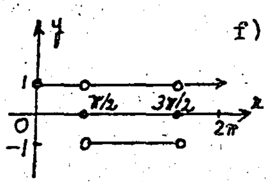
c)



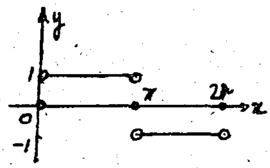
d)



e)



f)



74. a) $(-\infty, -3)$, $(-3, \infty)$, b) $[3\pi/4, 5\pi/4]$, $[5\pi/4, 7\pi/4]$,

c) $(-\infty, -2)$, $[-2, 0]$, $[0, 2]$, $[2, \infty)$

76. $$y = \begin{cases} (x + 1)/3 & \text{when} \quad x \leq -4, \\ 2x/(3 - x) & \text{when} \quad -3 < x < 3 \end{cases}$$

80. a) $\pi/2$, b) 6π, c) 2π, d) $\pi/3$

e) 2, f) 1 g) π h) π

82. $(f \circ g)(x) = \sqrt{(x^2 - 3x + 3)/(x^2 - 4x + 3)}$, $[-1, 1) \cup (3, \infty)$

$(g \circ f)(x) = \sqrt{x + 1} \,/\, (x + 4 - \sqrt{x + 1})$, $\mathbb{R} - \{1, 3\}$

84. $(-\infty, -2]$

1. 6 (p. 85)

88. a) -1, 1, b) 4, 4, c) 6, 4, d) 0, 0

90. a) No limit, b) No limit, c) 0, d) No limit, e) 0

92. a) f: $[\![x]\!]$, g: $-[\![x]\!]$ at $x_0 \in \mathbb{Z}$

b) f: $x + [\![x]\!]$, g: $[\![x]\!]$ at $x_0 \in \mathbb{Z}$

c) f: $[\![x]\!]$, g: $[\![-x]\!]$ at $x_0 \in \mathbb{Z}$

d) f: $\tan x$, g: $\tan x$ at $x_0 \in \mathbb{Z}$

94. a) $\sqrt{8}$, b) 3, c) $\sqrt{7}$, d) -4

96. a) 6, b) 2

104. a) -1, b) 1, c) No limit, d) -1

106. a) 3/2, b) 1/3

1. 7 (p. 99)

112. a) Discontinuous, since $f(2)$ is not defined.

b) continuous, c) discont. at $x = 2$ and $x = 1$.

116. a) Discont. at $x = 0$, at $x = 1$, finite jump at $x = 0$, infinite jump at $x = 1$.

b) Infinite discontinuity at $x = k\pi$ $(k \neq 0)$, and removable discont. at $x = 0$.

c) Removable discont. at $x = 0$, infinite jump at $x = k\pi$ $(k \neq 0)$.

d) Removable discont. at $x = 0$, infinite discont. at $x = (2k + 1)\pi/2$.

118. a) At $x = (2k + 1)\pi/2$, finite jump.

b) At $x = 0, \pm 1$, infinite jump.

c) At x such that $x^2 \in N$, finite jump.

d) At $x = 0$, infinite jump.

120. a) No point of discontinuity.

b) No point of discontinuity.

122. a) $f(x)$ is discontinuous on $[-1, 0]$

b) $m = 3/4$, $M = 7/8$, $x = 5$.

c) $m = -1/4$, $M = 3/40$, $x = 2$

d) $m = 2\sqrt{2}$, $M = 2 + 2/\sqrt{3}$; No x since $1 < m < M$.

124. 7/16, error 1/16.

Chapter 1 (p. 104)

130. a) $x = 0$, $y = -2/5$, b) $x = 1$, $y = 2$; $x = -1/3$, $y = -6$

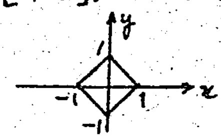
c) $x = 7/4$, $y = -1$, d) $x = -1/16$, $y = -3/16$.

132. a) $y(x - 1) = 0$, b) $x^2 + 2y^2 - x = 0$

134. a) max. 13, min. 8, b) max. 5, min. 0

138. a), b), c), e).

140. a) $[1, 5]$, b) $(-5, 1)$, c) $(-16, 2)$, d) $[2, 16]$

148. a) 

b) $|x| + |y| = 1$.

158. $\ell(x) = \frac{1}{2}(S + \pi x^2)\ell_o$

160. $d(m) = 4m/\sqrt{1 + m^2}$; $D_d = \mathbb{R}$, $R_d = [0, 4]$.

162. $f(x) = (x^4 - 12x^3 + 56x^2 - 120x + 99)/4$.

168. a) $\sqrt{2}$, b) $\sqrt{37}$, c) $\tan^2 16$, d) $16/3$

172. a) $\pi/2$, b) 2π, c) 3, d) $\pi/3$.
Hint: Transform first the given expression into linear form such as $3 \tan 2x - \sin 5x$, and then find the period.)

174. a) 0, b) 0.

176. 5

178. a) 7, b) 0, c) No limit d) No limit

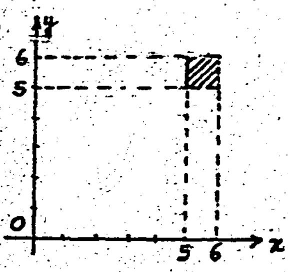
180. a) $\mathbb{R}$, b) $\mathbb{R} - \{x: x = k/2, k \in \mathbb{Z}\}$,
c) yes, d) $x = [2y - 1]$.

182. $x = -R \cos 2\alpha$, $y = R \sin 2\alpha$; $\alpha = 3\pi/8$.

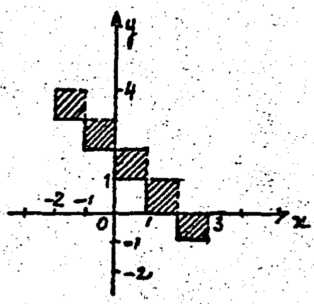
184. $A(\alpha) = \frac{1}{2}(-\sin\alpha)\cos\alpha$; $m = 0$, $M = 1/2$.

150. a). b), d), e).

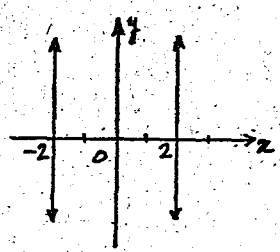
152. a)



b)



154. a)



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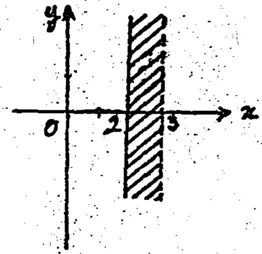
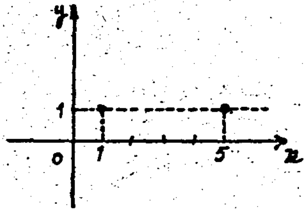


c)



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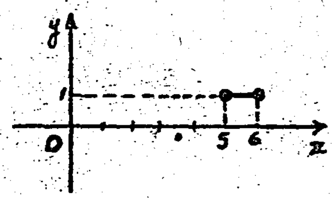
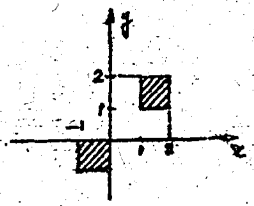
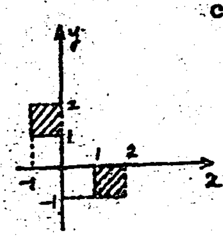


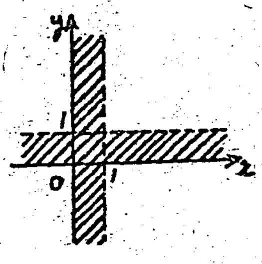
156.



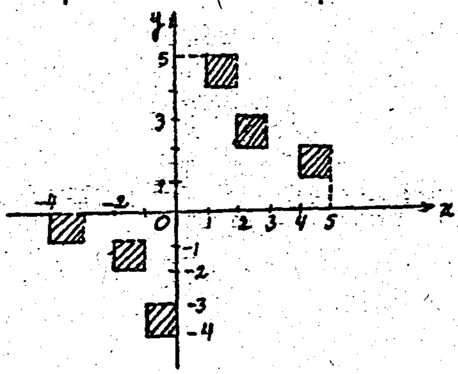
b)



c)



d)



CHAPTER 2

DIFFERENTIATION

2.1 DERIVATIVE

A. Definitions:

Let $f: [a, b] \to R$, $y = f(x)$ be a function defined on a closed interval $[a, b]$ and let $x_0 \in (a, b)$. Letting x be a point in a deleted neighborhood $N(x_0)$ of x_0 we call the difference $\Delta x = x - x_0$ an increment of x, which is positive or negative according as x is on right or left of x_0. If $\Delta x < 0$, it is called a decrement. The increment Δx is a new variable independent of x which may also be denoted by h.

For our function $y = f(x)$, consider the difference

$$\Delta y = f(x_0 + \Delta x) - f(x_0),$$

which may be negative, zero or positive, it is called an increment of y.

Now we form the ratio of increments, namely

$$\frac{\Delta y}{\Delta x} = \frac{\Delta f(x)}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}, (x \neq x_0)$$

called a difference quotient or Newton quotient, and consider limits as Δx tends to 0^-, 0^+ and 0, if any, and if all exist they are respectively called the left derivative the right derivative and the derivative of f at x_0. So in case

$$\lim_{x \to 0^-} \frac{\Delta y}{\Delta x}, \quad \lim_{x \to 0^+} \frac{\Delta y}{\Delta x}$$

exist and equal to each other, the common limit

$$\lim_{x \to 0} \frac{f(x_o + \Delta x) - f(x_o)}{\Delta x}$$

is the <u>derivative</u> of f at x_o, and f is said to be <u>differentiable</u> at x_o.

If h is used for Δx, that is, if $h = \Delta x = x - x_o$, the derivative is also expressed as

$$\lim_{h \to 0} \frac{f(x_o + h) - f(x_o)}{h} \quad \text{or} \quad \lim_{x \to x_o} \frac{f(x) - f(x_o)}{x - x_o}$$

<u>Example</u>. Find the derivative, if any, of the given function at indicated point:

1) $y = x^2$, $x_o = 2$ 2) $f(x) = |x|$, $x_o = 0$

<u>Solution</u>:

1) $\lim_{h \to 0} \frac{(2 + h)^2 - 2^2}{h} = \lim_{h \to 0} \frac{4h + h^2}{h} = \lim_{h \to 0} (4+h) = 4,$

2) $\lim_{h \to 0} \frac{|0 + h| - 0}{h} = \lim_{h \to 0} \frac{|h|}{h} = \begin{cases} -1 & \text{when } h > 0 \\ 1 & \text{when } h < 0 \end{cases}$

The left derivative is -1, right derivative is 1 at $x_o = 0$ but there is no derivative at that point.

If $f: [a, b] \to \mathbb{R}$ has derivative at every point x of (a, b) is said to be <u>differentiable</u> in (a, b). The differentiability of f at the end points a, b is defined by the right and left derivatives respectively.

Notation:

$D[a, b] = \{f: f \text{ is differentiable on } [a, b]\}$,

$D(a, b) = \{f: \ f \text{ is differentiable in } (a, b)\}$

If $y = f(x) \in D[a, b]$, the derivative (the <u>first derivative</u>) of f at $x \in [a, b]$ is denoted in various ways:

y' (y prime), f', $\frac{dy}{dx}$ (dy by dx), $\frac{df}{dx}$, $\frac{d}{dx} y$ (d by dx, y), $\frac{d}{dx} f$,

$D_x y$ (D sub x, y), $D_x f$, f_1 (f one). ($D_x = d/dx$)

and the derivative at a particular point x_0 is written as follows:

$$(f'(x))_{x = x_0} \quad \text{or} \quad f'(x)\Big|_{x = x_0}, \ f'(x_0), \ D_x f(x)_{x = x_0}, \ \ldots$$

<u>Example</u>. Find the derivative function of the following functions and find the set on which derivatives exist.

a) $y = x^2$ b) $y = \sqrt{x}$ c) $y = x^n$, $n \in N$

<u>Solution</u>.

a) $\Delta y = (x + \Delta x)^2 - x^2 = 2x\,\Delta x + (\Delta x)^2$

$$\Rightarrow \quad \frac{\Delta y}{\Delta x} = 2x + \Delta x \Rightarrow \frac{dy}{dx} = 2x \ ; \quad (-\infty, \infty)$$

b) $\dfrac{\Delta f}{\Delta x} = \dfrac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} = \dfrac{(\sqrt{x + \Delta x} - \sqrt{x})(\sqrt{x + \Delta x} + \sqrt{x})}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})}$

$$\frac{df}{dx} = \frac{1}{2\sqrt{x}} \ ; \quad (0, \infty)$$

c) $\Delta y = (x + \Delta x)^n - x^n = nx^{n-1}\Delta x + (\Delta x)^2(\ldots) \Rightarrow$

$$\frac{\Delta y}{\Delta x} = nx^{n-1} + \Delta x(\ldots) \Rightarrow \frac{dy}{dx} = nx^{n-1} \ ; \ (-\infty, \infty)$$

<u>Example</u>. Same question for:

$$\lim_{x\to x_o} f(x) = \lim_{x\to x_o} f(x_o) + \lim_{x\to x_o} \frac{f(x) - f(x_o)}{x - x_o} .$$

$$\lim_{x\to x_o} (x - x_o) = f(x_o) + f'(x_o) \cdot 0 = f(x_o)$$

and it follows from the definition of continuity of f at x_o, that f is continuous at x_o. Since x_o is taken arbitrarily on $[a, b]$, f is continuous on $[a, b]$.

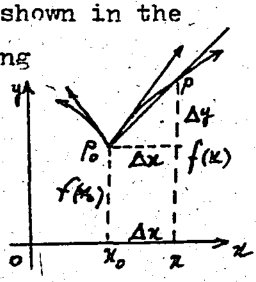
We note that the converse of this theorem is not true, that is, a function continuous at a point may not be differentiable at that point. Indeed, $f(x) = |x|$ is continuous at 0 without being differentiable at 0.

B. <u>Geometric and physical interpretations of the first derivatives.</u>

<u>Geometric interpretation:</u>

Let $f: \mathbb{R} \to \mathbb{R}$ be a function with graph as shown in the figure. The right derivative of $f(x)$ at x_o being

$$\lim_{\substack{x\to x_o \\ (x > x_o)}} \frac{f(x) - f(x_o)}{x - x_o} = \lim_{\substack{\Delta x\to 0 \\ (\Delta x > 0)}} \frac{\Delta y}{\Delta x},$$



where $\frac{\Delta y}{\Delta x}$ is the slope of the line passing through $P_o = (x_o, f(x_o))$ and nearby point $P = (x, f(x))$ on the curve, it represents the slope of the limiting line through P_o as P approaches P_o. This limiting line is the <u>right tangent line</u> of $f(x)$ at x_o.

Similarly the left derivative of $f(x)$ at x_o is the slope of the <u>left tangent line</u> at x_o. If both left and right derivatives exist and equal to each other at x_o the derivative

a) $f(x) = 3x\,|x + 1|$ b) $g(x) = [\![x]\!]\, x^2$

Solution.

a)

$$f(x) = \begin{cases} -3x^2 - 3x & \text{when } x \leqslant -1 \\ 3x^2 + 3x & \text{when } x > -1 \end{cases} \Rightarrow f'(x) = \begin{cases} -6x-3 & \text{when } x \leqslant -1 \\ 6x + 3 & \text{when } x > -1 \end{cases}$$

$f'(x)$ exists when $x < -1$ or $x > -1$, since they are polynomial functions of x, but at $x = -1$, $\lim\limits_{x \to (-1)^-} f'(x) = 3$ and $\lim\limits_{x \to (-1)^+} f'(x) = -3$ limit does not exist. Then $f(x)$ is differentiable on $(-\infty, -1) \cup (-1, \infty)$.

b)

$$g(x) = nx^2 \quad \text{when} \quad n \leqslant x < n + 1, \quad n \in \mathbb{Z}$$

$$g'(x) = 2nx \quad \text{when} \quad n \leqslant x < n + 1 \quad n \in \mathbb{Z}$$

Derivative exists when $n < x < n + 1$, $n \in \mathbb{Z}$. but

$$\lim_{x \to n^-} g'(x) = 2(n - 1)x, \qquad \lim_{x \to n^+} g'(x) = 2n\,x$$

are not the same except at $x = 0$.

Then derivative exists on the set $\mathbb{R} - \mathbb{Z} + \{0\}$

<u>Theorem. $f \in D[a, b] \Rightarrow f \in C[a, b]$, that is, a differentiable function is a continuous function.</u>

<u>Proof.</u> Let $x_0 \in [a, b]$ and $x \in N(x_0)$. Since $x \neq x_0$, the identity

$$f(x) = f(x_0) + \frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0)$$

holds. Taking limit of each side when $x \to x_0$, we have exists at x_0 and it is the slope of the unique tangent line at x_0. Consequently, if $f(x)$ has derivative at x_0, then the curve represented by this function has (unique) tangent line with slope $f'(x_0)$, and normal line with slope $-1/f'(x_0)$. Then we have the equations of tangent and the normal lines through $(x_0, f(x_0))$ with known slope :

$$y - f(x_0) = f'(x_0).(x - x_0)$$

and

$$y - f(x_0) = -\frac{1}{f'(x_0)} \cdot (x - x_0)$$

The slope of the tangent line at a point is the slope of the curve $y = f(x)$ at the same point. Let α be the angle between the positive x-axis and the tangent line at x_0. Then $\tan\alpha = f'(x_0)$ where $\alpha = \arctan f'(x_0)$ is the slope angle of the curves at x_0.

The angle between two curves at a certain common point is defined to be the angle between the tangent lines at this point. Two curves intersect each other orthogonally at a point if the angle between them is 90^0 at that point, and they are tangent at x_0 if the angle of intersection is zero at x_0.

Example. Find the equations of tangent and normal lines at the points of intersects of the following curves of $y = f(x) = x^2$, $y = g(x) = \sqrt{8x}$, and the angles between their point of intersection.

Solution. Equating y's we have $x^4 = 8x \Rightarrow x_1 = 0$, $x_2 = 2 \Rightarrow y_1 = 0$, $y_2 = 4$ and the points of intersection are $O = (0, 0)$, $A = (2. 4)$.

$$f'(x) = 2x \Rightarrow f'(0) = 0, \quad f'(2) = 4$$

$$g'(x) = \frac{\sqrt{2}}{\sqrt{x}} \Rightarrow f' \text{ is not defined at } 0, \text{ but}$$

for $x \to 0^+$, having $g'(x) \to \infty$ we say that the slope of $f(x)$ at 0 is ∞ meaning geometrically that the tangent line at $(0, 0)$ is perpendicular to x-axis. $g'(2) = 1$.

Tangent line at $(0, 0)$ to $f(x)$: $y = 0$

Normal line at $(0, 0)$ to $f(x)$: $x = 0$

Tangent line at $(2, 4)$ to $g(x)$: $y - 4 = 1.(x-2)$

Normal line at $(2, 4)$ to $g(x)$: $y - 4 = -(x - 2)$

As to the angles, at the origin $\alpha_1 = \pi/2$, and at A

$$\tan \alpha_2 = \frac{4 - 1}{1 + 4} = 3/5 \Rightarrow \alpha_2 = \arctan 3/5$$

Physical interpretation (Rectilinear motion):

Let $s = s(t)$ be a function of time t representing the distance s travelled of a moving particle on a straight line. Then in a time interval $[t_1, t_2] \in D_s$ (domain of), we have the average velocity:

$$\bar{v} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

To find the instantaneous velocity $v(t_o)$ at $t_o \in D_s$ we form the Newton quotient

$$\frac{\Delta s}{\Delta t} = \frac{s(t_o + \Delta t) - s(t_o)}{\Delta t},$$

then

$$v(t_o) = \lim_{\Delta t \to 0} \frac{s(t_o + \Delta t) - s(t_o)}{\Delta t}$$

if exists. At any instant t the velocity is

$$v(t) = s'(t) = \frac{ds}{dt}$$

The absolute value $|v(t)|$ is the speed of the particle. In the same manner we have

$$\bar{a} = \frac{v(t_2) - v(t_1)}{t_2 - t_1}, \quad a(t) = v'(t)$$

as the average acceleration in time interval $[t_1, t_2]$ and instantaneous acceleration at time t.

Example. If the equation of motion on a line is given by

$$s = t^3 - 27t \text{ cm/sec},$$

find the following:

a) average velocities in the time intervals $[0, 3]$ and $[0. 3\sqrt{3}]$.

b) $v(t)$ at $t_1 = 3$, $t_2 = 3\sqrt{3}$,

c) s from $t = 0$ to $t = 3$, and from $t = 0$ to $t = 3\sqrt{3}$,

d) absolute distance travelled by the particle in the intervals $[0. 3]$, $[0, 3\sqrt{3}]$.

e) Find the time where the acceleration is 12 cm/sec^2.

Solution:

a) $s(0) = 0$ cm, $s(3) = -54$ cm, $s(3\sqrt{3}) = 0$ cm $\Rightarrow$

$$\bar{v} = \frac{-54 - 0}{3 - 0} = -18 \text{ cm/sec} \quad \text{in} \quad [0, 3],$$

$$\bar{v} = \frac{0 - 0}{3\sqrt{3} - 0} = 0 \text{ cm/sec} \quad \text{in} \quad [0, 3\sqrt{3}].$$

b) $v(t) = 3t^2 - 27 \Rightarrow v(3) = 0$ cm/sec, $v(3\sqrt{3}) = 54$ cm/sec.

c) $s = -54 - 0 = -54$ cm from 0 to 3; $s = 0$ cm from 0 to $3\sqrt{3}$.

d) $|s| = 54$ cm from 0 to 3; $|s| = 108$ cm from 0 to $3\sqrt{3}$.

e) $a = \frac{dv}{dt} = 6t \Rightarrow 6t = 12 \Rightarrow t = 2$ sec.

Heigher order derivatives:

Let $f: [a, b] \to R$ be a differentiable function. If f' is differentiable, its derivative is the second (order) derivative of f, written:

y'' (y double prime), f'', $\frac{d^2y}{dx^2}$ (d^2y by dx square),

$\frac{d^2f}{dx^2}$, $\frac{d^2}{dx^2}y$ (d^2 by $(dx)^2$, y), $\frac{d^2}{dx^2}f$,

$D_{xx}y$ (D sub xx),

$D_{xx}f$, f_{11}(f one one). $\left(D_x^2 = \frac{d}{dx}\frac{d}{dx} = \frac{d^2}{dx^2}\right)$

If f'' is differentiable one talks about the third (order) derivative f''' of f. In general n^{th} order derivative of f is denoted by the symbols

$$y^{(n)},\ f^{(n)},\ \frac{d^ny}{dx^n},\ \frac{d^nf}{dx^n},\ \frac{d^n}{dx^n}y,\ \frac{d^n}{dx^n}f.$$

In analogy with $f \in D[a, b]$ the symbols

$$f \in D^n[a, b], \quad f \in D^n(a, b)$$

have obvious meanings.

C. Theorems on derivatives.

Theorem 1. If $f \in D[a, b]$ and c is a constant, then

1) $(cf(x))' = c\,f(x)$ 2) $\left(\frac{1}{f(x)}\right)' = -\frac{f'(x)}{-f^2(x)}$

3) $|f(x)|' = \begin{cases} -f'(x) & \text{when } f(x) \leqslant 0 \\ f'(x) & \text{when } f(x) \geqslant 0 \end{cases}$ or $|f(x)|' = f'(x)\frac{|f(x)|}{f(x)}$

4) $[\![f(x)]\!]' = \begin{cases} 0 & \text{when } n < f(x) < n+1, \quad n \in \mathbb{Z} \\ \text{Does not exist when} & f(x) = n \in \mathbb{Z} \end{cases}$

Proof.

1) It is a direct consequence of the definition f derivative

2) $\left(\frac{1}{f(x)}\right)' = \lim\limits_{h \to 0} \frac{1}{h}\left(\frac{1}{f(x+h)} - \frac{1}{f(x)}\right)$

$$= -\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \lim_{h \to 0} \frac{1}{f(x+h)f(x)} =$$

$$= -\frac{f'(x)}{f^2(x)}.$$

3) $|f(x)| = \begin{cases} -f(x), & f(x) \leqslant 0 \\ f(x), & f(x) > 0 \end{cases} \Rightarrow |f(x)|' = \begin{cases} -f'(x), & f(x) \leqslant 0 \\ f'(x), & f(x) > 0 \end{cases}$

$$\Rightarrow |f(x)|' = \frac{|f(x)|}{f(x)}\,f'(x).$$

4) $[\![f(x)]\!]' = 0$ when $n < f(x) < n + 1$, $n \in \mathbb{Z}$, since $[\![f(x)]\!]$ = constant under the conditions.

In case $f(x) = n$, $n \in \mathbb{Z}$, the derivative does not exists, since $[\![f(x)]\!]$ is discontinuous at $x = n$.

Theorem 2. If f, g are differentiable functions on a common domain, then

1) $(f + g)' = f' + g'$ 2) $(f - g)' = f' - g'$

3) $(f \cdot g)' = f'g + fg'$ 4) $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ $(g(x) \neq 0)$

Proof.

1) $$(f + g)'(x) = \lim_{h \to 0} \frac{(f + g)(x + h) - (f + g)(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x + h) + g(x + h) - f(x) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) + g'(x) = (f' + g')(x).$$

$$(f + g)' = f' + g'$$

2) It can be proved as in (1).

3) $$(fg)'(x) = \lim_{h \to 0} \frac{(fg)(x + h) - (fg)(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x + h)\ g(x + h) - f(x)\ g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x + h)\ g(x + h) - f(x)\ g(x + h)}{h} +$$

$$\lim_{h \to 0} \frac{f(x)\ g(x + h) - f(x)\ g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \lim_{h \to 0} g(x+h) +$$

$$\lim_{h \to 0} f(x) \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x)\, g(x) + f(x)\, g'(x) = (f'g)(x) + (fg')(x)$$

$$= (f'g + fg')(x).$$

4. $(\frac{f}{g})'(x) = (f \cdot \frac{1}{g})'(x) = (f' \frac{1}{g} + f(\frac{1}{g})')(x)$

$$= (\frac{f'}{g} - \frac{fg'}{g^2})(x) = \frac{f'g - fg'}{g^2}(x)$$

$$(\frac{f}{g})' = \frac{f'g - fg'}{g^2}. \blacksquare$$

This theorem, if expressed in terms of the symbol $D = D_x = d/dx$, takes the form:

1) $D(f + g) = Df + Dg$ 2) $D(f - g) = Df - Dg$

3) $D(fg) = (Df)g + f\,Dg$ 4) $D\,\frac{f}{g} = \frac{(Df)g - f\,Dg}{g^2}$

where D is called the <u>derivative operator</u>. An operator in general is a mapping from functions to functions so that the operator D maps differentiable functions to functions. For instance if $f(x) = x^2$ and $g(x) = 2x$, then $Df(x) = g(x)$.

The derivative operator D has, in particular, the two properties

i) $D(f + g) = Df + Dg$ ii) $D(cf) = c\,Df$ (c is a constant)

An operator having these two properties is called a <u>linear operator</u>. Hence D is a linear operator. If D^0 is defined to be the identity operator I with the property $I(f) = f$, the following are all linear operators:

$$D^0(=I),\ D,\ D^2,\ D^3,\ \ldots,\ D^n,\ D^{n+1} = D\, D^{n-1},\ \ldots$$

We define the sum $D^p + D^q$ and $c\, D^p$ by the equalities

$$(D^p + D^q)f = D^p f + D^q f \quad \text{for all} \quad f \in D^\infty$$

$$(cD)f = c\, Df$$

The operators $D^p + D^q$ and $c\, D^p$ salisfy (i) and (ii) and hence are linear operators.

<u>Example</u>. Apply the linear operator $2D^2 - D + 3$ to the function $f(x) = \sqrt{x} - x^2$. (Here 3 means $3I$).

$$(2D^2 - D + 3)(\sqrt{x} - x^2) = (2D^2)(\sqrt{x} - x^2) - D(\sqrt{x} - x^2) + 3(x - x^2)$$

$$= 2D^2\sqrt{x} - 2D^2 x^2 - D\sqrt{x} + Dx^2 + 3\sqrt{x} - 3x^2$$

$$= 2\, D\left(\frac{1}{2\sqrt{x}}\right) - 2.2 - \frac{1}{2\sqrt{x}} + 2 + 3\sqrt{x} - 3x^2$$

$$= -\frac{1}{2}\ \frac{1}{x\sqrt{x}} - 2 - \frac{1}{2\sqrt{x}} + 3\sqrt{x} - 3x^2.$$

<u>Theorem 3</u>. (<u>Chain rule</u>) <u>Let $f \circ g$ be a composite function. If f, g are differentiable on their domains, then $f \circ g$ is differentiable on its domain and the derivative is given by</u>

$$(f \circ g)'(x) = f'(g(x))\ g'(x)$$

<u>Proof</u>. Setting $y = (f \circ g)(x) = f(g(x))$ we have

$$\frac{dy}{dx} = \lim_{x \to 0} \frac{f(g(x + \Delta x)) - f(g(x))}{\Delta x} = \lim_{x \to 0} \frac{f(g(x) + \Delta g(x)) - f(g(x))}{\Delta x}$$

$$= \lim_{x \to 0} \left(\frac{f(g(x) + \Delta g(x)) - f(g(x))}{\Delta x} \frac{\Delta g(x)}{\Delta g(x)}\right.$$

$$= \lim_{x \to 0} \left(\frac{f(g(x) + \quad g(x)) - f(g(x))}{\Delta g(x)} \frac{\Delta g(x)}{\Delta x}\right)$$

where setting $u = g(x)$, we have $\Delta g(x) = \Delta u \to 0$ as $\Delta x \to 0$ from the differentiability of $g(x)$, and

$$\frac{dv}{dx} = \lim_{\Delta u \to 0} \frac{f(u + \Delta u) - f(u)}{\Delta u} \lim \frac{\Delta u}{\Delta x}$$

$$= f'(u) \cdot \frac{du}{dx} = f'(g(x)) \cdot g'(x). \blacksquare$$

As a first extension of this theorem we have the chain rule for $y = (f \circ g \circ h)(x) = f(g(h(x))$, namely

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

where $y = f(u)$, $u = g(x)$, $v = h(x)$, or

$$(f \circ g \circ h)'(x) = f'(g(h(x)) \cdot g'(h(x)) \cdot h'(x)$$

This theorem and its generalitations are extremely useful in differentiating some functions of complicated nature.

Example. Differentiate the functions

a) $y = \sqrt{x^2 - 3x + 2}$ b) $y = (x^2 - 3x + 2)^3$

Solution.

a) We think of y as the function of function $f(g(x))$ where $y = f(x) = \sqrt{u}$, $u = x^2 - 3x + 2$. Then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{u}} \cdot (2x - 3) = \frac{2x - 3}{2\sqrt{x^2 - 3x + 2}}$$

b) Again setting $y = u^3$, $u = x^2 - 3x + 2$ we have

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= 3u^2 \cdot (2x - 3) = 3(x^2 - 3x + 2)^2 \cdot (2x-3)$$

Corollary. If $f: \mathbb{R} \to \mathbb{R}$ is differentiable and invertible in an interval, then f^{-1} is differentiable and

$$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$$

Proof. Since $(fof^{-1})(x) = I(x) = x$, we have by above theorem

$$1 = ((fof^{-1})(x))' = f'(f^{-1}(x)) \cdot (f^{-1}(x))',$$

and then the desired result.

Example. If $f(x) = \frac{x}{x + 3}$, find $\left.\frac{d}{dx} f^{-1}(x)\right|_{x = 1/4}$

Solution 1. The inverse of the given function being

$$f^{-1}(x) = \frac{3x}{1 - x}$$

by direct differentiation we have

$$\frac{d}{dx} f^{-1}(x) = \frac{3}{(1-x)^2}, \quad \frac{16}{3} \quad \text{at} \quad x = 1/4.$$

Solution 2. By the use of corollary: $\frac{x}{x + 3} = \frac{1}{4} \Rightarrow x = 1,$

$f: 1 \to \frac{1}{4}$ $\qquad f^{-1}: \frac{1}{4} \to 1.$

$$f'(x) = \frac{3}{(x+3)^2} \Rightarrow f'(f^{-1}(\tfrac{1}{4})) = f'(1) = \frac{3}{16} \Rightarrow \frac{d}{dx} f^{-1}(x)\Big|_{x=\frac{1}{4}} = 16/3$$

Implicit differentiation.

If a function $y = y(x)$ is implicitly defined as in

$$x^2y^2 - 5xy + (x^2 - 2x) = 0, \quad x \sin y + \cos(x + y) = 0$$

the derivative y' of a theoretically existing unknown function is obtained by term-by-term differentiation yieding a relation solvable for y' as $y'(x, y)$.

Example. Find the derivative of the functions

$x^2y^2 + 5xy - 2(x^2 - 6x) = 0$ at $A = (1, -2)$, and find $y'(x, y)$.

Solution. Differentiating every term with respect x keeping in mind that y is function of x we have

$$2x\, y^2 + x^2(2yy') + 5y + 5xy' - 2(2x - 6) = 0$$

$$(2x^2y + 5x)y' + (2xy^2 + 5y - 4x + 12) = 0.$$

Setting now $x = 1$, $y = -2$, we get

$$(-4 + 5)\, y'(1) + (8 - 10 - 4 + 12) = 0 \Rightarrow y'(1) = -6$$

For $y'(x, y)$ we have

$$y' = -\frac{2xy^2 + 5y - 4x + 12}{2x^2y + 5x}$$

Corollary. $D x^r = r x^{r-1} \quad (r \in \mathbb{Q})$

Proof. This was proved earlier for $r \in Z_o^+ = N$.
If $r \in Z^-$, set $r = -n$ with $n \in N$. Then

$$D x^r = D x^{-n} = D \frac{1}{x^n} = \frac{-n x^{n-1}}{x^{2n}} = -n x^{-n-1} = r x^{r-1}.$$

Let now $r = p/q \in \mathbb{Q}$ with $q > 0$. Setting $y = x^{p/q}$ we have $y^q = x^p$ which when differentiated implicitly gives $q y^{q-1} y' = p x^{p-1}$ or

$$Dx^r = y' = \frac{p}{q} \frac{x^{p-1}}{y^{q-1}} = \frac{p}{q} \frac{x^{p-1} y}{y^q} = \frac{p}{q} \frac{x^{p-1} \cdot x^{p/q}}{x^p}$$

$$= \frac{p}{q} x^{p/q-1} = r x^{r-1} . \blacksquare$$

That the corollary holds also for $r \in \mathbb{R}$ will be proved in a later chapter.

Derivative of a parametric function.

Consider two functions $x = x(t)$, $y = y(t)$. When t is eliminated between x and y (which is theoretically possible) one obtains a relation between x and y giving rise to an implicitly defined function. Such a function $y = y(x)$, defined by the pair

$$\begin{aligned} x &= x(t) \\ y &= y(t), \end{aligned} \qquad \text{(a)}$$

where t is a parameter, is called a parametric function. The pair (a) may define one or more functions each of which is a parametric function.

Theorem. If $x(t)$, $y(t)$ are differentiable functions, then the parametric function defined by (a) is differentiable and the derivative is given by

$$\frac{dy}{dx} = \frac{\dot{y}(t)}{\dot{x}(t)}, \qquad (\dot{x}(t) \neq 0)$$

where $\dot{x}(t)$, $\dot{y}(t)$ are the derivatives of $x(t)$, $y(t)$ with respect to their arguments t.

Proof. Since $x(t)$, $y(t)$ are differentiable with respect to t we have

$$\dot{x}(t) = \lim_{t\to 0} \frac{\Delta x}{\Delta t}, \quad \dot{y}(t) = \lim_{t\to 0} \frac{\Delta y}{\Delta t}$$

so that $\Delta x \to 0$, $\Delta y \to 0$ when $\Delta t \to 0$, and

$$\frac{dy}{dx} = \lim_{x\to 0} \frac{\Delta y}{\Delta x} = \lim_{t\to 0} \frac{\Delta y/\Delta t}{\Delta x/\Delta t} = \frac{\lim\limits_{t\to 0} \Delta y/\Delta t}{\lim\limits_{t\to 0} \Delta x/\Delta t} = \frac{\dot{y}(t)}{\dot{x}(t)}. \quad \blacksquare$$

Example. Write the equation of tangent line to the curve of the parametric function $x = t^2 - 2t$, $y = 3t + 2$ at the point $A = \left(-\frac{3}{4}, 7/2\right)$ corresponding to $t = \frac{1}{2}$.

Solution:

$$\dot{x} = 2t - 2, \quad \dot{y} = 3 \Rightarrow dy/dx = \dot{y}/\dot{x} = \frac{2}{3}(t - 1) \Rightarrow \left.\frac{dy}{dx}\right|_{t=1} = -1/3$$

Since $m = -1/3$ the equation of the tangent line at A is

$$y - \frac{7}{2} = -\frac{1}{3}\left(x + \frac{3}{4}\right)$$

D. Derivatives of trigonometric and inverse trigonometric functions.

Theorem. The derivatives of trigonometric functions on their respective domains are

1. $D \sin x = \cos x$ 2. $D \cos x = -\sin x$

3. $D \tan x = \sec^2 x$ 4. $D \cot x = -\csc^2 x$

5. $D \sec x = \sec x \tan x$ 6. $D \csc x = -\csc x \cot x$

Proof. $$D \sin x = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{2 \sin \frac{h}{2} \cos(x + \frac{h}{2})}{h}$$

$$= \lim_{h \to 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}} \lim_{h \to 0} \cos(x + \frac{h}{2}) = 1.$$

$$= 1.\cos x = \cos x.$$

2. $D \cos x = D \sin(\frac{\pi}{2} - x) = \cos(\frac{\pi}{2} - x).D(-x) = -\sin x.$

3. $$D \tan x = D \frac{\sin x}{\cos x} = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x.$$

4. Proved as in (3).

5. $D \operatorname{Sec} x = D \frac{1}{\cos x} = -\frac{-\sin x}{\cos^2 x} = \frac{1}{\cos x} \frac{\sin x}{\cos x} = \sec x \tan x.$

6. Proved as in (5). ■

Theorem. The derivatives of the inverse trigonometric functions on their respective fundamental domains and rastricted ranges are

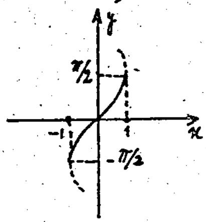
1. $D \arcsin x = \dfrac{1}{\sqrt{1-x^2}}$ 2. $D \arccos x = -\dfrac{1}{\sqrt{1-x^2}}$

3. $D \arctan x = \dfrac{1}{1+x^2}$ 4. $D \operatorname{arccot} x = -\dfrac{1}{1+x^2}$

5. $D \operatorname{arcsec} x = \dfrac{1}{|x|\sqrt{x^2-1}}$ 6. $D \operatorname{arccsc} x = -\dfrac{1}{|x|\sqrt{x^2-1}}$

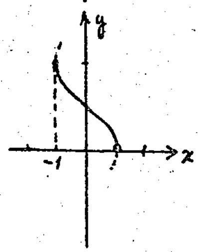
Depending on the restriction, the above fractions involving radical may have opposite sign. Discuss the cases.

<u>Proof</u>. 1. Setting $y = \arcsin x$ with $x \in [-1, 1]$, $y \in [-\pi/2, \pi/2]$, we have $x = \sin y$ which when differentiated implicitly with respect to x gives $1 = (\cos y)y'$ $(\cos y > 0)$ or



$$y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

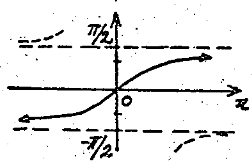
2. $y = \arccos x$ $(x \in [-1, 1],\ y \in [0, \pi])$



$$\Rightarrow x = \cos y \Rightarrow 1 = (-\sin y)y', \ (\sin y > 0)$$

$$y' = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-\cos^2 y}} = -\frac{1}{\sqrt{1-x^2}}$$

3. $y = \arctan x$ $(x \in (-\infty, \infty),\ y \in (-\frac{\pi}{2}, \frac{\pi}{2}))$



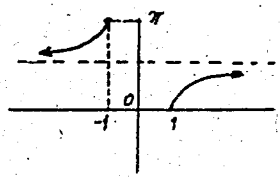
$$\Rightarrow x = \tan y \Rightarrow 1 = (\sec^2 y)\, y'$$

$$\Rightarrow y' = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

4. Proved similarly.

5. $y = \operatorname{arcsec} x \quad (x \in (-\infty, -1] \cup [1, \infty),$

$\qquad y \in [0, \pi/2) \cup (\pi/2, \pi])$



$\Rightarrow \; x = \sec y \Rightarrow 1 = (\sec y \tan y)\, y'$

$(\sec y = x, \quad \tan y > 0)$

$$y' = \frac{1}{\sec y \tan y} = \frac{1}{x\sqrt{\sec^2 x - 1}} = \frac{1}{|x|\sqrt{x^2 - 1}}$$

6. Proved similarly. ■

<u>Example.</u> Find the derivatives of the following functions at indicated points:

a) $y = \sin \dfrac{x}{x+2}, \quad x = 0$

b) $y = \arccos \dfrac{x}{2}, \quad A = (0, \dfrac{\pi}{2}), \quad B = (0, \; 3\pi/2)$

<u>Solution:</u>

a) $y' = (\cos \dfrac{x}{x+2}) \dfrac{2}{(x+2)^2} \Rightarrow y'(0) = 1/2$

b) $$y' = -\frac{1}{\sqrt{1 - (\frac{x}{2})^2}} \cdot \frac{1}{2} = \begin{cases} -\dfrac{1}{2} & \text{at} \quad A \\[2ex] \dfrac{1}{2} & \text{at} \quad B \end{cases}$$

E X E R C I S E S (2. 1)

1. Find the derivative, if any, of the following functions at the indicated point by Newton quotient:

   a) $y = x^3, \quad x = -2$ b) $y = \sqrt{2x}, \quad x = 0$

   c) $y = [\![x^2 + 5x]\!], \quad x = 3$ d) $y = [\![x^2 + 5x]\!], \quad x = 2/3$

2. Same question for:

a) $y = \dfrac{x}{x^2+1}$, $x = 1$ b) $y = \dfrac{x^2}{x-1}$, $x = 0$

c) $y = |x^2 - 5|$, $x = 2$ d) $y = |x^2 - 5|$, $x = \sqrt{5}$

3. Find the derivative of each function by difference quotient, and discuss their existence:

a) $y = \begin{cases} x^2 & \text{when } x < 1 \\ 2x & \text{when } x \geq 1 \end{cases}$ b) $y = \begin{cases} x+1 & \text{when } x \leq 0 \\ x+2 & \text{when } x > 0 \end{cases}$

4. Express the following limits in terms of $f'(x)$:

a) $\lim\limits_{h \to 0} \dfrac{f(x-h) - f(x)}{h}$ b) $\lim\limits_{h \to 0} \dfrac{f(x+2h) - f(x)}{h}$

c) $\lim\limits_{h \to 0} \dfrac{f(x+2h - f(x+h)}{h}$ d) $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x-h)}{h}$

e) $\lim\limits_{h \to 0} \dfrac{f(x+h+h^2) - f(x)}{h}$ f) $\lim\limits_{h \to 0} \dfrac{f(x+ah) - f(x+bh)}{h}$

5. Let $f(x) = x^2 + |x|$. Does $f'(0)$ exist? Does $f'(x)$ for $x \neq 0$ exist?

6. Find a polynomial of least degree with the given conditions:

a) $P(1) = 0$, $P'(1) = 2$, $P''(0) = 2$

b) $Q(-1) = 1$, $Q'(0) = 1$, $Q''(2) = 0$, $Q'''(0) = -6$

7. Find y' by Newton quotient if $y = \arcsin x$. (Hint. use $x = \sin z$).

8. If f $D^2(-\infty, \infty)$, g $D^3(-\infty, \infty)$, then prove

a) $\lim\limits_{h \to 0} \dfrac{f(x+2h) - 2f(x+h) + f(x)}{h^2} = f''(x)$

b) $\lim\limits_{h \to 0} \dfrac{g(x + 3h) - 2g(x + 2h) + 2g(x + h) - g(x)}{h^3} = g'''(x)$

9. Find, if any, the equation of the tangent and normal line to the curve of,

a) $y = x^2 + |x|$ at $x = -2$, $x = 0$

b) $y = [\![x]\!] \; x^2 - 3x$ at $x = 0$, $x = 3/2$

c) $y = \sqrt{x}$, at $x = 1$, $x = 0$

d) $y = \sqrt[3]{x}$, at $x = -8$.

10. Find constants a, b for $f(x) = \dfrac{ax + 5}{x + b}$ if $f(1) = 2$ and $f'(2) = -2/3$

11. Let the equation of motion of a particle be $s = 10\, t^2 - 5t + 3$.

a) If particle starts to more at $t = 0$, at what time t_1 it stops? What is the direction of motion? Does the motion continue after it stops?

b) When the position of the particle will be symmetrical of the position at $t = t_1$, with respect initial position?

c) What is the average velocity in the time interval $[0, 1]$?, and at what time the velocity will be 5 unit sec?

d) Show that the motion has constant acceleration.

12. Find $(3f)'$, $(1/f(x))'$, $|f(x)|'$, $[\![f(x)]\!]'$ if

a) $f(x) - x^2$ b) $f(x) = \sqrt{x}$ c) $f(x) = 2x + 3$.

13. If $f(x) = \sqrt{x}$, $g(x) = x^2 - 2x$ find the derivatives of

a) $2f(x) - \frac{1}{2} g(x)$ b) $f(x)\, g(x)$

c) $\frac{f(x)}{g(x)}$ d) $\frac{g(x)}{f(x)}$

14. Let f be a differentiable function. Show that the tangent line to f and $1/f$ at the point x_0 intersect the x-axis at points which are symmetrically placed with respect to x_0.

15. If $u = u(x)$, $v = v(x)$ are functions differentiable up to any order, there is a formula for the n th derivative $(uv)^{(n)}$ established by Leibniz. For $n=2$ and 3 the formulas are

a) $(uv)'' = u''v + 2u'v' + uv''$

b) $(uv)''' = u'''v + 3u''v' + 3u'v'' + uv'''$

Prove these and write the formula for $n = 4$.

c) $(uv)^{(4)} = \dots$

16. Apply the linear operator $1 + D + 2D^2 - D^3$ to

a) $x^4 - 3x^2$ b) $8x\sqrt{x}$

17. Prove by induction

a) $D\,x^n = n\,x^{n-1}$, $n \in \mathbb{Z}$,

b) $D^m x^n = n \dots (n - m + 1)x^{n-m}$, $m \in \mathbb{Z}_1$ $(m \leq n)$, n is fixed.

18. Find the derivatives of the following functions

a) $(x + 2)(x^3 + 7x^{5/2})$ b) $(5x^2 + x)\left(\frac{2}{x} - \frac{3}{x^2}\right)$

c) $\sqrt{5x^3 - x}$ d) $\sqrt[3]{7x + x^2}$

19. Find $f'(x)$ if $f(x) = |3x^2 - 4x + 5| + \frac{1}{x}$

20. Prove the following:

a) Derivative of a periodic function is a periodic function,

b) Derivative of an even function is an odd function,

c) Derivative of an odd function is an even function.

Give an example of a nonperiodic function whose derivative is periodic.

21. Find $(f \circ g)'(x)$ and $(g \circ f)'(x)$ for the following pair of functions f and g, and write the set of x's on which $f \circ g$ and $g \circ f$ are differentiable:

a) $f(x) = x^2 - 3x$, $g(x) = \sqrt[3]{x}$

b) $f(x) = \sqrt{x^2 + 1}$, $g(x) = \sqrt{x + 2}$

22. If $f(x) = x^3 - 5x^2 + 7x + 8$, $g(x) = x^3 + 2x$, find $(f \circ g)'(x)$ at $x = -2$

23. Find f and g if $(f \circ g)'_x = 1 + \frac{3}{2\sqrt{x+1}}$,
$f'(x) = 2x + 3$ and $g'(x) = \frac{1}{2\sqrt{x+1}}$

24. If $f(x) = x^3 + 8$, find $(f^{-1}(0))'$

25. Find the derivative of

a) $y = \sin \frac{x}{x^2 + 1}$ b) $y = \sin(x \cos x)$

b) $y = \tan^2 x - \sec^2 x$ d) $y = \sec^3(x^2 + 3x)$

26. Find the equation of the tangent line to the curve at indicated point:

a) $y = \sqrt{\sin x}$, $A = (\frac{\pi}{6}, \frac{\sqrt{2}}{2})$

b) $y = \sin (\frac{\pi}{2} \cos x)$, $B = (\frac{\pi}{2}, 0)$

27. Find the derivatives of the following functions:

a) $y = \cos 5x - \tan^2 x$ b) $y = \arcsin (x \cos 3x)$

c) $y = \tan(\arcsin \frac{x^2-1}{x^2})$ d) $y = \arcsin x + \arcsin\sqrt{3x}$.

e) $y = \arcsin (\sin x)$ f) $y = \cos (\arccos x)$

28. Prove by induction $(n \in N)$:

a) $D^n \sin x = \sin(x + n\frac{\pi}{2})$

b) $D^n \cos x = \cos(x + n\frac{\pi}{2})$

c) $D^n \frac{1}{x} = (-1)^n \frac{n!}{x^{n+1}}$

d) $D^n \sqrt{x} = \frac{1}{2} \prod_{i=1}^{n-1} (\frac{1}{2} - i)\, x^{\frac{1}{2} - n}$

29. Evaluate dy/dx and simplify

a) $\sqrt{x^2 - a^2} - a \operatorname{arcsec} \frac{x}{a}$ b) $\frac{a^2}{2} \arcsin \frac{x}{a} + \frac{1}{2} x\sqrt{a^2 - x^2}$

c) $2 \arcsin \sqrt{\frac{x - a}{b - a}}$ d) $\frac{1}{a - b} \arctan (\frac{b}{a} \tan x)$

e) $\frac{2}{\sqrt{-\Delta}} \arctan \frac{2ax + b}{\sqrt{-\Delta}}$ where $\Delta = b^2 - 4ac < 0$

30. Write the equation of normal line

a) to the <u>cycloid</u> $\begin{cases} x = \theta - \sin\theta \\ y = 1 - \cos\theta \end{cases}$ at $\theta = \pi/2$

b) to the <u>ellipse</u> $\begin{cases} x = a\cos\theta \\ y = b\sin\theta \end{cases}$ at $A = (x, y)$

31. Prove

a) $D^{2n} \sin\theta = (-1)^n \sin\theta$ b) $D^{2n} \cos\theta = (-1)^n \cos\theta$

c) $D^{2n+1} \sin\theta = (-1)^n \cos\theta$ d) $D^{2n+1} \cos\theta = (-1)^n \sin\theta$

32. Find $\frac{d^n}{dx^n} \frac{1}{x}$

33. Find the equation of the tangent and normal line to the curves of the following functions at the indicated points:

a) $y = x \sin x^2$, $x = \sqrt{\pi/2}$

b) $y = \arctan(x^2 + 1)$. $A = (0, \pi/4)$

c) $\begin{cases} x = 5 \sec t \\ y = 4 \tan t \end{cases}$ at $t = \frac{\pi}{3}$

d) $x \sin y - y^2 = 0$ at $A = (\frac{\pi^2}{18}, \frac{\pi}{6})$

34. A moving particle has coordinates $x = 4 \cos t$, $y = 3 \sin t$ as functions of time t. Find the following:

a) v_x, v_y (components of veclocty)

b) speed at time t

c) a_x, a_y (components of acceleration)

d) amount of acceleration a

35. Find the points where the tangent line are horizontal:

a) $y = x^3 - 12x$ b) $y = \frac{x}{x^2 + 1}$

c) $y = 3 \sin x + \sqrt{3} \cos x$ in $[0, 2\pi]$

d) $x^2 - 2xy - y^2 + 4y - 2x + 1 = 0$

e) $\begin{cases} x = t^2 + 2t \\ y = t^3 - 3t \end{cases}$

Answers to even numbered exercises

2. a) 0, b) 0, c) 1, d) Does not exist

4. a) $-f'(x)$, b) $2f'(x)$, c) $f'(x)$ d) $2f'(x)$

e) $f'(x)$, f) $(a - b)\ f'(x)$

6. a) $x^2 - 1$. b) $-x^3 + 6x^2 + x - 5$

10. $a = -1$, $b = 1$; $a = -19/4$, $b = 7/8$

12. a) $6x$; $-2/x^3$, $2x$; (if $x^2 \notin N$) 0

b) $3/(2\sqrt{x})$; $-(1/2)x^{-3/2}$; $1/(2\sqrt{x})$; (if $\sqrt{x} \notin N$) 0

c) 6; $-2/(2x + 3)^2$; $2|2x + 3| / (2x + 3)$; (if $2x \notin N$) 0

16. a) $x^4 + 4x^3 + 21x^2 - 18x - 12$,

b) $8x\sqrt{x} + 12\sqrt{x} + 12/\sqrt{x} + 3/(x\sqrt{x})$

18. a) $4x^3 + (49/2)x^{5/2} + 6x^2 + 35x^{3/2}$. b) $10 + 3/x^2$.

c) $(15x - 1)/(2\sqrt{5x^3 - x})$, d) $(7 + 2x)/(3(7x + x^2)^{2/3})$.

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26. a) $y = \sqrt{6}x + (3\ 2 - \pi)/6$. b) $y = -\pi x/2 + \pi^2/4$.

30. a) $x + y = \pi/2$, b) $(a \cos \theta)x - (b \cos \theta)y = (a^2 - b^2)\cos\theta \sin\theta$

32. $(-1)^n \dfrac{n!}{x^{n+1}}$ $\dfrac{x}{|x|}$

34. a) $-4 \sin t$, $3 \cos t$, b) 5, c) $-4 \cos t$, $-3 \sin t$, d) 5.

2. 2 DIFFERENTIAL

A. Definitions and a Theorem:

Here we shall attach meanings to the symbols "dx" and "dy", and to the bar "—" appearing in the symbol $\frac{dy}{dx}$ for derivative of a differentiable function $y = f(x)$

The symbol dx appearing in dy/dx is any real number whatever in $(-\infty, \infty)$ and it is called the differential of the independent variable x. It is another independent variable like x (and dx is not a function of x)

The symbol dy appearing in the derivative dy/dx is the real number defined by

$$dy = f'(x)\, dx \; (=df) \qquad (1)$$

and it is called the differential of the dependent variable y. The differential dy is a function of both x and dx.

In view of (1) the bar "—" in $\frac{dy}{dx}$ indicates the ratio sign, and we read dy/dx as dy over dx. When $dx = 0$, then $dy = 0$.

Example.

1. $y = \tan x^2 \Rightarrow dy = (\sec^2 x^2)\, 2x\, dx$

2. $y = \sec^2 x \Rightarrow dy = 2 \sec x \cdot \sec x \tan x\, dx = 2 \sec^2 x \tan x\, dx$

3. $v = \arccos\sqrt{t} \Rightarrow dv = -\frac{1}{\sqrt{1-t}} \cdot \frac{1}{2\sqrt{t}}\, dt$

Writing the theorems on derivative with D replaced by d/dx, and multiplying every term by dx, we have:

Theorem. If u, v are differentiable functions, then

1. $d(u + v) = du + dv$ 2. $d(u - v) = du - dv$

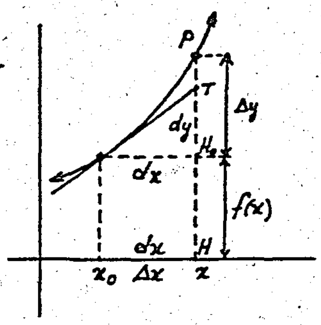
3. $d(uv) = vdu + udv$ 4. $d\,\frac{u}{v} = \frac{vdu - udv}{v^2}$ $(v \neq 0)$

B. Interpretations of differential:

Geometric interpretation:

Let $y = f(x)$ be a function with graph as shown in the figure, and differentiable at a point x_0.

Then $\frac{dy}{dx} = \tan\alpha$, $dy = \tan\alpha\ dx = \tan\alpha\ \Delta x$. Therefore for the given figure the differential for the given function dy represents $\overline{H_0T}$.



Although the exact incremant is

$$y = \overline{H_0P} = dy + \overline{TP} = dy + \varepsilon\,\Delta x$$

where $\varepsilon \to 0$ as $\Delta x \to 0$, it can be approximated by dy when Δx is small.

Example. Use differential to compute $\sqrt{23}$ approximately.

Solution. $\sqrt{23}$ suggests considering the function $y = \sqrt{x}$ with $x_0 = 25$ (for immediate evaluation). Then $dx = -2$, and

$$\Delta y \cong dy = \frac{1}{2\sqrt{x}}\,dx\Big|_{\substack{x=25\\dx=-2}} = \frac{1}{2.5}(-2) = -0.2$$

$$\sqrt{23} = 5 - 0.2 = 4.8$$

Error. Let $y = f(x)$ be a differentiable function relating quantities x and y. If x is measuring with an error Δx $(= dx)$, it is required to find error made in y.

The error in y is $y = f(x + \Delta x) - f(x) = dy + \varepsilon\Delta x$. Neglecting the term $\varepsilon\Delta x$ we have as the error:

$$\Delta y = dy = f'(x)\Delta x = f'(x)dx$$

Δy or dy is the error in y corresponding to the error $\Delta x = dx$ in x.

When measuring the distance between two cities with correct distance 500 km, suppose one makes an error of 2 km, and when measuring the distance between two trees with correct distance 50 m, one makes an error of $\frac{1}{2}$ m. We may discuss as to which measurement is more accurate. Since

$$\frac{2}{500} < \frac{1/2}{50},$$

the first measurement is more accurate then the second.

Therefore the relative error defined by dy/y is more important than the error dy itself.

The abs lute error $|\Delta y|$ is defined by $|y_c - y_m|$ where y_c, y_m are the correct and measured values of y respectively.

For simplicity one uses percent (percentage) error defined by $100\frac{dy}{y}$. In practice y_c is unknown. Then for relative error and percentage error are taken

$$\frac{dy}{y_m} \quad \text{and} \quad 10\,\frac{dy}{y_m}$$

For a given relation $y = f(x)$ between two quanties x and y the error Δx in x can be determined by the accuracy of the measuring instrument, and the error dy in y is obtained by differential.

Example 1. A square land has a measured side 8,23 m. If the error in measurement is 0,5 cm, what will the error be in the (computation of) area?

Solution. Let x be the side of the square. The area is $A = x^2$. Then $dA = 2x\,dx$ where dx and dA are errors in x and A respectively. Then

$$dA = 2.\ 823.\ 0{,}5 = 823\ cm^2 = 8.23\ dm^2$$

$$(A_m = 8{,}23^2m^2 \Rightarrow |A_c - A_m| < 8{,}23\ dm^2)$$

Example 2. The volume V of a spherical ball is determined as $920\ cm^3$ with an error of $dV = 5\ cm^3$. Find the error and relative error in radius r.

Solution. To find r_m from V_m, we solve $\frac{4}{3}\pi\ r_m^3 = 920$: $r_m = \sqrt[3]{\frac{690}{\pi}}$. Then the error can be obtained as follows:

$$V = \frac{4}{3}\ \pi\, r^3 \Rightarrow dV = 4\pi r^2 dr \Rightarrow 5 = 4\pi\left(\frac{690}{\pi}\right)^{2/3} dr$$

$$\Rightarrow \quad dr = \frac{5}{4}\ \left(\frac{\pi}{690}\right)^{2/3} cm,$$

$$\Rightarrow \quad \frac{dr}{r} = \frac{5}{4}\left(\frac{\pi}{690}\right)^{2/3} \cdot \left(\frac{\pi}{690}\right)^{1/3} = \frac{5\pi}{4.690} = \frac{\pi}{8.69} = \frac{\pi}{552}.$$

C. Related rates:

The derivative

$$\frac{dy}{dx} = f'(x) \qquad (a)$$

is the rate (amount) of change of y with respect to the variable x. Writing (a) in differentiable form

$$dy = f'(x)\ dx \qquad (b)$$

and supposing that x and y are both functions of a parameter t, usually time, we have from (b)

$$\frac{dy}{dt} = f'(x)\,\frac{dx}{dt} \qquad (c)$$

where now dy/dt and dx/dt are the rates of change of y and x with respect to t, and (c) establishes therefore a related rates.

In solving problems on related rates, one must find the relation between the quantities involved, and then differentiate that relation with respect to the parameter.

Example 1. Let A be the area of a circle of radius r and suppose r varies with time t. How the rates dA/dt and dr/dt are related to each other?

Solution: Having $A = \pi r^2$ or $A(t) = \pi r^2(t)$ we get by differentiation

$$\frac{dA}{dt} = 2\ r\ \frac{dr}{dt}$$

which is the required relation between the rates.

Example 2. If the radius r of a sphere decreases 7 cm/sec, how fast the volume V decreases when r = 5 cm?

Solution.

$$V = \frac{4}{3}\pi r^3 \qquad \frac{dV}{dt} = 4\pi r^2\ \frac{dr}{dt}$$

where dr/dt = -7 and r = 5. Then

$$\frac{dV}{dt} = 4\ .\ 5^2.(-7) = -700\ \ cm^3/sec.$$

Example 3. Sand falls to form a conical pile at the rate of 10 m^3/min. The height of the pile is always equals the radius of the base. How fast the altitude of the pile increases when it is 5 m?

Solution. Let at the time t the radius be r and altitude h. By hypothesis $r = h$. Then

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi h^3 \Rightarrow \frac{dV}{dt} = \pi h^2 \frac{dh}{dt}$$

$$\Rightarrow 10 = \pi . 5^2 . \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{2}{5\pi} \text{ m/sec.}$$

Example 4. A particle is forced be move in the positive (counter clockwise) sense on the parabola $y = 4 - x^2$ with a constant speed of 2 units/sec. Find the components v_x, v_y of the velocity at $A = (0, 4)$ and $B = (2, 0)$.

Solution. $v_x = dx/dt$. $v_y = dy/dt$ they are related by $\frac{dy}{dt} = -2x \frac{dx}{dt}$. Then we have

$$v_y = -2xv_x \quad \text{and} \quad v_x^2 + v_y^2 = 4.$$

At A: $u_x^2 + v_y^2 = 4$; $v_y = 0$, $v_x = -2$ units/sec.

At B: $v_x^2 + v_y^2 = 4$; $v_y = -4\, v_x \Rightarrow v_x = -2/\sqrt{17}$,

$v_y = 8/\sqrt{17}$.

E X E R C I S E S (2. 2)

36. Find the differentials of the following functions:

a) $y = x^3 - 3x^2 + 7x + 5$ b) $y = \dfrac{x^2}{1 + x}$

c) $v = \arcsin (u^2 - 1)$ d) $s = t \sin \dfrac{t}{1 + t^2}$

37. Find the approximate values of the following, using differentials:

a) $\sqrt[3]{1333}$ b) $\sin 32^{o}$

c) $(3,2)^4 - 5\,(3,2)^3 + 2(3,2)$ d) $\tan 42^{o}$

38. Find the error in the area of a circle with radius 11 cm when the error in the radius in 2 mm.

39. Equation of motion of a particle is given by $s = 4t^3 - 5t^2 + 3$ km. If percent error in s is 2 when $t = 10$ sec, what is the percentage error in t?

40. If a side of a cube is 12 cm with an error 0,2 cm, what is the error and relative error in the volume?

41. A particle is forced to move on the ellipse $4x^2 + 9y^2 = 36$ in the positive sense with $v_x = 2$ m/sec. Find v_y, and the speed of the particle when $x = 2$, $y > 0$.

42. Find the rate of change of volume of a sphere of radius R with respect to the area of the sphere when $R = 10$ cm.

43. A water tank has a shape of circular cone with vertical axis and vertex at bottom. The radius r is 10 m and height h is 12 m. The water running into the tank with the rate of 250 lt/min. How fast the level of water is rising when the level is 8 m above the vertex?

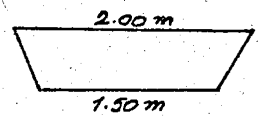
44. If a cone has base radius R equal to its height, what is the rate of change of volume with respect to the lateral area of the cone when $R = 6\sqrt{2}$?

45. In a rectangular coordinate system Oxy two particles A and B are moving. A on positive x-axis and B on y-axis, initially 10 and 2 km distant from the origin respectively. If A moves

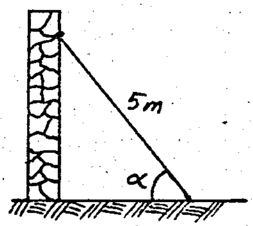
toward O and B away from O, with velocities 2 and 3 km/min respectively, what is the rate of change of the distance AB with respect to time t when t = 3 min?

46. A boy 1,60 m tall is standing on the ground, and a light source is rising up vertically with a constant velocity of 2 m/sec. The head of the boy and the light source are initially in the same level and 3 m apart from each other. Find the velocity of the end point of the shadow of the head of the boy when t = 4 sec. How fast the distance between the head of the boy and the end point of its shadow decreases at t = 4 sec?

47. A trough of 4 m length has cross section as shown in the figure. Some number of animals consume of 1 ton of water per minute. How fast the level of water, decreases when deepness of water is 20 cm?



48. A ladder of 5 m length leans against a vertical wall. If the upper end slides down at the rate of 20 cm/sec,



a) What is the velocity of lower end (on the ground),

b) How fast the area of OAB is changing,

c) How fast α is changing?

when the lover end is 3 m away from the wall?

49. Given a triangle ABC with b = 60 m, c = 40 m. If the angle A increases at the rate of 0,1 radian/sec,

a) How fast is the area changing when $A = \pi/2$ and $A = \pi/3$,

b) How fast is the side a increasing when $A = \pi/3$.

50. A point in an angle of $\pi/3$ is moving to a side in perpendicular direction with the velocity of 5 km/hr. How fast its distance from the other side is changing at any time?

Answers to even numbered exercises

36. a) $dy = (3x^2 - 6x + 7)dx$, b) $dy = (2x + x^2)dx/(1 + x)^2$

c) $dv = 2u\ du/\sqrt{2u^2 - u^4}$

d) $ds = (\sin \dfrac{t}{1 + t^2} + \dfrac{t(1 - t)}{(1 + t^2)^2} \cos \dfrac{t}{1 + t^2})\ dt$

38. $44\pi/10$ cm^2.

40. 86,4; 1/20.

42. 5

44. 3

46. -3/20 m/sec; $-1{,}2/\sqrt{320}$.

48. a) 4/15 m/sec, b) 7/15 m^2/sec, c) -1/15 rad/sec.

50. 5/2.

2. 3 PROPERTIES OF DIFFERENTIABLE FUNCTIONS

A. Rolle's Theorem and the Mean Value Theorem

Theorem (ROLLE). Let $f(x)$ be defined on a closed interval $[a, b]$. If $f(x)$ satisfies the conditions, called the Rolle's condition,

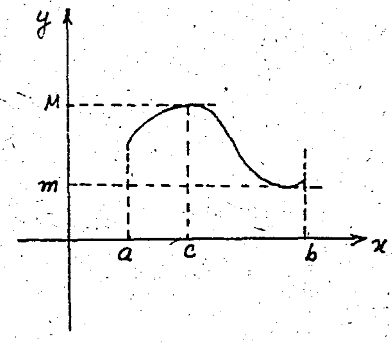
1) $f(x) \in C[a, b]$, 2) $f(x) \in D(a, b)$

3) $f(a) = f(b)$,

then there exists an interior point $c \in (a, b)$ such that $f'(c) = 0$

Proof. If $f(x)$ is a constant function, then the Rolle's conditions are satisfied and $f'(c) = 0$ for any point $c \in (a, b)$ and the theorem is established.

Let $f(x)$ be a non constant function. From the continuity over $[a, b]$ it attains its minimum m and maximum M with $m < M$. Since $f(x)$ is non constant, from (3), m or M corresponds to a point $c \in (a, b)$.



Let M be attained at $c \in (a, b)$. We prove that for this c one has $f'(c) = 0$.

Since $f(x) \in D(a, b)$, the derivative $f'(c)$ exists at c as the common value of left and right derivatives at c:

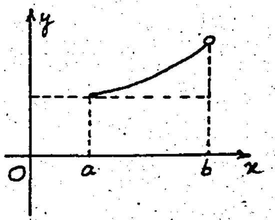
$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{\substack{x \to c \\ x < c}} \frac{f(x) - M}{x - c} \geqslant 0, \text{ since } f(x) \leqslant M,$$

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = \lim_{\substack{x \to c \\ x > c}} \frac{f(x) - M}{x - c} \leqslant 0, \text{ since } f(x) \leqslant M.$$

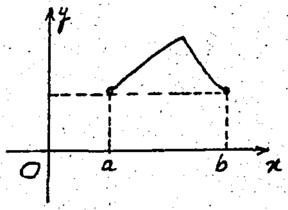
Hence $0 \leqslant f'(c) \leqslant 0 \Rightarrow f'(c) = 0.$

Similar proof can be given for m.

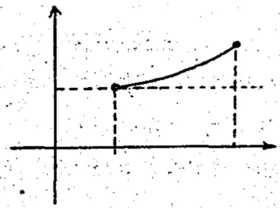
If one of the Rolle's conditions fails to exist, the theorem may not hold. We give below some examples of functions for which the theorem does not hold:



cond. (1) fails



cond. (2) fails



cond. (3) fails

<u>Example</u>. For which ones of the functions the Rolle's theorem is applicable at indicated interval; if so, find a point implied by the three Rolle's conditions.

a) $f(x) = |x|,\ [-2,\ 2]$, b) $g(x) = x^3 - 2x^2 - 4x,\ [-2,\ 2]$

<u>Solution</u>.

a) f is continuous on $[-2,\ 2]$. But f being not differentiable at $x = 0$, the condition (2) fails to exist and the Rolle's theorem may not hold in $[-2,\ 2]$.

b) Since $g(x)$ is a polynomial, it is continuous and differentiable at every $x \in \mathbb{R}$, and the first two Rolle's condition hold. As to the third one it also holds, since $g(-2) = -8 = g(2)$. Hence $g'(x) = 0$ is satisfied for some $c \in (-2,\ 2)$. Indeed

$$g'(x) = 3x^2 - 4x - 4 = 0 \Rightarrow c_{1,\ 2} = \frac{2 \pm \sqrt{4 + 12}}{3} = \frac{2 \pm 4}{3}$$

$$= \begin{cases} 2 & \notin \ (-2,\ 2) \\ -2/3 & \in \ (-2,\ 2) \end{cases} \Rightarrow c = -2/3$$

In case the solution of $f'(x) = 0$ is too difficult to obtain, one has to show the existence of c in the given open interval:

Example. If Rolle's Theorem is applicable to the function $f(x) = x \sin x$ in $[0, \pi]$, find a number $c \in (0, \pi)$ as implied by the Rolle's conditions.

Solution. This function is differentiable at any $x \in \mathbb{R}$ and hence continuous at any $x \in \mathbb{R}$, and in particular on $[-2, 2]$ and $(-2, 2)$. The third Rolle's conditions holds also. Hence there exists $c \in (-2, 2)$ at which $f'(x) = \sin x + x \cos x = 0$. The transcondental equation $\sin c + c \cos c = 0$ cannot be solved exactly for c.

The existence of a root of $f'(x)$ is shown from its continuity in $(-2, 2)$ by finding a subinterval $[\alpha, \beta]$ such that $f'(\alpha)\ f'(\alpha) < 0$. Since

$$f'\left(\frac{\pi}{2}\right) = 1, \quad f'\left(\frac{-\pi}{2}\right) = -1$$

have opposite signs, there is a root c in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \subset (-2, 2)$.

The Mean Value Theorem (MVT). Omitting the third Rolle's condition we get a theorem as a first extension of Rolle's Theorem, called the Mean Value Theorem (MVT):

Theorem (MVT). Let $f(x)$ be a function defined on a closed interval $[a, b]$. If

1) $f(x) \in C[a, b]$, 2) $f(x) \in D(a, b)$,

then there exists an interior point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{a}$$

Note. If $f(a) = f(b)$ then the MVT reduce to the Rolle's theorem. The equality (a) means the existence of $c \in (a, b)$ such that at $(c, f(c))$ the tangent line is parallel to the chord of the curve joing its end points A and B.

<u>Proof.</u> Consider the function φ whose graph is the segment $[AB]$. Having

$$\varphi(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a),$$

the function $f - \varphi$ is seen to satisfy the three Rolle's conditions:

1) $f - \varphi \in D\,(a, b)$

2) $f - \varphi$ is continuous at a and b

3) $(f - \varphi)(a) = (f - \varphi)(b) = 0$

Hence by Rolle's theorem, there exists $c \in (a, b)$ such that $(f - \varphi)'(c) = 0$ or $f'(c) = \varphi'(c)$. Hence

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \blacksquare$$

The equality (a) is valid for $a < b$ or $b < a$, since these inequalities are not considered in the proof.

Let $x_0, x \in [a, b]$. The theorem holds for the interval with end points x_0, $x = x_0 + h$ and (a) becomes

$$f(x) = f(x_0) + (x - x_0)\, f'(c), \quad c \in (\overline{x, x_0})$$

$$f(x_0 + h) = f(x_0) + h\, f'(c), \quad c \in (\overline{x, x_0})$$

$$f(x_0 + h) = f(x_0) + h\, f'(x_0 + \theta h), \quad 0 < \theta < 1$$

where the symbol $(\overline{x, x_0})$ denotes the interval (x, x_0) if $x < x_0$ or (x_0, x) if $x_0 < x$.

Example. Given $f(x) = x^3 - 9x + 1$ on $[-1, 4]$ apply the MVT to determine an interior point c as implied by the MVT.

Solution. Since for the polynomial function $x^3 - 9x + 1$ the two conditions (1) and (2) hold, such a point c exists in $(-1, 4)$. To determine it, from $f'(x) = 3x^2 - 9$, we have

$$3c^2 - 9 = \frac{f(4) - f(-1)}{4 - (-1)} = \frac{64 - 36 + 1 - (-1 + 9 + 1)}{5} = 9 + 4 = 13$$

$$\Rightarrow \quad c^2 = \frac{13}{3} = \frac{39}{9} \quad \Rightarrow \quad c = \pm\sqrt{39}/3$$

$$\Rightarrow \begin{array}{l} c_1 = \sqrt{39}/3 \cong 2 \in (-1, 4) \\ \\ c_2 = -\sqrt{39}/3 \cong -2 \notin (\overline{-1, 4}) \end{array} \quad \Rightarrow c = \sqrt{39}/3 \in (\overline{4, -1})$$

Corollary.

1. $f'(x) = 0$ on $[a, b] \Rightarrow f(x) = $ constant
2. $f'(x) = g'(x)$ on $[a, b] \Rightarrow f(x) = g(x) + c$, c is any constant

Proof.

1. Applying the MVT to $f(x)$ on $[a, x] \subset [a, b]$, we have

$$f(x) = f(a) + (x - a)\, f'(c), \quad c \in (\overline{a, x})$$
$$= f(a) = \text{constant, since } f'(c) = 0.$$

2. $f'(x) \equiv g'(x) \Rightarrow f'(x) - g'(x) \equiv 0$

$$\Rightarrow [f(x) - g(x)]' = 0 \Rightarrow f(x) - g(x) = c. \ \blacksquare$$

This corollary can be used to prove identities, an illustrative examples of which are:

Example. By differentiation, prove the identities

a) $\cos^2 x + \sin^2 x = 1$, b) $(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3$

Proof.

a) We transpose the right hand side term(s) to the left, and set $f(x) = \cos^2 x + \sin^2 x - 1$. Derivative being $f'(x) = 2\cos x(-\sin x) + 2\sin x\cos x = 0$, we conclude that $f(x) = c$ (constant). c is determined by a special value of x, say 0: $c = f(0) = 1+0-1 = 0$. Hence $\cos^2 x + \sin^2 x - 1 = 0$ or $\cos^2 x + \sin^2 x = 1$.

b) Setting

$$f(x) = (a+x)^3 - a^3 - 3a^2x - 3ax^2 - x^3,$$

we have

$$f'(x) = 3(a+x)^2 - 3a^2 - 6ax - 3x^2$$
$$= 3(a+x)^2 - 3(a+x)^2 = 0$$

$$\Rightarrow f(x) = c \Rightarrow c = f(0) = a^3 - a^3 = 0 \Rightarrow f(x) = 0.$$

The MVT for two functions. l'Hospital's Rule

Theorem (CAUCHY). Let $f(x)$, $g(x)$ be two functions defined on a closed interval $[a, b]$. If $f(x)$, $g(x)$ satisfy the four conditions

1) $f(x), g(x) \in C[a, b]$ 2) $f(x), g(x) \in D(a, b)$

3) $g(b) \neq g(a)$

4) $f(x)$, $g(x)$ do not vanish simultaneously in (a, b).

Then there exists an interior point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)} \qquad \text{(a)}$$

Proof. Consider the function

$$F(x) = f(x)\left[g(b) - g(a)\right] - g(x)\left[f(b) - f(a)\right].$$

It satisfies the first two Rolle's conditions and also the third one, namely, $F(a) = F(b)$. Hence there is an interior point $c \in (a, b)$ such that

$$F'(c) = f'(c)\left[g(b) - g(a)\right] - g'(c)\left[f(b) - f(a)\right] = 0.$$

Since $g(b) - g(a) \neq 0$ by (3), and since $f'(c)$, $g'(c)$ do not vanish simultaneously by (4), we have the required result (a). ∎

We remark that (a) reduces to the MVT for one function as seen by taking $g(x) = x$.

An important use of (a) is in the evaluation of indeterminate form 0/0. Thus if in (a) one sets $f(a) = 0$, $g(a) = 0$ and if the theorem is applied to the interval $[a, x]$ one obtains

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} \quad \text{with} \quad c \in (\overline{a, x})$$

which we state as the l'Hospital's rule:

Rule (l'HOSPITAL). If

1. $f(x), g(x) \in C[a, x]$,
2. $f(x), g(x) \in D(a, x)$
3. $f(a) = 0, \quad g(a) = 0$
4. $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$ exists,

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

This rule can also be applied to indeterminate forms

∞/∞ for $x \to a$ or $x \to \infty$, which we state as corollary 1 and 2.

Corollary 1. $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = [\frac{0}{0}] = \lim\limits_{x\to\infty} \frac{f'(x)}{g'(x)}$ if the latter limit exist.

Proof. Setting $x = 1/t$ we have

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{t\to 0} \frac{f(1/t)}{g(1/t)} = \lim_{t\to 0} \frac{D_t\ f(1/t)}{D_t\ g(1/t)}$$

$$= \lim_{t\to 0} \frac{f'(1/t) - (-1/t^2)}{g'(1/t) - (-1/t^2)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} .$$

Corollary 2. $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \left[\frac{\infty}{\infty}\right] = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$ if the latter limit exists.

Proof. Let $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lambda\ (\neq 0)$

$$\lambda = \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{1/g(x)}{1/f(x)} = \left[\frac{0}{0}\right]$$

$$= \lim_{x\to a} \frac{D\ 1/g(x)}{D\ 1/f(x)} = \lim_{x\to a} \frac{-g'(x)/g^2(x)}{-f'(x)/f^2(x)} = \lim_{x\to a} \left[\frac{f(x)}{g(x)}\right]^2 \lim_{x\to a} \frac{g'(x)}{f'(x)}$$

$$\Rightarrow \quad \lambda = \lambda^2 \lim_{x\to a} \frac{g'(x)}{f'(x)} \Rightarrow \lim_{x\to a} \frac{f'(x)}{g'(x)} = \lambda .$$

Example. Find limits by l'Hospital's Rule, when possible.

a) $\lim\limits_{x\to 5} \frac{\sqrt{x+4} - 3}{x - 5}$ b) $\lim\limits_{x\to\infty} \frac{x^2}{x - \sin x}$

c) $\lim\limits_{x\to 0} \frac{x^2 + \sin x^2}{\tan^2 x}$ d) $\lim\limits_{x\to 0} \left[\frac{1}{x} - \cot x\right]$

e)* $\lim\limits_{x\to 2} \frac{5x^2 - 6}{8x + 5}$

Solution.

a) $\lim\limits_{x \to 5} \dfrac{\sqrt{x+4} - 3}{x - 5} = \left[\dfrac{0}{0}\right] = \lim\limits_{x \to 5} \dfrac{\dfrac{1}{2\sqrt{x+4}}}{1} = \dfrac{1}{6}$

b) $\lim\limits_{x \to \infty} \dfrac{x^2}{x - \sin x} = \left[\dfrac{\infty}{\infty}\right] = \lim\limits_{x \to \infty} \dfrac{2x}{1 - \cos x}$

$= \lim\limits_{x \to \infty} \dfrac{2}{\dfrac{1}{x} - \dfrac{\cos x}{x}} = \infty$ (No limit)

c) $\lim\limits_{x \to 0} \dfrac{x^2 + \sin x^2}{\tan^2 x} = \lim\limits_{x \to 0} \dfrac{2x + (\cos x^2)2x}{2 \tan x \sec^2 x}$

$= \lim\limits_{x \to 0} \dfrac{x}{\tan x} \quad \dfrac{1 + \cos x^2}{\sec^2 x} = \lim\limits_{x \to 0} \dfrac{x}{\tan x} \lim\limits_{x \to 0} \dfrac{2}{\sec^2 x} = 2$

d) $\lim\limits_{x \to 0} \left(\dfrac{1}{x} - \cot x\right) = \lim\limits_{x \to 0} \left(\dfrac{1}{x} - \dfrac{\cos x}{\sin x}\right) =$

$= \lim\limits_{x \to 0} \dfrac{\sin x - x \cos x}{x \sin x} = \lim\limits_{x \to 0} \dfrac{\cancel{\cos x} - \cancel{\cos x} + x \sin x}{\sin x + x \cos x} = \left[\dfrac{0}{0}\right]$

which is an indeterminate form. Applying again the same rule we have

$\lim\limits_{x \to 0} \dfrac{x \sin x}{\sin x + x \cos x} = \lim\limits_{x \to 0} \dfrac{\sin x + x \cos x}{\cos x + \cos x - x \sin x} = 0.$

e)* $\lim\limits_{x \to 2} \dfrac{5x^2 - 6}{8x + 5} = \dfrac{14}{21} = \dfrac{2}{3}$. Apply l'Hospital's rule

and compare the results. Why the results are distinct? In this problem l'Hospital's rule is not applicable

Remark. Since in the statement of the theorem, $[a, b]$ is an interval, having $a<b$ the equality (a) holds for $a<b$. But since the proof is valid for an interval with end points a and b (with no restriction $a<b$) the equality (a) is valid also if $b<a$, but c may be different:

$$f(a) = f(b) + (a - b)\, f'(b) + \frac{1}{2} f''(\gamma)(a-b)^2 \qquad (a')$$

Let x_0, $x \in [a, b]$. The theorem holds for the interval with end points x_0, $x = x_0 + h$:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(c)(x - x_0)^2, \quad c \in (\overline{x_0, x}) \qquad (a'')$$

$$f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{1}{2} f''(c)h^2, \qquad c = x_0 + \theta h,\ 0<\theta<1 \quad (a''')$$

where the third terms on the righ hand sides in (a), (a'), (a") and (a"') are called the remainders, denoted by R_2. Why $R_1 = f'(c)(x - x_0)$ in MVT?

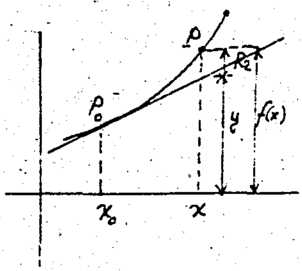
The remainder and its geometric interpretation

Consider the remainder

$$R_2 = \frac{1}{2} f''(c)\ (x - x_0)^2$$

in the MVT formula (a"), namely in

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(c)(x - x_0)^2 \qquad (a'')$$



where $x_0 \in [a, b]$, $x \in N(x_0)$, $c \in (\overline{x_0, x})$

Since the equation of tangent line to f at x_0 is

$$y - f(x_0) = f'(x_0)(x - x_0)$$

since there is no indeterminate form.

The extended MVT

Theorem. Let $f(x)$ be defined on a closed interval $[a, b]$. If $f(x)$ satisfies the two conditions

1. $f(x) \in C[a, b]$, 2. $f(x) \in D^2(a, b)$,

then there exists an interior point $c \in (a, b)$ such that

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \frac{1}{2} f''(c).(b-a)^2 \qquad (a)$$

Proof. We consider the function

$$\varphi(x) = f(x) - f(a) - f'(a)(x-a) - \frac{1}{2}\lambda(x-a)^2.$$

It satisfies certainly the first two conditions of Rolle's Theorem. We determine the constant λ such that it also satisfies the third condition $\varphi(a) = \varphi(b)$. λ is then given by

$$0 = f(b) - f(a) - f'(a)(b-a) - \frac{1}{2}\lambda(b-a)^2 \qquad (b)$$

For the λ given by (b) there exists $c_1 \in (a, b)$ such that $\varphi'(c_1) = 0$.

Consider now the function

$$\varphi'(x) = f'(x) - f'(a) - \lambda(x-a)$$

defined on $[a, c_1]$. It satisfies the first two conditions of Rolle's Theorem as well as the third one, since $\varphi'(a) = 0 = \varphi'(c_1)$. Hence there exists $c \in (a, c_1)$ such that $\varphi''(c) = 0$:

$$0 = \varphi''(c) = f''(c) - \lambda \Rightarrow \lambda = f''(c)$$

and (a) follows. ∎

the relation (a") takes on the form

$$f(x) = y + R_2$$

so that R_2 means the difference $f(x) - y$ of the ordinates of f and tangent line at x, and may be taken as a measure of derivation of the curve from the tangent line.

Example. Given $f(x) = x^3 - 3x$, find the interval for R_2 when $x = 5/2$ and $x_o = 2$.

Solution. The extended MVT for $f(x)$ at $x_o = 2$ is:

$$f(x) = 2 + 9\ (x - 2) + \frac{1}{2} f''(c)(x - 2)^2$$

$$R_2 = \frac{1}{2} f''(c)\ (x - 2)^2$$

$$= \frac{1}{2} 6c(\frac{5}{2} - 2)^2 = \frac{3}{4} c, \quad (2 < c < 5/2)$$

$$R_2 \in (3/2,\ 15/8).$$

B. Increasing and decreasing functions.

By the use of the MVT we obtain a creteria for a differentiable function to be increasing or decreasing at a point and on an interval.

Let $f(x) \in D[a, b]$. A point $x_o \in [a, b]$ is said to be a critical point of f if $f'(x) = 0$ at x_o. At such a point the tangent line is horizontal.

Theorem. A function $f(x) \in D[a, b]$ is increasing (decreasing) on $[a, b]$ if $f'(x) > 0$ $(f'(x) < 0)$ for all $x \in [a, b]$.

Proof. Let x_1, x_2 $(x_1 < x_2)$ be any two points on $[a, b]$. Since $f(x) \in D[a, b] \Rightarrow f(x) \in C[a, b]$, the MVT is applicable on the subinterval $[x_1, x_2]$ and we have, for some $c \in (x_1, x_2)$,

$$f(x_2) - f(x_1) = (x_2 - x_1)f'(c) \begin{cases} > 0 & \text{when } f'(x) > 0 \\ < 0 & \text{when } f'(x) < 0. \blacksquare \end{cases}$$

If $f'(x) \geqslant 0$ $(f'(x) \leqslant 0)$ for all $x \in [a, b]$ then $f(x)$ is said to be <u>non decreasing</u> (<u>non increasing</u>) function on $[a, b]$.

<u>Example</u>. Find the cretical points and determine the intervals of increase and decrease of the functions

a) $f(x) = x^3 - 3x + 2$ b) $g(x) = \arctan x$

<u>Solution</u>.

a) $f(x)$ is differentiable on $(-\infty, \infty)$ and we have $f'(x) = 3x^2 - 3 = 3(x^2 - 1)$. Critical points: $x_1 = -1$, $x_2 = 1$.

Interval(s) of increase:

$$f'(x) = 3(x^2 - 1) > 0 \Rightarrow (-\infty, -1) \quad \text{or} \quad (1, \infty)$$

Interval(s) of decrease: $(-1, 1)$

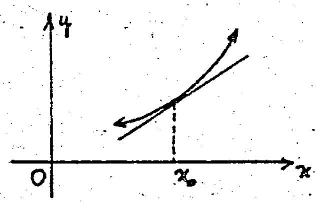
b) $g'(x) = \dfrac{1}{1 + x^2} > 0$. There are no critical points, and g is an increasing function on $(-\infty, \infty)$.

If $x_o \in (a, b)$ is not a critical point, the derivative of f at x_o is either positive or else negative; in the former case f is said to be <u>increasing at x_o</u>, in the latter case <u>decreasing at x_o</u>.

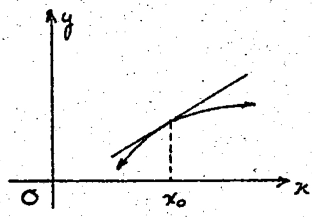
C. <u>Concavity and extrema:</u>

A differentiable function $f: \mathbb{R} \to \mathbb{R}$ is said to be concave upward (downward) at a point x_0 if in a neighborhood of x_0 the curve lies above (below) the tangent line at x_0.

The terms concave up (down) are also used.



Graph of a function concave up at x_0



Graph of a function concave down at x_0

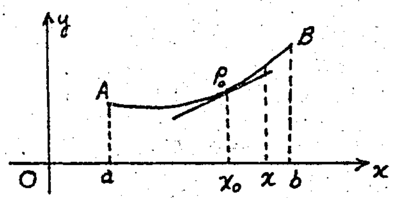
The concept of concavity is related to second order derivative. Indeed, the following theorem holds:

Theorem. A function $f \in D^2[a, b]$ is concave up (down) on the interval $[a, b]$ if $f''(x) > 0$ $(f''(x) < 0)$ for all $x \in [a, b]$.

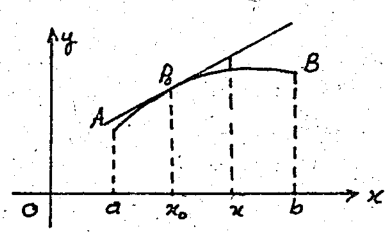
Proof. In view of the definition of concavity at $x_0 \in [a, b]$ we form the difference between $f(x)$ and the ordinate $y(x)$ of the tangent line at x_0, for any $x \in [a, b]$ namely

$$f(x) - y = R_2 = \frac{1}{2} f''(c) (x - x_0)^2$$

where $(x - x_0)^2 > 0$ in $N(x_0)$. Hence the result follows by the sign of $y''(x)$ in $[a, b]$. ∎



Graph of a concave up function on $[a, b]$



Graph of a concave down function on $[a, b]$

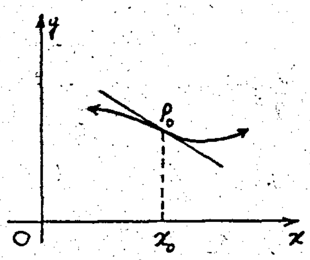
Examining the above figures, we see that in the concave up case the slope $\tan\alpha$ of the tangent line at x_0 increases as x_0 varies from a to b, in other words, the derivative function $f'(x)$ is increasing on $[a, b]$, while in the concave down case $f'(x)$ is a decreasing function.

It is seen that the tangent line at P_0 turns in counter clockwise (clock-wise) sense as P_0 moves from A to B when the curve is concave up (down).

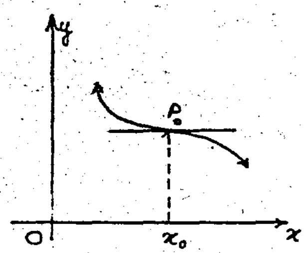
From this observation we conclude that at a point x_0 where $f(x)$ is concave up (down) the second derivative $f''(x)$ is positive (negative) at x_0.

A point $(x_0, f(x_0))$ on the curve of $f(x)$ is called a <u>point of inflection</u> (or <u>inflection point</u>) if at that point the concavity changes sense (up $\rightarrow$ down, or down $\rightarrow$ up).

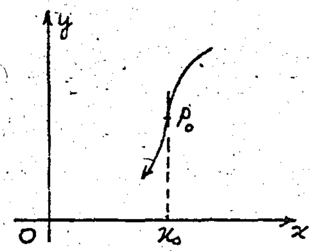
At such a point the tangent line is changing from rotating in one direction to rotating in the opposite direction, in other words the curve lies above the tangent line in one side and below in the other side. Again at such a point the tangent line (with slope $f'(x)$, finite or infinite) crosses the curve.



An inflection point with arbitrary tangent line



An inflection point with horizontal tangent line



An inflection point with vertical tangent line

<u>Corollary</u>. If for a function $f(x) \in D^2[a, b]$,

$(x_o, f(x_o))$ is a point of inflection, then $f''(x_o) = 0$.

Proof. Since $f(x) \in D^2[a, b] \Rightarrow f''(x) \in C[a, b]$, and since $f''(x)$ changes sign as x increases through x_o it must vanish at x_o. ∎

The converse of this corollary is not true, that is, at a point x_o where $f''(x_o) = 0$ the function may not have an inflection point, as the point $x_o = 0$ for $f(x) = x^{2n} (n \in N_2)$.

Example 1. Given $y = x^4 - 2x^3 - 12x^2 - 4$ find the intervals in which the curve is concave up (down), and find inflection points, if any.

Solution. $y(x)$ being differentiable we examine the sign of $y'' = 12x^2 - 12x - 24 = 12(x^2 - x - 2) = 12(x+1)(x-2)$

Intervals of concave up: $(-\infty, -1)$, $(2, \infty)$

Interval of concave down: $(-1, 2)$

Since $y'' = 0$ at -1 and 2 and concavity changes at them, the points of inflection are -1 and 2.

Example 2. Same question for

a) $y = \dfrac{x+1}{x-3}$, b) $y = x^5 - 10x^3 - 20x^2 - 15x - 4$.

Solution.

$$\text{a) } y' = \frac{1(x-3) - 1(x+1)}{(x-3)^2} = \frac{-4}{(x-3)^2} = -4(x-3)^{-2} \Rightarrow$$

$$y'' = 8(x-3)^{-3} = \frac{8}{(x-3)^3}$$

Interval of concave down: $(-\infty, 3)$

Interval of concave up: $(3, \infty)$

Since there no point at which $y'' = 0$, there is no inflection point.

b) $y' = 5x^4 - 30x^2 - 40x - 10 \Rightarrow$

$y'' = 20x^3 - 60x - 40 = 20(x^3 - 3x - 2)$ admitting the rational root 2. Hence

$$y'' = 20(x-2)(x^2+2x+1) = 20(x-2)(x+1)^2$$

Interval of concave down: $(-\infty, 2)$

Interval of concave up: $(2, \infty)$

Since concavity changes at $x = 2$, this point the inflection point.

Example 3. Sketch the curve of

$$x = 5 \sec t$$

$$y = 4 \tan t$$

in a neighborhood of the point for $t = \pi/6$ (showing tangent line and concavity).

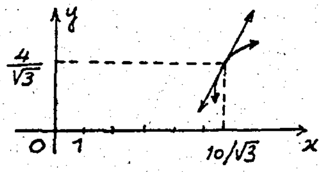
Solution.

$$y' = \frac{dy}{dx} = \frac{4 \sec^2 t \, dt}{5 \sec t \tan t \, dt} = \frac{4}{5} \csc t \Rightarrow y'(\pi/6) = \frac{8}{5}$$

$$y'' = \frac{d}{dx} \frac{4}{5} \csc t = \frac{4}{5} \frac{d}{dt} \csc t \cdot \frac{dt}{dx}$$

$$= \frac{4}{5}(-\csc^2 t \cot t) \frac{1}{5 \sec t \tan t} \Rightarrow y''(\tfrac{\pi}{6}) = -\frac{32}{25} < 0$$

$x = \frac{10}{\sqrt{3}}$, $y = \frac{4}{\sqrt{3}}$ when $t = \pi/6$.



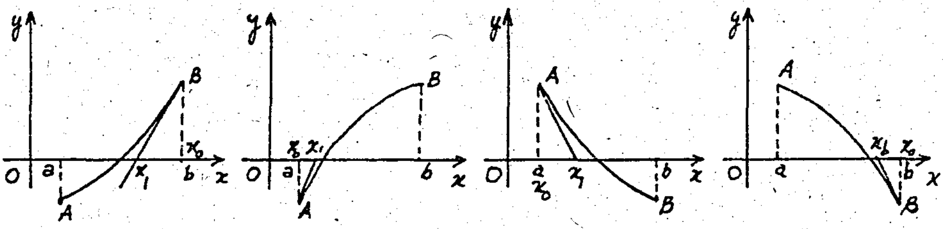
Newton's Method for finding approximate root.

Let for a twice differentiable function $f(x)$, a root of the equation $f(x) = 0$ be determined by successive approximations. One has to find, in the first step, an interval [a, b] containing a single root c in its interior. Existance of a single root in (a, b) is quaranteed by the conditions:

(1) $f(a) \cdot f(b) < 0$

(2) $f'(x) \neq 0$ in (a, b), (f is increasing or else decreasing). Imposing the third condition

(3) $f''(x) \neq 0$ in (a, b), (f is concave up or else concave down), we have the following exheustive cases as to the shape of graph:



As the initial approximation x_0 one takes that end point among a, b at which the tangent line intersects x-axis at a point x_1 which is nearer to the root c, than one obtained by the other end. This proper end x_0 is b, a, a, b in above figures. There is a rule for determining the proper end without use of the graph. The rule is: <u>the proper end x_0 is that end among a, b on which f(x) and f"(x) have the same sign</u> which can be verified by examining above figures.

Having determined the initial approximation x_0 by above rule the first approximation x_1 is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

as the x-intercept of the tangent line at $(x_0, f(x_0))$

The root c is contained then in a smaller interval with x_1 as proper end and the second approximation x_2 is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

as the x-intercept of tangent line at $(x_1, f(x_1))$.

By the same argument the next approximation is

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

and so on.

If we summarize, find an interval satisfying the three conditions (1), (2) and (3) and determine by the above rule the proper end as initial approximation x_0; the next approximations $x_1, x_2, \ldots$ being given by above formulas.

Example 1. Find the approximate positive root x_2 of

$$f(x) = x^2 - 3x - 1 = 0$$

Solution. We determine an interval containing the positive root c of the equation by the condition(a):

$$f(0) = -1,\ f(1) = -3,\ f(2) = -3,\ f(3) = -1,\ f(4) = 3$$

$$\Rightarrow \quad [a, b] = [3, 4].$$

$f'(x) = 2x - 3$, $f''(x) = 2$ are positive on $[3, 4]$. The interval containing then a single root c.

The proper end x_0: $b = 4$ is the proper end, since $f(x)$, $f''(x)$ have the same sign there.

$x_0 = 4,$

$$x_1 = 4 - \frac{f(4)}{f'(4)} = 4 - \frac{3}{5} = \frac{17}{5}.$$

$$f(17/5) = \frac{289}{25} - \frac{51}{5} - 1 = \frac{289 - 255 - 25}{25} = \frac{9}{25}$$

$$f'(17/5) = \frac{34}{5} - 3 = \frac{19}{5}$$

$$x_2 = \frac{17}{5} - \frac{9/25}{19/5} = \frac{17}{5} - \frac{9}{19.5} = \frac{323 - 9}{95} = \frac{314}{95}.$$

Example 2. Same question for $\tan x - 2x = 0$ in $[0, \pi/2]$

Solution. We find a smaller interval $[a, b]$ on which the conditions 1, 2 and 3 are satisfied. By trial we find $[\pi/4, \ 3\pi/8]$. By use of tables,

1) $f(\pi/4) = 1 - \pi/2 = 1 - 1,570 = -0,570 < 0,$

   $f(3\pi/8) = 0,069 > 0.$

2) $f'(x) = \sec^2 x - 2 > 0$ in $(\pi/4, \ 3\pi/8),$

3) $f''(x) = 2\sec^2 x \tan x > 0$ in $(\pi/4, 3\pi/8).$

The initial root: $x_o = 3\pi/8 \cong 1,179.$

$$x_1 = x_o - \frac{f(x_o)}{f'(x_o)} = 1,179 - \frac{0,069}{4,890} = 1,165,$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1,165 - \frac{0,098}{4,319} = 1,142.$$

E X E R C I S E S (2. 3)

51. For each of the following functions find a point, if

any, for the given interval that is required by the Rolle's Theorem.

a) $f(x) = x^4 - 9x^2$, $[0, 3]$

b) $f(x) = \dfrac{x^2 - 4}{x^2 + 4}$, $[-1, 1]$

c) $f(x) = \sin (x^2 - \frac{\pi}{6})$, $[-\sqrt{\frac{\pi}{3}}, \sqrt{\frac{\pi}{3}}]$

d) $f(x) = |x^2 - 4| - x$, $[-2, 3]$

52. Same question for

a) $y = 2|x| - 3$, $[-2, 2]$, b) $y = |x^2 - 4| + |x|$, $[-2, 2]$

c) $y = \sin x - \cos x$, $[-\frac{\pi}{2}, 0]$ d) $y = \dfrac{\sin x}{\sin x + 2}$, $[0, \pi]$

53. For each of the following functions find a point for the given interval that is required by the MVT:

a) $f(x) = x^3 - 2x + 4$, $[0, 2]$

b) $f(x) = \dfrac{x}{1 + x}$, $[0, 3]$

c) $f(x) = \sin x + \sqrt{3} \cos x$, $[0, \pi/3]$

54. Find θ in $f(b) = f(a) + f'(a + \theta h).(b - a)$ for each of the following function for the given interval as required by the MVT:

a) $f(x) = x^3 - 2x$, $[1, 2]$ b) $f(x) = \dfrac{x}{1 - x}$, $[2, 4]$

c) $f(x) = \sin x$, $[\frac{\pi}{6}, \frac{\pi}{3}]$ d) $f(x) = |x - 3| - x$, $[-1, 3]$

55. Prove by differentiation that the following functions are constant and find these constants.

a) $\cos^2\theta + \sin^2\theta$ b) $\arccos x + \arcsin x$

c) $\sin 3\theta - 3\cos^2\theta \sin\theta + \sin^3\theta$

d) $y^2 - x^2$ where $x = t + \frac{1}{t}$, $y = t - \frac{1}{t}$

56. For each of the following pair of functions find a point for the given interval that is required by the Cauchy's MVT:

a) $f(x) = x^3 - x$, $g(x) = x^2 + 3x$, $[0, 2]$

b) $f(x) = 3 \sin x$, $g(x) = \cos x$, $[\frac{\pi}{6}, \frac{\pi}{3}]$

57. Apply l'Hospital's Rule to evaluate limits

a) $\lim\limits_{x \to a} \dfrac{\sin x - \sin a}{x^2 - a^2}$ b) $\lim\limits_{x \to \frac{1}{2}} \dfrac{\sqrt{2x - 1}}{\sqrt{2x} - 1}$

c) $\lim\limits_{x \to y} \dfrac{\sin(x - y)}{x - y}$ d) $\lim\limits_{x \to \pi/2} \dfrac{\sin x - 1}{\cot x}$

58. Same question for

a) $\lim\limits_{x \to 1} \dfrac{x^3 - x^2 - x + 1}{x^3 - 1}$ b) $\lim\limits_{x \to 2} \dfrac{\arcsin (x - 2)}{\sqrt{2x} - 2}$

c) $\lim\limits_{x \to \pi/2} (x - \frac{\pi}{2})\sec x$ d) $\lim\limits_{x \to -2} \dfrac{\sqrt[3]{x^2 + 4} + x}{\sqrt{6 + x} - 2}$

59. Evaluate the following limits. (When you use l'Hospital's rule, why the results are not the same?)

a) $\lim\limits_{x \to -1} \dfrac{x^3 + 1}{x - 1}$ b) $\lim\limits_{x \to 0} \dfrac{\tan x + 2}{1 + \sin x}$

60. Given $f(x) = x^4 - 4x^2 + 5$, find the interval for the remainder R_2, in the extended MVT when $x_o = 2$ and $x = 5/2$.

61. Find the interval in which the following functions are increasing (decreasing):

a) $y = |x|$ b) $y = \frac{x}{1 + x}$

c) $y = \frac{1}{1 + x^2}$ d) $y = \sin x - \cos x$

62. In the following, determine the conditions for the constants a, b for f(x) to be increasing (decreasing) function.

a) $y = \frac{ax + 3}{x + b}$ in $\mathbb{R}$, b) $y = \frac{ax^2 + 2x}{bx + 3}$ in $\mathbb{R}$,

c) $y = \frac{ax + 4}{bx + 2}$ in $\mathbb{R}$, d) $y = \cos bx$ in $[0, 2\pi]$

63. For the following parametric functions find an interval in which the functions increase (decrease):

a) $\begin{cases} x = 4 \cos 3t \\ y = 4 \sin 3t \end{cases}$ b) $\begin{cases} x = 4 \cos t \\ y = 3 \sin t \end{cases}$

64. Find the interval in which the following functions are increasing (decreasing):

a) $y = \arctan (\sin x)$, b) $y = \sqrt{1 - x^4} + \arcsin x^2$,

c) $y = |x^2 - 5x + 4| - x$, d) $y = \arctan x + \frac{x^2}{1 + x^2}$

65. Find the interval in which the curves of the following functions are concave up (down) and find inflection points, if any:

a) $y = x^3 - 8x^2 + 4$ b) $y = \frac{x^2 + 3}{x + 1}$

c) $y = \tan (x - 2)$ d) $y = 2 \sec x$

66. Sketch the graph of the following functions in a neighborhood of the indicated point:

a) $f(x) = 2x^3 - 7x + 3$, $x=2$, b) $g(x) = \sqrt{x}$, $x = 0$

c) $h(x) = \frac{2}{x+3}$, $x = -1$ d) $h(x) = [\![2x - 1]\!] - x$, $x = \sqrt{3}$.

67. Given $y = (x - a)(x - b)/(x - c)$ evaluate y'' and then find the intervals of concavity for the cases.

a) $c < a < b$ b) $a < c < b$ c) $a < b < c$

68. By Newton's method find an approximate value of the root in the given interval:

a) $x^3 - 5x + 1 = 0$ in $[0, 1]$,

b) $\sin x + \frac{1}{4} = 0$ in $[-\frac{\pi}{6}, 0]$.

69. Find the relation between the constants a, b for the given functions to have no point of inflection:

a) $y = x^4 + ax^3 + 2bx^2 + x + 2$

b) $y = \frac{x^2 + ax}{x + b}$

70. Find the constants a, b for the given functions to have a point of inflection with horizontal tangent at the indicated point:

a) $y = ax^3 + bx^2 + x + 3$, $x=1$,

b) $y = \frac{a \sin x}{5+\cos x}$, $x = \pi/3$.

Answers to selected even numbered exercises

52. a) No point, b) No point, c) $-\pi/4$, d) $\pi/2$

54. a) $\sqrt{7/3} - 1$, b) $(\sqrt{3}-1)/2$, c) $1/2$, d) any θ.

56. a) $(3 + \sqrt{219})/15$, b) $\pi/4$

58. a) 0, b) 2, c) -1, d) $8/3$

60. $(5, 67/8)$.

62. a) $b > 3$, $b < 3$ for any $a \in \mathbb{R}$,

b) $ab > 0$ and $a(3a - 2b) < 0$, $ab < 0$ and $a(3a-2b) > 0$.

c) $a - 2b > 0$, $a - 2b < 0$

d) $(0, \pi/(2b)) \cup (2\pi/(2b), 2\pi/b)$, $(\pi/(2b), 3\pi/(2b))$

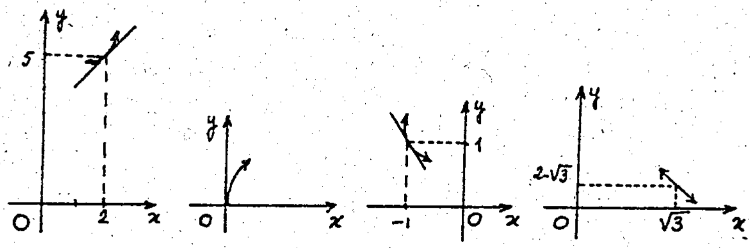
64. a) $(-\pi/2, \pi/2)$, $(\pi/2, 3\pi/2)$,

b) $(-\infty, -1) \cup (0, 1)$, $(-1, 0) \cup (1, \infty)$,

c) $(1, 2) \cup (4, \infty)$, $(-\infty, 1) \cup (2, 4)$

d) $(-\infty, \infty)$, Non decreasing.

66.



68. a) $x_2 = 0{,}216$, b) $x_1 = -0{,}250$

70. a) $a = -b = 1/3$ b) No a and b.

2.4 EXTREMA (MAXIMA - MINIMA)

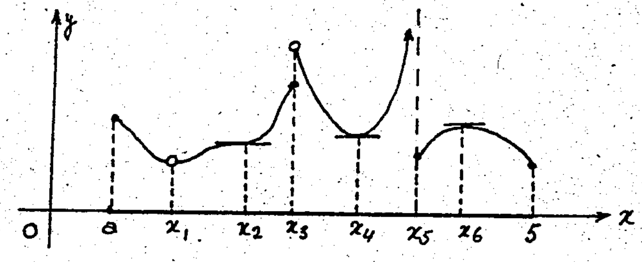
A. <u>Local Extrema</u>:

A function $f(x)$ is said to have a <u>local maximum</u> (<u>local minimum</u>) at x_0 if there exists a deleted neighborhood $N(x_0)$ of x_0 such that $f(x_0) > f(x)$ $(f(x_0) < f(x))$ for all $x \in N(x_0)$.

If x_0 is an end point of the interval $[a, b]$ of definition, the above inequalities are taken for all $x \in N(x_0) \cap [a, b]$.

The term <u>relative extrema</u> is also used to mean local extrema.

The following curve of a certain function has local maximum at a, x_6, and local minimum at x_1, x_4, x_5, b.



<u>Theorem</u>. Let $f(x)$ and $f'(x)$ be a differentiable functions in a domain D. If $f'(x_o) = 0$ with $x_o \in D$, then x_o is a

a) maximum point of f when $f''(x_o) < 0$,

b) minimum point of f when $f''(x_o) > 0$,

c) inflection point with horizontal tangent when $f''(x_o) = 0$ with $f''(x_o - h) \cdot f''(x_o + h) < 0$ for sufficiently small h.

<u>Proof</u>. We have, from the extended MVT

$$f(x) = f(x_o) + f'(x_o).(x - x_o) + \frac{1}{2} f''(x_o + \theta h).(x - x_o)^2$$

and the hypothesis $f'(x_o) = 0$,

$$f(x) - f(x_o) = \frac{1}{2} f''(x_o + \theta h).(x - x_o)^2$$

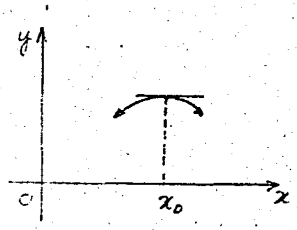
a) $f''(x_o) < 0 \Rightarrow f''(x_o + \theta h) \quad 0$ for all sufficiently small h

$$\Rightarrow \quad f(x) - f(x_o) < 0 \Rightarrow f(x_o) > f(x).$$

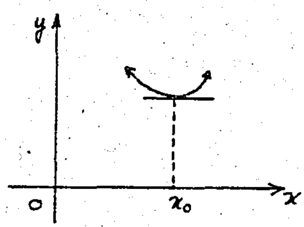
b) In the same manner, one obtains $f(x_0) < f(x)$.

c) It is a consequence of the definition of the inflection point. ■

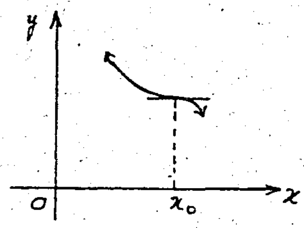
The following figures illustrate an intuitive proof of the same theorem:



A maximum (relative)



A minimum (relative)

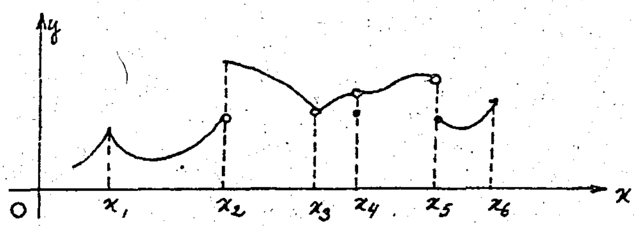


An inflection point

Corollary. Let $f(x)$, $f'(x)$ be differentiable functions on D. If $f(x_0)$ is a local maximum (minimum), then $f'(x_0) = 0$ and $f''(x_0) < 0$ $(f''(x_0) > 0)$.

Proof. The reader can prove the corollary by contradiction (indirectly). ■

There may exist other relative (local) extrema for a function $f(x)$ other than ones determined by $f'(x) = 0$. The following figure illustrate such type of extrema:



The graph has such relative maximum at x_1, x_2 and relative minimum at x_3, x_4 (at x_5 there is neither relative maximum nor minimum).

To find the points where $f(x)$ has local extrema one proceeds as follows:

1. According to related Theorems, solving $f'(x) = 0$,
2. Finding points where $f'(x)$ does not exist, (See x_1, x_2, x_3, x_4 above)
3. Finding points of discontinuities and examining them, (See x_2, x_4, x_5)
4. Examining $f(a)$, $f(b)$ when $f(x)$ is considered in a closed interval $[a, b]$.

Example 1. Find and identify all critical points:

a) $y = x^3 - 6x^2$, b) $y = \dfrac{2}{x^2 - 4x + 8}$,

c) $y = \sin x + \sqrt{3} \cos x$, d) $x = 4t^2 + \frac{1}{t}$, $y = 4t^2 - \frac{1}{t}$

Find also local extrema, if any.

Solution.

a) $y' = 3x^2 - 12x = 3x(x - 4) = 0 \Rightarrow x_1 = 0, \quad x_2 = 4$

$y'' = 6x - 12 = 6(x - 2)$

$y''(0) = -12 < 0$. There is a local max at 0, $f(0) = 0$

$y''(4) = 12 > 0$. There is a local min at 4, $f(4) = -32$

b) $$y' = -\frac{2(x - 2)}{(x^2 - 4x + 8)^2} = 0 \quad \Rightarrow x_1 = 2$$

$$y'' = -\frac{2(x^2 - 4x + 8) - 2(x - 2).2(x^2 - 4x + 8).(2x - 4)}{(x^2 - 4x + 8)^4}$$

$$y''(2) = -\frac{2.(4 - 8 + 8)}{4^4} < 0.$$ $x_1 = 2$ is a relative max.

$f(2) = 1/2.$

c) $y' = \cos x - \sqrt{3}\sin x = 0 \Rightarrow \tan x = \frac{1}{\sqrt{3}}$

$$x_k = \frac{\pi}{6} + k\pi, \quad k \in Z.$$

$$y''(x) = -\sin x - \sqrt{3}\cos x$$

$$y''(x_k) = -\sin\left(\frac{\pi}{6} + k\pi\right) - \sqrt{3}\cos\left(\frac{\pi}{6} + k\pi\right)$$

$$= -\frac{1}{2}\cos k\pi + 0 - \sqrt{3}\left[\frac{3}{2}\cos k\pi - 0\right]$$

$$= -2\cos k\pi \begin{cases} < 0 & \text{when } k \text{ is even (relative max),} \\ > 0 & \text{when } k \text{ is odd (relative min).} \end{cases}$$

$$y(x_k) = \sin\left(\frac{\pi}{6} + k\pi\right) + \sqrt{3}\cos\left(\frac{\pi}{6} + k\pi\right)$$

$$= \frac{1}{2}\cos k\pi + \sqrt{3}\ \frac{\sqrt{3}}{2}\cos k\pi$$

$$= 2\cos k\pi = \begin{cases} 2 & \text{when max} \\ -2 & \text{when min.} \end{cases}$$

d) $\frac{dy}{dx} = \frac{8t + \frac{1}{t^2}}{8t - \frac{1}{t^2}} = \frac{8t^3 + 1}{8t^3 - 1} = 0 \Rightarrow t_1 = -1/2$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\ \frac{dy}{dx} = \left(\frac{d}{dt}\ \frac{dy}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}\ \frac{8t^3 + 1}{8t^3 - 1}\ \frac{1}{dx/dt}$$

$$= \frac{-48t^2}{(8t^3-1)^2} \cdot \frac{1}{8t - \frac{1}{t^2}} = -\frac{48t^4}{(8t^3-1)^3}$$

$\Rightarrow \quad y''(-\frac{1}{2}) > 0$. A local min with $y(-1/2) = 3$.

Example 2. Find the relative extrema of the function

$$f(x) = \begin{cases} |x^2 - 4| & \text{when} \quad x < 3 \\ 8 & \text{when} \quad x = 3 \\ x + [\![x-1]\!] & \text{when} \quad x > 3 \end{cases}$$

on the closed interval $[-3, 5]$.

Solution.

Step 1. Since

$$f'(x) = \begin{cases} \dfrac{|x^2 - 4|}{x^2 - 4} \, 2x & \text{when} \quad x < 3 \\ \text{No derivative when} \quad x = 3 \\ 1 \quad \text{when} \quad x > 3, \quad x \notin \mathbb{Z} \end{cases}$$

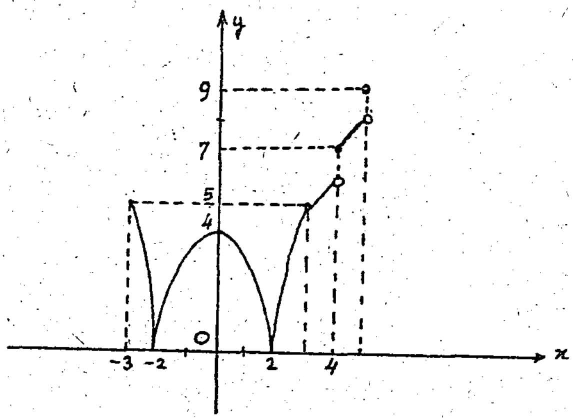
$$= \begin{cases} 2x \ \text{in} \ [-3, -2) \cup (2, 3) \\ -2x \ \text{in} \ (-2, 2) \\ 1 \ \text{in} \ (3, \infty) - \mathbb{Z} \end{cases}$$

we have $f'(x) = 0 \Rightarrow x = 0$. Now, since $0 \in (-2, 2)$, $f''(0) = -2 \Rightarrow f(x)$ has a relative max at $x = 0$.

Step 2. $f'(x)$ does not exist at the points $x = -2$, $2, 3, 4, 5$ and we have $f(-2) = 0$, $f(2) = 0$, $f(3) = 8$, $f(4) = 7$, $f(5) = 9$. Since $f(x) \geq 0$ when $x < 3$ it follows that at $x = -2, 2$ there are relative min (Sketch is helpful).

Step 3. Points of discontinuities are $x = 3, 4, 5$ which were discussed in Step 2.

Step 4. $f(-3) = 5$, $f(5) = 9$ are relative max.



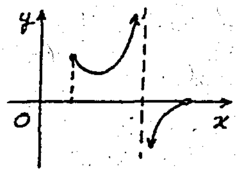
B. The absolute extrema

The largest (smallest) value of a function in a closed interval [a, b] is called the absolute maximum (absolute minimum) or simply maximum (minimum) of the function on that interval.

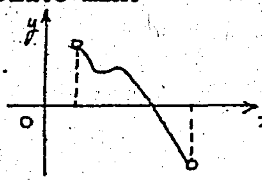
Comparing the definitions of absolute and local extrema we see that the concept of absolute extrema refers to the whole interval, while the concept of relative (local) ones reffer to neighborhoods.

The above definition implies that the absolute maximum (minimum) of a function defined on [a, b] is the max (min) of the set of all local maxima (minima), and this gives the method of finding such maximum and minimum. We remark that a function which is not continuous on a closed interval, or one defined on an open interval may not have the absolute maximum or absolute minimum.

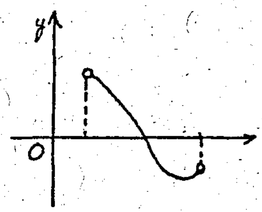
The following graphs of certain functions have no absolute max or no absolute min.



No absolute extrema



No absolute extrema



No absolute maxima

The problem of determining the absolute max (min) of a function defined on a closed interval reduces therefore to the problem of determining the local ones.

Example. Find the absolute maximum M and the absolute minimum m of the following functions:

a) $f: [-3, 5] \to \mathbb{R}, \quad f(x) = c$

b) $f: [-2, \frac{1}{2}] \to \mathbb{R}, \quad f(x) = \dfrac{x^2 - 1}{x^2 + 1}$

c) $f: [-\frac{1}{4}, 3] \to \mathbb{R}, \ f(x) = |x - 1| - 3|x| + 2x$

Solution.

a) $f'(x) = 0$ for all $x \in [-3, 5]$ and $f''(x) = 0$. Then no point in the interval is a local extrema. But for all $x \in [-3, 5]$ the absolute max and absolute min are the constant c.

First we find all relative extrema in the given closed interval:

$$f'(x) = \frac{4x}{(x^2 + 1)^2} = 0 \Rightarrow x = 0 \Rightarrow f(0) = -1$$

(It is not necessarity to identify this point to be max or min).

$$f(-2) = \frac{3}{5}\ , \ f(\tfrac{1}{2}) = -\frac{3}{5}$$

$$\text{Abs max} = \max\{3/5, \ -3/5, \ -1\} = 3/5$$
$$\text{Abs min} = \min\{3/5, \ -3/5, \ -1\} = -1$$

b) $f \in C[-1/4, 3]$, but $f \in D[-1/4, 3]$ except at $x = 0$ and $x = 1$.

$$f(x) = \begin{cases} 4x + 1 & \text{if } -\frac{1}{4} \leqslant x < 0 \\ 1 - 2x & \text{if } 0 \leqslant x < 1 \\ -1 & \text{if } 1 \leqslant x < 3 \end{cases} \Rightarrow f'(x) = \begin{cases} 4 & \text{if } -\frac{1}{4} \leqslant x < 0 \\ -2 & \text{if } 0 \leqslant x < 1 \\ 0 & \text{if } 1 \leqslant x < 3 \end{cases}$$

The value -1 of the function may be the absolute extrema. $f'(x)$ does not exist at $x = 0$ and $x = 1$. Then $f(0) = 1$ is a relative max, and $f(1) = -1$ is neither max nor min.

$$f(-\tfrac{1}{4}) = 0, \quad f(3) = -1$$

$$\text{Abs max} = \max\{-1, 1, 0\} = 1$$
$$\text{Abs min} = \min\{-1, 1, 0\} = -1$$

EXERCISES (2. 4)

71. Find and identify the relative extrema of the following functions:

    a) $f(x) = |x^2 - 16| - 6x$ b) $f(x) = \dfrac{x}{1 + x^2}$

72. Same question for the functions given in Exercise 65.

73. Same question for the functions given in Exercise 66.

74. Find the values of a, if any, for the following functions to have a relative extrema:

    a) $y = \dfrac{ax}{x^2 + 1}$ at $x = -1$ b) $y = \dfrac{x}{x^2 + a}$ at $x = \sqrt{2}$

75. Find the absolute extrema of the following functions at the given intervals:

    a) $y = x^3 - 4x$, $[-3, 1]$, b) $y = \dfrac{\sin x}{2 + \cos x}$, $[\frac{\pi}{2}, \frac{3\pi}{2}]$

76. Same question for

    a) $y = \sin x + \sqrt{3}\cos x$, $[\pi/6, 7\pi/6]$

    b) $y = 1 + \tan x$, $[\pi/6, \pi/3]$

77. Prove the following inequalities by the use of

absolute extrema:

a) $x^2 - x + 1 \geqslant \frac{3}{4}$ b) $x^4 - 4x^3 \geqslant -27$

c) $\frac{x^2 + 4x + 1}{x^2 + 4x + 8} < 8$ d) $\frac{2}{3} \leqslant \frac{x^2 + 1}{x^2 + x + 1} \leqslant 2$

78. Find the absolute maximum and minimum of the following functions in the given intervals

a) $y = \frac{1}{\sqrt{x}} + \frac{\sqrt{x}}{4}$, $[1, 9]$, b) $y = \frac{x^4}{x^3 + 1}$, $[0, 2]$

c) $y = 3x^{2/3} + 2x$, $[0, 8]$, d) $y = 8\sqrt[4]{x} + x^2$, $[1, 16]$

79. Same question for

a) $f(x) = x|x - 1| + 2x$, $[-1, 2]$

b) $f(x) = \frac{3}{x} + 3x$, $[-3, -1]$

80. Prove the inequalities under the given condition:

a) $x^{\alpha} + (1 - x)^{\alpha} \leqslant 1$, $(0 \leqslant x < 1, \alpha > 1)$

b) $2x^3 + 3x^2 - 12x + 7 > 0$, $(x > 1)$

c) $x^{2n-1} + x < x^{2n} + 1$, $(x \neq 1, x > 0, n \in \mathbb{N})$

d) $x^{n-1} + \frac{1}{x^{n+1}} < x^n + \frac{1}{x^n}$ $(x \neq 1, x > 0, n \in \mathbb{N})$

81. Prove the inequality for $\theta \in [0, \pi/2]$

$$\frac{2}{\pi} \leqslant \frac{\sin \theta}{\theta} \leqslant 1 \qquad \text{(JORDAN)}$$

82. Prove:

a) $\cos x \geqslant 1 - \frac{1}{2}x^2$ for $x \in \mathbb{R}$

b) $\tan x \geqslant x + \frac{1}{3}x^3$ for $0 \leqslant x < \pi/2$

c) $\sin x + \tan x > 2x$ for $0 < x < \pi/2$

d) $\cos x < \frac{\sin^2 x}{x^2}$ for $0 < x < \pi/2$

83. Prove

a) $\left(\frac{x+1}{a+1}\right)^{a+1} \quad \left(\frac{x}{a}\right)^{a}$ if $x > 0$, $a > 0$, $x \neq a$

b) $|x + a|^{\alpha} \leqslant |x|^{\alpha} + |a|^{\alpha}$ if $0 \leqslant \alpha < 1$

84. Prove:

a) $\cos^4 x + \sin^4 x \geqslant \frac{1}{2}$

b) $\dfrac{(x+a)^2}{x^2 + x + 1} \leqslant \frac{4}{3}\ (a^2 - a + 1)$

c) $|a \cos x + b \sin x| \leqslant \sqrt{a^2 + b^2}$

85. Prove

a) $\arccos x \geqslant \sqrt{1 - x^2}$ if $-1 \leqslant x \leqslant 1$

b) $\arcsin x \leqslant x + \sqrt{x}$ if $0 \leqslant x \leqslant 1$

c) $\arctan x \leqslant x - \frac{1}{6} x^3$ if $0 \leqslant x \leqslant 1$

d) $\arctan x \geqslant x - \frac{1}{3} x^3$ if $x \geqslant 0$

Answers to even numbered exercises

72. a) Rel.max: $(0, y(0))$, Rel.min: $(16/3, y(16/3))$

b) $(1, 2)$, $(-3, -6)$, c) No rel.extr.

d) $(2k\pi, 1)$, $((2k+1)\pi, -1)$.

74. a) Any a, b) 2

76. a) $M = 2$, $m = -2$, b) $M = \sqrt{3} + 1$, $m = (\sqrt{3}+1)/\sqrt{3}$

78. a) $M = 5/4$, $m = 1$, b) $M = 16/9$, $m = 0$

c) $M = 28$, $m = 0$, d) $M = 272$, $m = 9$.

2.5 CURVE SKETCHING

A general procedure fbr sketching the curve of a given function $f: \mathbb{R} \to \mathbb{R}$, $y = f(x)$ is the following:

1. Determine the domain D_f:

If f admits a period T, sketching is done on $[0, T]$ followed by translations along x-axis.

If f is an even (odd) function sketching is done on $[0, \infty)$ followed by symetry with respect to y-axis (origin)

2. Find points of infinite jumps and indicate them by dotted vertical lines as vertical asymptotes,

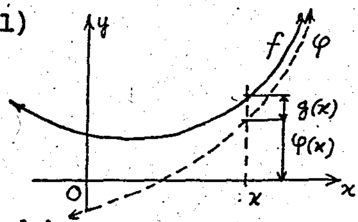
3. Determine the asymptotes (horizontal, oblique, curvilinear)

If $f(x) \to \infty$ (or $-\infty$) as $|x| \to \infty$, then the curve has an infinite branch and there may exists a curve approaching indefinitely to the curve of the function. This is the case when $f(x)$ is of the form or can be point in the form

$$f(x) = \varphi(x) + g(x) \text{ with } g(x) \to 0 \text{ as } |x| \to \infty, \quad (a)$$

and the graph of $y = \varphi(x)$ is called a curvilinear asymptote including the special cases $\varphi(x) = ax + b$ called an inclined asymptote if $a \neq 0$, and a horizontal imptote if $a = 0$.

If there is an inclined (horizontal) asymptote, $\varphi(x) = ax + b$, then from



$$f(x) = ax + b + g(x)$$

the unknown constants a, b are determined by

$$a = \lim_{x \to \infty} \frac{f(x)}{x}, \qquad b = \lim_{x \to \infty} [f(x) - ax]$$

4. Determine the critical points and intervals of increase and decrease,

5. If necessary, determine the inflection points and intervals of concavity,

6. If necessary, find the intercepts and some special points,

7. Arrange a table showing the above data for x, y', y and if necessary for y'', and then sketch the curve.

Sketching the graph of a polynomial function.

Let $P(x) = a_0x^n + \dots + a_{n-1}x + a_n \quad (a_0 \neq 0)$

be a polynomial function of degree $n \geqslant 2$.

1. $D_p = \mathbb{R}$. No period. For symmetry compare $P(x)$ and $P(-x)$,

2. No infinite jump (no vertical asymptote). Discuss behavior as $x \to \infty$ $(-\infty)$,

3. No horizontal asymptote (since $P(x)$ has no limit as $x \to \infty$ $(-\infty)$). No inclined asymptote (Since $P(x)/x$ has no limit as $x \to \infty$ $(-\infty)$). No curvilinear asymptote too,

4. $P'(x) = 0$ gives critical points. Examine the sign of $P'(x)$ in a table,

5. Examine $P''(x)$ for inflection points and concavity,

6. $P(0)$ is the y-intercept; x-intercepts are determined by $P(x) = 0$.

Example. Sketch the graph of $P(x) = x^3 - 5x^2 + 2x + 8$.

Solution. $P(x)$ is neither even nor odd.

$$P'(x) = 3x^2 - 10x + 2 = 0 \Rightarrow x_1 = \frac{5 - \sqrt{19}}{3},\ x_2 = \frac{5 + \sqrt{19}}{3}$$

Let $P(x_1) = \alpha_1, \quad P(x_2) = \alpha_2.$

$P''(x) = 6x - 10 \Rightarrow P''(x_1) < 0, \quad P''(x_2) > 0, \quad P''(\frac{5}{3}) = 0$

$P(0) = 8$ (y-intercept)

$x^3 - 5x^2 + 2x + 8 = 0 \Rightarrow x_3 = -1 \Rightarrow x_4 = 2, \quad x_5 = 4.$
(x-intercepts)

$P(x) \to \infty \quad (-\infty) \quad$ as $\quad x \to \infty \, (-\infty)$

x	$-\infty$	-1	0	x_1	$5/3$	2	x_2	4	∞
y'	+	+	+	+ 0	−	−	− 0	+	+
y	$-\infty$	$\nearrow$ 0	$\nearrow$ 8	$\nearrow$ α_1	$\searrow$		α_2	$\nearrow$	∞
y''			−		0		+		

<u>Sketching the graph of a rational function:</u>

Let

$$R(x) = \frac{P(x)}{Q(x)}$$

be a rational function with

$$P(x) = \sum_{k=0}^{n} a_k x^k, \qquad Q(x) = \sum_{k=0}^{m} b_k x^k$$

1. $D_R = \mathbb{R} - \{x: \ Q(x) = 0\}.$

2. $Q(x) = 0$ gives vertical asymptotes at x_i's except $\lim\limits_{x \to x_i} P(x)/Q(x) \neq \infty.$

3. Asymptotes (other than vertical):

$y = 0$ is the horizontal asymptote if $n < m,$

$y = \frac{a_n}{b_n}$ is the horizontal asymptote if $n = m$,

$y = Ax + B$ is the inclined asymptote if $n = m + 1$
(Obtainable by ordinary division)

$y = \sum_{k=0}^{n} A_k x^k$ is the curvilinear asymptote if $n=m+r$, $r>1$
(Obtainable by ordinary division)

4. $R'(x) = 0$ gives critical points (If, in Step 2, we observe that $P(x)$ and $Q(x)$ have a common factor $x - x_i$, differentiate after concellation keeping in mind that the obtained derivative is not defined at x_i). Examine the sign of $R'(x)$ in a table.

5. If necessary, examine $R''(x)$ for inflection points and sign.

6. $R(0)$ is the y-intercept and $P(x_i) = 0$ $(Q(x_i) \neq 0)$ gives x_i as x-intercepts.

<u>Example</u>. Sketch the graph of $y = \frac{x^3 - x}{x^2 + x - 2}$ (Do not consider inflection points and concavity).

$$D_y = R - \{-2, 1\}$$

$$P(x) = x^3 - x \Rightarrow P(-2) = -8 + 2 = -6, \quad P(1) = 0$$

$\Rightarrow$ the line $x = -2$ is the vertical asymptote, but the line $x = 1$ is not.

$$y = \frac{(x - 1)(x^2 + x)}{(x - 1)(x + 2)} \Rightarrow y = \frac{x^2 + x}{x + 2} \text{ provided } x \neq 1.$$

$$y = \frac{x^2 + x}{x + 2} = x - 1 + \frac{2}{x + 2} \Rightarrow y = x - 1$$ is the inclined asymptote.

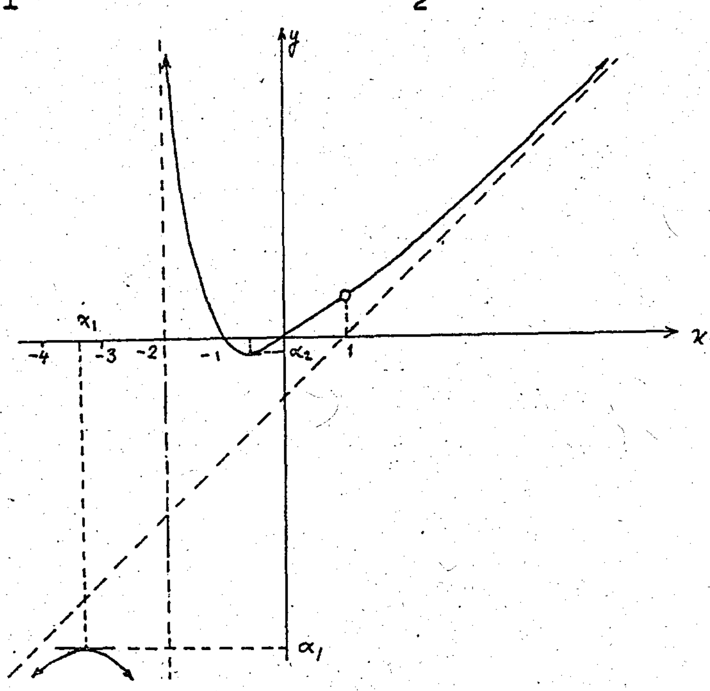
$$y' = \frac{x^2 + 4x + 2}{(x+2)^2} = 0 \Rightarrow x_1 = -2 - \sqrt{2}, \quad x_2 = -2 + \sqrt{2}.$$

Let

$$y(x_1) = \alpha_1, \qquad y(x_2) = \alpha_2.$$

$y(0) = 0$ is the y-intercept. $x^2 + x = 0 \Rightarrow x_3 = -1$, $x_4 = 0$ are the x-intercepts.

x	$-\infty$		$-2-\sqrt{2}$		-2		-1		$-2+\sqrt{2}$		0		1		∞
y'		$+$	0	$-$	$\vert$	$-$	0	$-$	0	$+$		$+$		$+$	
y	$-\infty$	$\nearrow$	α_1	$\searrow$	$-\infty \vert \infty$	$\searrow$	0	$\searrow$	α_2	$\nearrow$	0	$\nearrow$			∞
			$(\alpha_1 < 0)$						$(\alpha_2 < 0)$						



<u>Note</u>: Find the points of intersection of the curve and asymptotes, if any, when necessary.

Sketching the graph of an algebraic (non rational) function: An algebraic function was defined by the relation

$$P_o(x)\, y^n + P_1(x)\, y^{n-1} + \dots + P_{n-1}(x)y + P_n(x) = 0 \quad (1)$$

with polynomial coefficients and decreasing powers of y, or by

$$Q_o(y)x^m + Q_1(y)x^{n-1} + \dots + Q_{m-1}(y)x + Q_m(y) = 0 \quad (1')$$

obtained from (1) as decreasing powers of x. Polynomials and rational functions are some special cases of algebraic functions.

The determination of horizontal, vertical and inclined (oblique) asymptotes, if any, may be obtained by the use of the following theorem:

Theorem. For an algebraic function given by (1) or (1')

(a) The vertical asymptotes are given by the real roots of $P_o(x) = 0$, where $P_o(x)$ is the leading coefficient in (1),

(b) The horizontal asymptotes are given by the real roots of $Q_o(y) = 0$, where $Q_o(y)$ is the leading coefficient in (1'),

(c) The slope a and the y-intercept b of an oblique asymptote $y = ax + b$ is given by real roots a_i of $R_o(a) = 0$, and $R_1(a_i, b) = 0$ (if $R_1(a_i, b) \neq 0$) respectively, (If $R_1(a_i, b) = 0$ use $R_2(a_i, b) = 0$, and so on), where $R_o(a)$, $R_1(a, b)$, $R_2(a, b)$, ... are the coefficients in

$$R_o(a)x^r + R_1(a, b)x^{r-1} + \dots + R_{r-1}(a,b)x + R_r(b) = 0 \quad (2)$$

which is obtained by substituting $y = ax + b$ in (1) or (1') and arranging in decreasing powers of x.

Proof.

a) Setting $y = 1/t$ in (1):

$$P_o(x) + P_1(x)t + \dots + P_n(x)t^n = 0$$

$$\Rightarrow P_o(x) = 0 \quad \text{when} \quad t \to 0 \quad (y \to \infty)$$

b) Setting $x = 1/t$ in (1), we get $Q_o(y) = 0$ when $t \to 0$ $(x \to \infty)$

c) From the definition, $a = \lim_{x \to \infty} y/x$, $b = \lim_{x \to 0} (y-ax)$.

The equation (2) gives the common points of the curve of (1) and of a line $y = ax + b$. For the latter to be an asymptote, from (b), in (2) we have $R_o(a) = 0$. When $R_o(a) = 0$, (2) starts with $R_1(a, b)x^{r-1}$. From (b) again, $R_1(a, b) = 0$. If it is not identically zero, we have b. Otherwise consider the next coefficient, and so on. ∎

Example 1. Find vertical, horizontal and oblique asymptotes, if any, of the function

$$y = x\sqrt{\frac{x + 2}{x - 2}}$$

Solution. The relation containing this function is

$$(x - 2)y^2 - x^2(x + 2) = 0 \qquad (1)$$

$x = 2$ is the vertical asymptote.

$$x^3 + 2x^2 - xy^2 + 2y^2 = 0 \qquad (1')$$

No horizontal asymptote.

Setting $y = ax + b$ in (1) we have

$$(a^2 - 1)x^3 + (-2a^2 + 2ab - 2)x^2 + (b^2 - 4ab)x - 2b^2 = 0$$

$$\Rightarrow \quad a = \pm 1 \Rightarrow -2 \pm 2b - 2 = 0 \Rightarrow b = \pm 2$$

$y = x + 2$ and $y = -x - 2$ are the inclined asymptotes.

Example 2. Sketch the curve of the algebraic function

$$y = f(x) = x\sqrt{\frac{x+2}{x-2}}$$

Solution. $\qquad D_f = R - (-2, 2]$

There is one vertical asymptote at $x = 2$.

In Example 1 asymptotes were obtained by the use of a Theorem. We obtain here the oblique (and horizontal) asymptote $y = ax + b$, by direct use of limits:

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \sqrt{\frac{x+2}{x-2}} = 1$$

$$b = \lim_{x\to\infty} [f(x) - x] = \lim_{x\to\infty} [\sqrt{\frac{x+2}{x-t}} - 1]$$

$$= \lim_{x\to\infty} \frac{(\frac{x+2}{x-2})^{1/2} - 1}{1/x} = \lim_{x\to\infty} \frac{\frac{1}{2}(\frac{x+2}{x-2})^{-1/2} \frac{-4}{(x-2)^2}}{-1/x^2} = 2$$

Oblique asymptote: $y = x + 2$

Remark. This unique result does not contradict the results obtained in Example 1, because, there the two asymptotes were those of the relation (1), while the function admits only one).

$$y' = \sqrt{\frac{x+2}{x-2}} + x \cdot \frac{1}{2} \frac{1}{\frac{x+2}{x-2}} \frac{-4}{(x-2)^2}$$

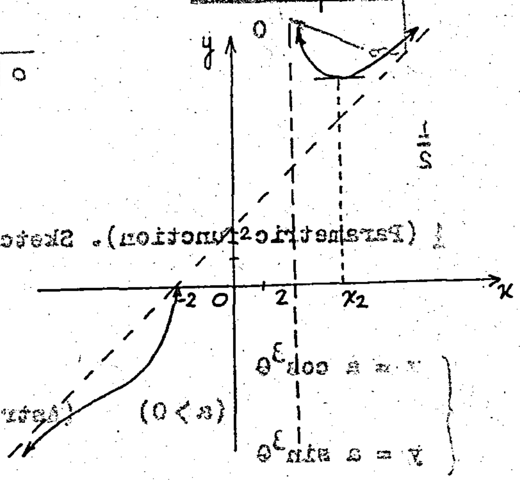
$$= \frac{\left(\frac{x+2}{x-2}\right)(x-2)^2 - 2x}{\sqrt{x^2-4}} = \frac{x^2 - 2x - 4}{(x-2)^2\sqrt{x^2-4}} = 0$$

$x = 1 \pm \sqrt{1+4} \Rightarrow x_1 = 1 - \sqrt{5}, \quad x_2 = 1 + \sqrt{5}.$

Let $f(x_1) = \alpha_1$, $f(x_2) = \alpha_2$.

~~x-intercept: ±2, y-intercept:~~

x	x_1		-2		2		x_2		∞
y'		$+$				$-$	0	$+$	
y	$-\infty$	$\nearrow$	0		∞	$\searrow$	α_2	$\nearrow$	∞



Explain why the curve is below the oblique asymptote in $(-\infty, -2)$, and above in $(2, \infty)$?

Example 3. (Transcendal function). Sketch the graph of

$$y = \frac{\sin x}{1 + \cos x}$$

<u>Solution</u>.

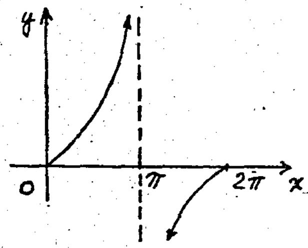
$$D_y = \mathbb{R} - \{x:\ x = (2k+1)\pi,\ \ k \in \mathbb{Z}\}$$

The function admits a period $T = 2\pi$. It will suffice to sketch the curve in $[0,\ 2\pi]$ followed by translations. At $x = \pi$ there is a vertical asymptote.

$$y' = \frac{\cos x(1 + \cos x) + \sin^2 x}{(1 + \cos x)^2} = \frac{1 + \cos x}{(1 + \cos x)^2} = \frac{1}{1 + \cos x} \quad \text{if}$$

$\cos x \neq -1$. There is no critical point, and $y' > 0$.

x	0		π		2π
y'		$+$		$+$	
y	0	$\nearrow$	$\infty \mid -\infty$	$\nearrow$	0



$$y'(0) = \frac{1}{2}$$

<u>Example 4</u> (Parametric function). Sketch the graph of the relation

$$\begin{cases} x = a\cos^3\theta \\ y = a\sin^3\theta \end{cases} \qquad (a > 0) \qquad \text{(Astroid)}$$

<u>Solution</u>.

$$|x| = a|\cos\theta|^3 \leqslant a, \quad \text{and} \quad y \leqslant a$$

$$\Rightarrow \quad x \in [-a,\ a], \quad y \in [-a,\ a] \quad \text{(bounded)}$$

$x(\theta)$, $y(\theta)$ are periodic with $T = 2\pi$. We only vary θ in $[-\pi,\ \pi]$.

When $\theta \to -\theta$, then $x \to x$, $y \to -y$. There is symmetry with respect to x-axis. Vary then θ in $[0, \pi]$.

When $\theta \to \theta + \pi$, then $x \to -x$ and $y \to -y$. There is symmetry with respect to the origin. Vary then θ in $[0, \pi/2]$.

$$\frac{dy}{dx} = \frac{3\ a \sin^2\theta \cos\theta}{-3\ a \cos^2\theta \sin\theta} = -\tan\theta < 0 \quad \text{for} \quad [0, \ \pi/2)$$

The function is decreasing in $[0, \ \pi/2]$.

θ	0		$\pi/2$
x	a	$\searrow$	0
y	0	$\nearrow$	a
y'	0	$-$	

<u>Example 5</u>. Determine the asymptotes of the curve given by

$$x = t + \frac{1}{t}, \qquad y = t - \frac{2}{t}.$$

<u>Solution</u>.

V.A. :

$$|y| \to \infty \Rightarrow t \to 0 \quad \text{or} \quad |t| \to \infty \Rightarrow |x| \to \infty \Rightarrow \text{no V.A.}$$

H.A. :

$$|x| \to \infty \Rightarrow t = 0 \quad \text{or} \quad |t| \to \infty \Rightarrow |y| \to \infty \Rightarrow \text{no H.A.}$$

I.A. : $\quad y = ax + b$

$$|x| \to \infty \Rightarrow t \begin{cases} 0 \\ \infty \end{cases} \Rightarrow \frac{y}{x} = \frac{t^2 - 2}{t^2 + 1} \to \begin{cases} -2 \\ 1 \end{cases} \Rightarrow \begin{matrix} a_1 = -2 \\ a_2 = 1 \end{matrix}$$

$$y - ax = \begin{cases} y + 2x = 3t \to \begin{cases} 0 \\ \infty & \text{no I.A.} \end{cases} \\ y - x = -\frac{3}{t} \to \begin{cases} \infty & \text{no I.A.} \\ 0 \end{cases} \end{cases}$$

$\Rightarrow$ $y = -2x$, $y = x$ are the inclined asymptotes.

E X E R C I S E S (2. 5)

86. Sketch the graphs of the polynomials

    a) $y = (x - 1)^2(x + 1)(x + 2)$

    b) $y = (x + 2)^2(x - 2)^2$

    c) $y = (x - 1)^3(x + 1)$

    d) $y = (x + 3)^2(x - 2)^3$

87. Sketch the graphs of the rational functions

    a) $y = \dfrac{12x}{x^2 + x}$ b) $y = \dfrac{x^2 + 3x + 1}{x}$

    c) $y = \dfrac{2x^2 - 2x + 5}{2x - 1}$ d) $y = \dfrac{8x}{x + 5} + \dfrac{18x}{x - 5}$

88. Apply the theorem in the text to determine horizontal, vertical and inclined asymptotes, if any, of the following algebraic relations:

a) $(x-1)y^3 + 2x^3y - 5x + y + 2 = 0$

b) $(y^2-2)x^2 - 2y^3x + 4x - 5y + 1 = 0$

89. Sketch the graphs of

a) $y = \dfrac{x}{\sqrt{x-1}}$ b) $y = \dfrac{\sqrt{x+1}}{x}$

c) $y = x\sqrt[3]{\dfrac{x-1}{x+1}}$ d) $y = x\sqrt[3]{x^2-8}$

90. Sketch the graphs of the relations

a) $\sqrt{x} + \sqrt{y} = \sqrt{a}$, $a > 0$, b) $\sqrt{x} - \sqrt{y} = \sqrt{a}$, $a > 0$

91. Find a rational function of x

a) with range $(-\infty, -1] \cup [\frac{1}{4}, \infty)$

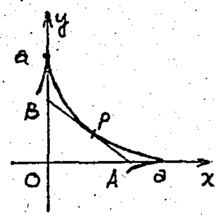
b) with range $[-1, 4)$

92. Sketch:

a) $\begin{cases} x = a \cos\theta \\ y = b \sin\theta \end{cases}$ (ellipse) b) $\begin{cases} x = a \sec\theta \\ y = b \tan\theta \end{cases}$ (hyperbola)

c) $\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}$ $t \in [0, 2\pi]$ (cycloid)

93. Consider the astroid determined by the parametric equation $x = a\cos^3 t$, $y = a\sin^3 t$. Let the tangent line at a point $P(t)$ meet the axes at A and B. Show that $|AB| = a$.

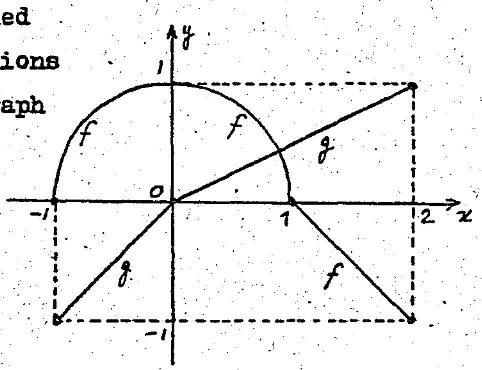


94. Given $y = x^{\alpha}$, sketch the graph when

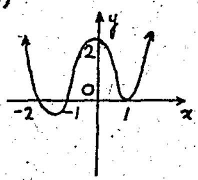
a) $\alpha = 0$ b) $\alpha = \dfrac{2}{3}$ c) $\alpha = 3/2$

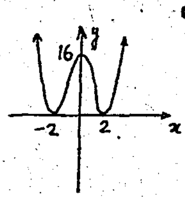
d) $\alpha = 2n$, $n \in N^+$ e) $\alpha = 2n + 1$ $(n \in N)$

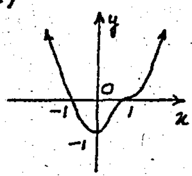
95. On the figure are sketched the curves of some functions f and g. Sketch the graph of f + g.

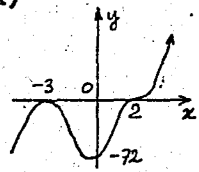


Answers to even numbered exercises

86. a)


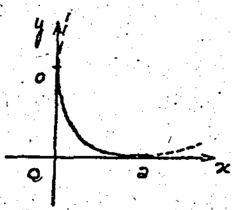
b)


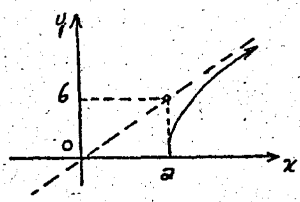
c)


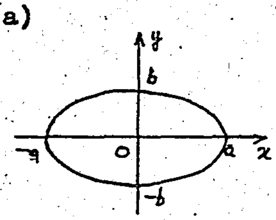
d)


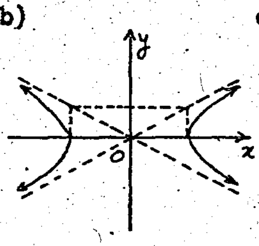
88. a) VA: x = 1, HA: y = 0, IA: No IA

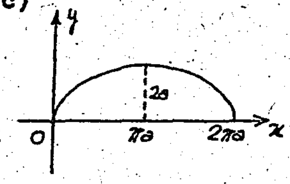
b) VA: x = 0, HA: y = $\pm\sqrt{2}$, IA: y = x/2.

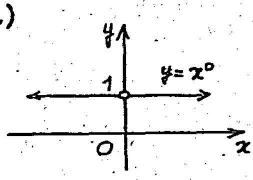
90. a)


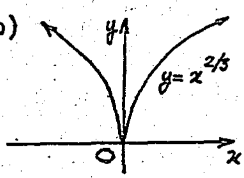
b)


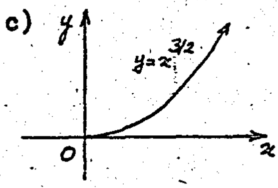
92. a)


b)


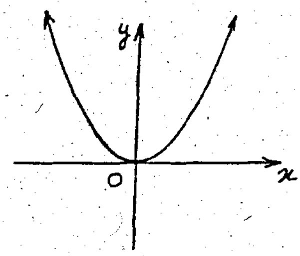
c)


94. a)


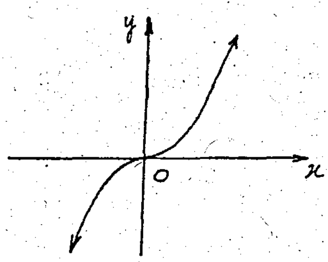
b)


c)


d)



e)



2. 6 PROBLEMS ON MAXIMA, MINIMA

Problems on maxima, minima in mathematics and physical sciences lead to the determination of a function on a certain interval on which it is usually differentiable. The general proceedure may be outlined as follows:

1. Express the quantity to be extremalized in terms of some quantities one of which is selected as independent variable, the others being written in terms of this variable.

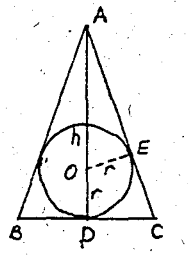
2. Determine the interval for the variable.

3. Set the derivative of the function equal to zero and solve for the independent variable. Consider only those which are in the interval.

4. If a single root is obtained, the value of the function at this point may be decided, by the nature of the problem, to be a max or min. Otherwise second derivative is needed.

5. If the interval is closed at one or two ends, then end points must be taken in consideration among local extrema obtained by derivative.

Example 1. Find the altitude h of an isoceles trangle of minimum area which can be circumscribed about a circle of given radius r.



Solution. Let ABC be the triangle with $|AB| = |AC|$ be dawn about the circle. Letting $h = |AD|$, $|DC| = a$, area A is given by

$$A = ah$$

where both a and h are variables. Selecting h as the variable we express a and then A as functions of h. From similarity of right riangles ADC and AEO, we have

$$\frac{h}{a} = \frac{|AE|}{r} \Rightarrow \frac{h}{a} = \frac{\sqrt{(h-r)^2 - r^2}}{r}$$

$$\Rightarrow \quad hr = a\sqrt{h^2 - 2rh} \Rightarrow a = \frac{rh}{\sqrt{h^2 - 2rh}}$$

and

$$A(h) = r\,\frac{h^2}{\sqrt{h^2 - 2rh}}, \quad h \in (2r, \infty).$$

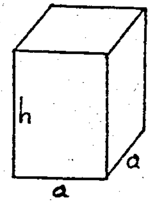
Differentiation gives

$$\frac{dA}{dh} = r\,\frac{2h\sqrt{h^2 - 2rh} - \frac{1}{2}\,\frac{2h - 2r}{\sqrt{h^2 - 2rh}}\,h^2}{h^2 - 2rh}$$

$$= r\,\frac{2h(h^2 - 2rh) - (h-r)h^2}{h(h-2r)\sqrt{h^2-2rh}} = r\,\frac{2(h^2 - 2rh) - (h-r)h}{(h-2r)\sqrt{h^2-2rh}} = 0$$

$$\Rightarrow h^2 - 4rh + rh = 0 \Rightarrow h^2 = 3rh \Rightarrow h = 3r, \text{ since } h \neq 0.$$

Example 2. A rectangular box with a square base is to be built to contain 20 lt of olive oil. If the material for the top and front costs 0,75 TL/dm^2 and for the bottom and lateral faces 0,50 TL/dm^2 find the minimum cost.



Solution.

Denoting a side of the bottom by a and altitude of the box by h, then the total cost is

$$C = 0,75(a^2 + ah) + 0,50(a^2 + 3ah)$$

where a, h are related by $a^2h = 20$ or $h = 20/a^2$. So

$$C(a) = 0,75(a^2 + \frac{20}{a}) + 0,50(a^2 + \frac{60}{a}) = 1,25\, a^2 + \frac{45}{a}$$

$$\frac{dC}{da} = 2,50\, a - \frac{45}{a^2} = 0 \Rightarrow a^3 = \frac{450}{25} = \frac{90}{5} = 18$$

$$\Rightarrow a = \sqrt[3]{18} \text{ dm}, \quad h = \frac{20}{\sqrt[3]{18}} = \frac{10}{9}\sqrt[3]{324}$$

$$C_{min} = 1,25\sqrt[3]{324} + \frac{45}{\sqrt[3]{18}} = 1,25\sqrt[3]{324} + \frac{45}{18}\sqrt[3]{324}$$

$$= 3,75\sqrt[3]{324} = 11,25\sqrt[3]{12} = 11,25 \,.\, 2,29 = 25,75 \text{ TL}.$$

Example 3. Find the point on the parabola $y^2 = 4x$ which is nearest to the point $A = (-3, 6)$

Solution. Let $P = (x, y)$ be any point on the parabola. Then the distance $d = |AP| = \sqrt{(x + 3)^2 + (y - 6)^2}$ is to be minimized. If d is minimum d^2 is also minimum, since $d > 0$.

So the function to be minimized is

$$u = (x + 3)^2 + (y - 6)^2$$

where x, y being related by $y^2 = 4x$, we have

$$u(y) = (\frac{y^2}{4} + 3)^2 + (y - 6)^2$$

$$\frac{du}{dy} = 2(\frac{y^2}{4} + 3) \cdot \frac{y}{2} + 2(y - 6) = 0$$

$$\Rightarrow y(y^2 + 12) + 8(y - 6) = 0 \Rightarrow y^3 + 20y - 48 = 0$$

$$\Rightarrow (y - 2)(y^2 + 2y + 24) = 0 \Rightarrow y_1 = 2 \Rightarrow x_1 = 1$$

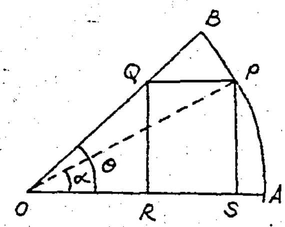
The required point is $(1, 2)$.

Example 4. Find the area of the rectangle of maximum area inscribed in a circular sector of given radius r and central angle $\theta \leqslant \pi/2$ when two vertices lie

a) on a terminal radius,

b) on the circular arc

and compare the two areas.



Solution.

a) The angle α determines P uniquely on the arc $\widehat{AB}$, and rectangle PQRS. The area of PQRS is

$$A_1 = |RS| \, |SP|$$

where

$$|SP| = r \sin\alpha , \qquad 0 < \alpha < \theta$$

$$|RS| = |OS| - |OR| = r\cos\alpha - \frac{RQ}{\tan\theta} = r\cos\alpha - \frac{r\sin\alpha}{\tan\theta}$$

Then

$$A_1(\alpha) = r^2 \sin\alpha\,(\cos\alpha - \frac{\sin\alpha}{\tan\theta})$$

$$= r^2\left[\sin\alpha\,\cos\alpha - \frac{\sin^2\alpha}{\tan\theta}\right]$$

$$\frac{dA_1}{d\alpha} = r^2\left[\cos 2\alpha - \frac{\sin 2\alpha}{\tan\theta}\right] = 0$$

$$\Rightarrow \quad \tan 2\alpha = \tan\theta \Rightarrow x = \frac{\theta}{2}$$

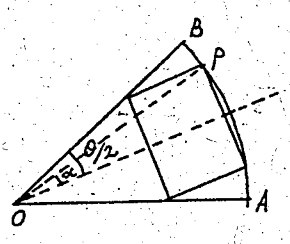
which corresponds obviously to maximum area.

$$A_1(\frac{\theta}{2}) = r^2\left[\sin\frac{\theta}{2}\cos\frac{\theta}{2} - \frac{\sin^2\frac{\theta}{2}}{\tan\theta}\right]$$

$$= r^2\left[\frac{1}{2}\sin\theta - \frac{1-\cos\theta}{2\tan\theta}\right] = \frac{1}{2}r^2\tan\frac{\theta}{2}$$

b) From (a) we have maximum shaded area

$$\frac{1}{2}r^2\tan\frac{\theta}{4}$$



which is $\frac{1}{2}A_2$. Then

$$A_2 = r^2\tan\frac{\theta}{4}$$

To compare, let $\theta = 4t \quad (0 < t < \pi/8)$.

$$\frac{A_1}{A_2} = \frac{1}{2}\,\frac{\tan 2t}{\tan t} = \frac{1}{1 - \tan^2 t} > 1 \Rightarrow A_1 > A_2.$$

E X E R C I S E S (2. 6)

96. Find a number

a) which exceeds its square by greatest amount,

b) such that itself plus nine time its reciprocal gives a maximum sum.

97. Divide the number 48 into two parts so that

a) the sum of the square of the parts is least,

b) the square of one part together with 3 times the square of the other part is the least possible.

98. Given an isocales right triangle ABC ($|AB| = |AC|$), find the point on the hypothenuse such that $|PA|^2 + |PB|^2 + |PC|^2$ is minimum.

99. Determine the square of least area inscribed in a given square of side a. (Each vertex is on each side).

100. An open box is to be made of a rectangular cardboard 10 x 16 cm by cutting congruent squares out of the corners and turning up to edges to form the lateral faces of the box. Determine the maximum capacity of the box.

101. Find the dimensions of the largest rectangle inscribed in an acute angled triangle with base on one side of the triangle.

102. The legs of an isoceles triangle are each 50 cm long. Find the length of the base, if the area of the triangle is maximum.

103. Find the area of the isoceles triangle of maximum area inscribed in a circle of given radius a.

104. A window has the shape of a rectangle surmounted by a semi-circle. If the perimeter is given, determine the shape of the window when area is maximum.

105. Of all right circular cylinders that can be inscribed in a given right circular cone, show that the one of greatest volume has altitude one third that of the cone. What is the volume of the cylinder?

106. A rectangular screen is hung on a vertical wall, with lower and upper edges being a and b meters above the ground respectively. How for away (from the wall) an observer should seat so that he sees the screen under greatest angle (observer's eye is h meters above the ground, $h < a < b$)

107. Find the volume of the right circular cone of max (min) volume inscribed (circumscribed) in (about) a sphere of given radius r.

108. Find the point on the given curve which is nearest to the given point:

a) $y = \sqrt{x}$, $(2, 0)$ b) $y^2 = 4x$, $(-3, 6)$

109. Find the points on $y^2 = 4x$ whose distances from $(6, -3)$ are maximum or minimum.

110. Show that the point $(2, 2)$ is the point on the graph of $y = x^3 - 3x$ that is nearest to the point $(11, 1)$.

111. Find the point on the given curve the sum of the squares of whose distances from the given points is a minimum.

a) $y = x$, (0, 0), (5, 0), (0, 4)

b) $y = 2x + 3$, (0, 0), (0, 9), (6, 0)

c) $x^2+y^2 = 25$, (3, 0), (0, 4), (0, 0)

112. A cone is to be made by cutting a sector from a circle and rolling up what is left. Determine the angle of the sector that is left so that the volume of the cone is a maximum.

113. A point A on a table is illumiated by a lamp on the wall which is a meters distant from A. How high above the table the lamp should be hung so that A has maximum illummation?

114. The square of the velocity of light of wave length ℓ on deep water is proportional to $\frac{\ell}{a} + \frac{a}{\ell}$ where a is a positive number. Find the minimum velocity.

115. What is the length of the shortest line segment tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$ and termінted by coordinate axes?

116. Given the parabola $y = 4 - x^2$, find the point on it in the first quadrant such that the tangent line drawn at that point forms with coordinate axes a triangle of minimum area.

117. A rectangular closed box with square base is to have a capecity of 27 liters. Determine the least amount of material réquired (in dm^3) .

118. A rectangular box with square base and with capacity of 30 liters is to be built. The material for the

top and front costs 15 TL/dm^2 and for bottom and other lateral faces 10 TL/dm^2. Find the minimum cost.

119. The material for the top of a cylindrical can costs twice as much per square unit area as that for the bottom and lateral faces. Find the proportions for the least cost.

120. Find the length of the shortest line segment not possing through the origin and ending in the coordinate axes and passing through the point $(a^{3/2}, b^{3/2})$.

121. A container open at the top is made in the form of a right prism where the base is an isoceles right triangle. If the capacity is given, what propositions will make the amount of material used a minimum?

122. Find the slope of the line through the point (4, -5) out of which the parabola $y^2 = 8x$ cuts the segment of shortest length.

123. Find the equation of the locus of points P such that of all chords of parabola $y^2 = 8x$ through P, the one of minimum length has slope of 1.

124. A boat manifacturer estimates that he can sell 5000 boats a year at 9000 TL each, and that he can sell 1500 more boats per year for each 1000 TL decrease in price. What price per boat will bring the greatest returns?

125. If the safe distance between the centers of two cars going v miles/hr is given by the relation

$$s = 0{,}05\ v^2 + 1{,}4\ v + 5,$$

what is the maximum number of cars that can sefely pass a given point in an hour if the cars are all going at the same speed? (Assume that there is no car at the point when the counting is started).

Answers to even numbered exercises

96. a) 1/2, b) 3

98. Midpoint

100. 144

102. $50\sqrt{2}$

104. $x = r = \ell/(\pi + 4)$

106. Distance from the wall: $\sqrt{(a - h)(b - h)}$

108. a) $(3/2, \sqrt{3/2})$, b) (1, 2)

112. $2\sqrt{6}$ $\pi/3$ rad $\cong 294^{\circ}$

114. $v = \sqrt{2k}$ where k is the constant of proportianality.

116. $(2/\sqrt{3},\ 8/3)$.

118. Dimension: $3 \times 3 \times \frac{10}{3}$, cest: 675 TL.

120. $(a + b)^{3/2}$

122. -1

124. $6166,\overline{6}$

A SUMMARY

(Chapter 2)

2\. 1. If $\lim [f(x+\Delta x) - f(x)]/\Delta x$ when $\Delta x \to 0$ exists, $y=f(x)$ is said to be <u>differentiable</u> at x, and this limit is called the <u>derivative</u> of $f(x)$ at x, some notations of it being dy/dx, y', $f'(x)$, Dy, $Df(x)$. $f \in D[a, b]$ represents a function f differentiable on $[a, b]$. We have $f \in D[a, b] \Rightarrow f \in C[a, b]$.
Chain rules: (1) $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$ or $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ where $y = y(u)$, $u = u(x)$.
(2) $x = x(t)$, $y = y(t) \Rightarrow dy/dx = \dot{y}(t)/\dot{x}(t) = (dy/dt)/(dx/dt)$

2\. 2. $dy = f'(x)dx$ is the <u>differential</u> of $y = f(x)$. $\Delta y \doteq dy$ is the <u>increment</u> of y, and dy/y_m, $100\, dy/y_m$ are <u>relative error</u>, <u>percentage error</u> respectively, where y_m is the measured y.

2\. 3. Rolle's Theorem: $f \in C[a, b]$, $f \in D(a, b)$, $f(a) = f(b) \Rightarrow f'(c) = 0$ for some c in (a, b).
Mean Value Theorem: $f \in C[a, b]$, $f \in D(a, b) \Rightarrow$

$$f(b) = f(a) + (b-a)\, f'(c) \text{ for some } c\ (a, b), \text{ or}$$

$$f(a+h) = f(a) + h\, f'(a+\theta h) \text{ for some } \theta \in (0, 1).$$

L'Hospital'Rule: $f, g \in C[a, x]$, $f, g \in D(a, x)$, $f(a)=g(a)=0$

$$\Rightarrow \lim_{x \to a} f(x)/g(x) = \lim_{x \to a} f'(x)/g'(x) \text{ if the latter exists.}$$

Extended MVT: $f \in C[a, x]$, $f \in D^2(a, x)$

$$f(x) = f(a)+f'(a)(x-a) + \frac{1}{2} f''(c)(x-a)^2 \text{ for some } c \in (a, x).$$

Newton's Approximation Method for roots of $f(x) = 0$ in $[a, b]$:

x_0 is that end for which $f(x_0) \cdot f''(x_0) > 0$. Then
$x_n = x_{n-1} - f(x_{n-1})/f'(x_{n-1})$ for $n = 1, 2, \ldots$

2\. 4. Relative extrema: If $f'(x_0) = 0$ and

$$f''(x_0) \begin{cases} <0, & f(x_0) \text{ is a relative max} \\ =0, & \text{changing sign, point of inflection} \\ >0, & f(x_0) \text{ is a relative min.} \end{cases}$$

2\. 5. Asymptotes: If $f(x) = \varphi(x) + g(x)$, where $g(x) \to 0$ as $x \to \infty$, then $y = \varphi(x)$ is a <u>curvilinear asymptote</u>. If $\varphi(x) = ax + b$ then one has an <u>inclined asymptote</u> if $a \neq 0$, an <u>horizontal one</u> if $a=0$. $x=x_0$ is a <u>vertical asymptote</u> if $\lim f(x) = \infty$ when $x \to x_0$.

MISCELLANEOUS EXERCISES
(Chapter 2)

126. Determine whether the following functions have a derivative at the indicated points and if so what is the derivative?

a) $f(x) = (x^3 + x)\,|x|$ at $x = 0$

b) $y = x[\![x]\!]$ at $x = 1$

c) $y = \begin{cases} x^2 + 1 & \text{when } x < 1 \\ 2x - 1 & \text{when } x \geqslant 1 \end{cases}$ at $x = 1$

d) $f(x) = \begin{cases} \dfrac{x - 2}{x^2 + 3x - 10} & \text{when } x \neq 2 \\ 1/x & \text{when } x = 2 \end{cases}$ at $x = 2$.

127. Find the derivative of:

a) $y = \arcsin x - \arctan \dfrac{x}{\sqrt{1-x^2}}$

b) $y = \arccos \left(\arctan \dfrac{x}{1 + x^2}\right)$

c) $y = x \cos x - \dfrac{\sin x}{x}$

d) $y = x \sin x + x \cos^2 x$

128. Find d^2y/dx^2 if $y = \sin(\pi \cos^2 x)$

129. Find y', y'' in terms of t of the following parametric functions:

a) $x = \dfrac{1}{t - 1}$, $y = \dfrac{1}{t^2 - 1}$ b) $x = \dfrac{t}{t-1}$, $y = \dfrac{t^2}{t^2 - 1}$

130. Same question for:

a) $x = \cos^3 t$, $y = \sin^3 t$ b) $x = t - \sin t$, $y = 1 - \cos t$

131. Find $f'(x_0)$ if:

a) $f(x) = 2\sqrt{x}\,(3x - 2)$, $x_0 = 4$

b) $f(x) = \sqrt[3]{ax^2} + \sqrt[3]{a^2x}$, $x_0 = a$

c) $f(x) = (x + 1)/\sqrt{x}$, $x_0 = 1/4$

d) $f(x) = (1 + \sqrt{x})(2 + \sqrt{x})/x$, $x_0 = 1$

132. Find the derivative of the following functions:

a) $y = \arctan x - \frac{1}{2}\arctan\frac{2x}{1 - x^2}$

b) $y = \arcsin x + \arccos x$

c) $y = \arcsin x - \frac{1}{2}\arcsin 2x\sqrt{1 - x^2}$

d) $y = \arccos x - \frac{1}{2}\arccos\,(2x^2 - 1)$

133. Given $y = 4 - x^2$, find the tangent of the angle α between the tangent line and the line joining the origin to the point of contact. Find also $\lim \tan\alpha$ as x tends to:

a) $-\infty$, b) -2 c) 0, d) (2), e) ∞

134. A particle moves along the curve $y = x^3 - x$. If $v_x = 2$ m/sec and $a_x = -3\text{m/sec}^2$ as particle passes through the point $(1, 0)$, find v and a at that instant.

135. Find $f(x)$, $g(x)$ if $(fg)(x) = 3x^3 + 3x - 6$, $f'(x) = 2x + 1$, and $g'(x) = 3$.

136. Show that the following functions have multiple roots; and find these roots with their multiplicity:

a) $x^4 - 2x^3 + 2x - 1 = 0$

b) $x^4 - x^3 - 7x^2 + 16x + 36 = 0$

c) $x^6 + 2x^5 - x^4 - 4x^3 - x^2 + 2x + 1 = 0$

137. Prove that if a polynomial equation $P(x) = 0$ has an r-fold root, then $P'(x) = 0$ has an $(r - 1)$-fold root and conversely.

138. Find the equations of the tangent and normal lines to the following curves at the indicated points:

a) $x = t^3 - 3t, \quad y = t^2 - 2t; \quad t = 2$

b) $x = \sin t, \quad y = \tan t; \quad t = \pi/4.$

139. For what value of x the derivative of $f(x)$ is equal to the derivative of $g(x)$ if

a) $f(x) = x^3 - 2x, \; g(x) = -x^2 + 3x + 8$

b) $f(x) = \dfrac{x}{x + 2}, \quad g(x) = x/2.$

140. Find the equation of the tangent line to $y = x^3 - x$

a) parallel to $y = 2x + 1$

b) perpendicular to $y = 3x - 1.$

141. $s = 15 + v_0 t - \frac{1}{2} g t^2$ is the equation of motion of a particle thrown upward with an initial velocity v_0. (s is given in meters and t in seconds)

a) Explain the meaning of 15,

b) find the largest height,

c) find the largest velocity,

d) find the velocity when it is turning back, at $s=15$.

142. If $f(x + y) = f(x) + f(y)$ and if $f(0) = 0$, find $f'(x)$ and prove that f is a linear function.

143. If $f(x + y) = f(x)/f(y)$, and if $f(0) = 1$, find $f'(x)$ and $f^{(n)}(x)$ and show that $f'(0) = 0$ or $f(x) = c$.

144. If $f(x + y) = f(x)\,f(y)$ and $f(0) = 1$, find $f'(x)$

and $f''(x)$, $f^{(n)}(x)$.

145. If $dy/dx = p(x)$, then evaluate dx/dy, d^2x/dy^2, d^3x/dy^3 and express them in terms if p, p', p''.

146. Compute approximate increment of the following functions when x increases (decreases) from x_1 to x_2:

a) $f(x) = x^3 - 4\sqrt[3]{x}$, $x_1 = 1$, $x_2 = 0{,}98$

b) $f(x) = \dfrac{x^2 + 3x}{x + 1}$, $x_1 = 0$, $x_2 = 0{,}01$

147. Compute the following approximately by the use of differential:

a) $\sqrt{67}$ b) $\sqrt[3]{67}$ c) $(2{,}02)^3 - 5(2{,}02)^2 + 7(2{,}02)$.

148. a) Find the area of a circle when its rate of change with respect to a diameter is 4π m^3/min,

b) Find the rate of change of the circumference of a circle with respect to the area when the area is equal to $4\pi\, m^2$.

149. a) Find the rate of change of the area of a circle with respect to its radius when radius is 3m,

b) Find the rate at which the volume of a right circular cylinder of constant altitude 10 m, changes with respect to its diameter when the radius is 5 m.

150. If $y = 3x - x^3$ and x increases at the rate of 1/3 units per second, how fast is the slope of the curve changing when $x = 3$?

151. The equal sides of an isoceles triangle are 10 m long and base angles are decreasing at a rate of 2^o/sec. Find the rate of change of the area when

the base angles are 45°.

152. Each of two sides of a triangle are increasing at the rate of 0,5 m/sec, and the included angle decreasing 2°/sec. Find the rate of change of the area when the two sides are 5 m, 8 m and the included angle is 60°.

153. A particle travels along the parabola $y = ax^2+x+b$. At what point its abscissa and ordinate change at the same rate?

154. A particle starts at the origin and travels up the line $y = \sqrt{3}x$ at the rate of 5 m/sec. Two seconds later another particle starting at the origin travels up the line $y = x/\sqrt{3}$ at a rate of 10 m/sec. At what rate are they separating 2 seconds after the last particle has started?

155. A clock has hands 5 and 7 cm long. At what rate are the ends of the hands approaching each other when the time is 2 o'clock?

156. Find the point required by the ROLLE's Theorem for the following function in the indicate interval:

a) $f(x) = \arctan x - \pi x/4$, $[-1, 1]$

b) $f(x) = \sqrt{3x - x^2}$, $[0, 3]$

c) $f(x) = (x + 2)^{1/3}(x - 1)^2 + 6$, $[-2, 1]$

d) $f(x) = x^{1/3} - x/4$, $[-8, 8]$

157. Find the value of x which satisfies the MVT for the following function for given interval:

a) $y = \tan x$, $[0, \pi/3]$, b) $y = \arcsin x$, $[0, 1]$

c) $y = |x^3 - 1|$, $[2, 4]$, d) $y = \sec x$, $[0, \pi/3]$

158. Evaluate:

a) $\lim\limits_{x\to\infty} \dfrac{x\sqrt{x} + \sqrt[3]{x + 1}}{\sqrt{x^3 - 1} + x}$

b) $\lim\limits_{x\to\infty} \dfrac{x^{-1/2} + x^{-3/2}}{(2x+1)^{-3/2} + x^{-1}}$

159. Evaluate:

a) $\lim\limits_{x\to 0} \dfrac{\sqrt{a + x} - \sqrt{a - x}}{\sqrt{(a^2 - x^2) + (x - a)}}$

b) $\lim\limits_{x\to 1} \dfrac{x - (n + 1)x^{n+1} + nx^{n+2}}{(1 - x)^2}$

c) $\lim\limits_{n\to\infty} (x - \frac{1}{n})(x - \frac{2}{n}) \ldots (x - \frac{r}{n})$,

d) $\lim\limits_{x\to a} \dfrac{x^{1/3} - a^{1/3}}{x^{1/2} - a^{1/2}}$

160. Evaluate:

a) $\lim\limits_{x\to\infty} [\sqrt[3]{x^3 + 2x^2} - \sqrt[3]{x^3 + 8}]$

b) $\lim\limits_{x\to 0} [\cot x - \frac{1}{x}]$

c) $\lim\limits_{x\to \frac{\pi}{2}} [\sec x - \csc 2x]$

d) $\lim\limits_{x\to 0} [\frac{1}{x} - \frac{1}{\sin x}]$

161. Find the intervals of increase and decrease of

a) $f(x) = \sqrt{1+x} + \sqrt{1-x}$ and find its inverse function

b) $f(x) = x^n + x^{-n}, \quad n \in N$

162. Determine the constants a, b, c such that

$$\lim_{x \to 0} \frac{\sqrt{1+2x+2x^2} - a - bx - cx^2}{x^4} \text{ exists.}$$

163. Evaluate:

a) $\displaystyle \lim_{x \to a} \frac{(x^2+a^2)(x^2-a^2)^{3/2}}{(3x^2-a^2)(x^3-a^3)^{2/3}}$

b) $\displaystyle \lim_{x \to a} \frac{\sqrt{a^2-x^2} + a - x}{\sqrt{a-x} + \sqrt{a^3-x^3}}$

c) $\displaystyle \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a} + \sqrt{x-a}}{\sqrt{x^2-a^2}}$

164. Evaluate:

a) $\displaystyle \lim_{x \to \pi/2} (\sec x - \tan x)$ b) $\displaystyle \lim_{x \to 3} \frac{\tan \frac{\pi x}{6}}{\sec(x + \frac{\pi}{2} - 3)}$

c) $\displaystyle \lim_{x \to -1} \frac{(x+1)\sin(x+1)}{\sin(x^2-1)}$ d) $\displaystyle \lim_{x \to \pi/3} \frac{\arctan(x - \pi/3)}{\tan x - \sqrt{3}}$

165. Evaluate:

a) $\displaystyle \lim_{x \to 0} \frac{\sin x - x}{x^3}$ b) $\displaystyle \lim_{x \to 0} \frac{\tan x - x}{x - \sin x}$

c) $\displaystyle \lim_{x \to 0} \frac{\arcsin x - x}{x^3}$ d) $\displaystyle \lim_{x \to -3} \frac{\sqrt{6-x} - \sqrt{1-x} - 1}{x+3}$

166. Evaluate:

a) $\lim\limits_{x \to 0} \dfrac{\arctan x - x}{x^3}$ b) $\lim\limits_{x \to 0} \dfrac{\arctan x - x}{x - \arcsin x}$

167. Evaluate:

a) $\lim\limits_{x \to \pi/2} (\sec x - \tan x)$ d) $\lim\limits_{x \to \pi} \left(\dfrac{1}{\sin x} - \dfrac{1}{\pi - x}\right)$

c) $\lim\limits_{x \to 0} x^2 \csc(3 \sin^2 x)$

168. Find an irrational roots of the following equations to two decimal places by NEWTON's Method:

a) $x^3 + 2x - 5 = 0$ b) $x^4 + x^3 + x^2 - 1 = 0$

c) $x^4 - 4x^2 - 4x - 8 = 0$ d) $x^5 + x^3 + 2x - 5 = 0$

169. Determine the constants a, b, c so that the curve $y = ax^3 + bx^2 + cx$ will have a slope of 4 at its points of inflection $(-1, -5)$.

170. Find the relative extrema of the following functions:

a) $f(x) = (x-2)^3(x+1)$ b) $f(x) = x^4 - 2x^2$

c) $f(x) = 2x^3 - 3x^2$ d) $f(x) = \sqrt[5]{x^2 - 2x}$

171. Find and identify the relative extrema:

a) $f(x) = 2x^2 + 2/x^2$ b) $f(x) = 2x^4 + 2/x^4$

c) $f(x) = x^{2/3}(x-2)$ d) $f(x) = x\sqrt[3]{x^2 - 4}$

172. Find the relation between the constants a, b such that the following functions have no extrema:

a) $y = x^3 + ax^2 + bx + c$ b) $y = a/x + bx$

173. For the functions in Exercise 172 find a and b to have critical point at

a) $(1, 0)$ b) $(2, 1)$ respectively.

174. Find the relative extrema of

$$f(x) = 2x^3 - 3(a + b)x^2 + 6abx$$

a) if $a < b$ b) if $a = b$

175. Find, if any, the value of t for which the given function has an extremum and identify.

a) $x = t^2 - t$, $y = 2t^4 - t$ b) $x = \sin t + \cos t$, $y = \sin t - \cos t$

176. Find the value of t for the function to have an extremum and identify:

a) $x = t^3 + 3t + 2$, $y = t^2 - 1$ b) $x = 3 \cos t$, $y = 4 \sin t$

177. Find the absolute maximum M and absolute minimum m of the following functions in the given interval.

a) $y = \dfrac{\tan x}{1 + \sin x}$, $[0, \pi/4]$

b) $y = \dfrac{\sin x}{2 + \sec x}$, $[0, \pi/3]$

178. Show that the function $f(x) = x^3(x + a)$ attains a minimum of $-27\, a^4/256$.

179. Find M and m on the given interval:

a) $y = x\sqrt{x^2-4}$, $[-3, 5]$ b) $y = [\![x]\!]\,|x^2-1|$, $[0, 2]$

c) $y = x \sin x$, $[0, \pi/4]$ d) $y = \dfrac{\cos x}{x}$, $[\pi/6, \pi/2]$

180. Find the minimum and maximum of

$$f(x) = \left(\frac{a}{x}\right)^k + \left(\frac{x}{b}\right)^k$$

where a, b, k, x are positive

181. Find m, M of the function $f(x) = x/(x+1)$

182. Same question for $f(x) = 3\sqrt{x/(x^2+1)}$

183. Same question for $f(x) = \sin x + \sqrt{3}\cos x$

184. If m, n are positive integers with $m > n$, prove

a) $\dfrac{x^m - 1}{m} > \dfrac{x^n - 1}{n}$, for $x > 1$

b) $\dfrac{x^m - 1}{m} > \dfrac{x^n - 1}{n}$, for $0 < x < 1$

185. Prove the inequalities is Exercise 184 for $m, n \in \mathbb{Q}$.

186. If $f(x) = x + 2$ and $g(f(x) = x^2 - 3x$, find $g(x)$.

187. Defining

$$\max\{f(x),\ g(x)\} = \begin{cases} f(x) & \text{when } f(x) \geqslant g(x) \\ g(x) & \text{when } g(x) \geqslant f(x), \end{cases}$$

$$\min\{f(x),\ g(x)\} = \begin{cases} f(x) & \text{when } f(x) \leqslant g(x) \\ g(x) & \text{when } g(x) \leqslant f(x), \end{cases}$$

prove that if $f(x)$, $g(x)$ are continuous, then

a) $\max\{f(x),\ g(x)\}$ b) $\min\{f(x),\ g(x)\}$

are continuous.

188. Sketch $\max\{f(x),\ g(x)\}$ where

a) $f(x) = x^2$, $g(x)=x$ b) $f(x) = 1$, $g(x) = x^2$

c) $f(x) = x^2$, $g(x) = x^2+1$ d) $f(x) = x^3-x$, $g(x)=2x-2$

189. For the given functions in Exercise 188, sketch $\min\{f(x),\ g(x)\}$

190. Sketch:

a) $\{(x, y)\colon \max\{x, y\}=2\}$ b) $\{(x, y)\colon \min\{x, y\} = 2\}$

c) $\{(x, y): \min \{x, y\} > 2\}$ d) $\{(x, y): \max \{x, y\} \leq 2\}$

191. Sketch:

a) $\{(x, y): \max \{x, y\} = x+y\}$

b) $\{(x, y): \min \{x, y\} = x-y\}$

192. Sketch:

a) $\{(x, y): \max \{|x|, |y|\} < 1\}$

b) $\{(x, y): \min \{|x|, |y|\} < 1\}$

193. Find the cartesion equation of the following parametric curves:

a) $x = \sin t + \cos t$, $y = \sin t$ b) $x = \sin t$, $y = \cos 2t$

194. Same question for:

a) $x = 3 - t$, $y = t^2 - 2$ b) $x = 1/(1 + t^2)$, $y = t/(1 - t^2)$

195. Same question for:

a) $x = \tan t$, $y = \tan 2t$ b) $x = 1 - t^2$, $y = t + t^3$

196. Sketch: $y = \dfrac{3x^2 + 10x + 7}{x^2 + 2x + 2}$

197. Sketch:

a) $x = t + 1/t$, $y = t - 1/t$ b) $y = \dfrac{\sin x}{1 + \tan x}$

198. Sketch:

a) $y = \dfrac{(x - 2)^2}{x}$ b) $y = \dfrac{x^3 + 4x}{x + 1}$

c) $y = \dfrac{x^2 + 1}{4x}$ d) $y = (x - 2)\sqrt{x^2 - 2x}$

199. Sketch:

a) $y = \dfrac{4x^3}{x^3 + 8}$ b) $y = \dfrac{4x^3}{x^2 + 1}$

200. Sketch $y = \sin \dfrac{\pi}{x + [\![\frac{1}{1 + x^2}]\!]}$

201. Find a rational function $y = (ax^2+bx+c)/(x+d)$ which cannot be between -2 and 3 for any real value of x, and graph it.

202. P lies on the semicircle with AC as diameter. OB is the radius perpendicular to AC and H is the production of P on AC. Determine the position of P

a) on the arc $\overset{\frown}{AC}$, b) on the arc $\overset{\frown}{AB}$

so that $|AP| + |PH|$ is a maximum.

203. A statue 7 m tall stands on a monuement 10 m heigh. Where should a boy stand on the ground so that the view of statue be greatest.
(Eye's level of the boy is 1 m).

204. The material for the top and bottom of a cylindrical can costs twice as much her square unit as the material for the sides. Find the ration of altitude to the radius for which the cost is the least.

205. Find the maximum area of a right trapezoid of given altitude "h" and given perimeter "ℓ".

206. Find the maximum area of an isoceles trapezoid of base "a" and perimeter "2ℓ".

207. If a sum of the areas of a sphere and a regular tatrahedron is constant, what is the ratio of the

diameter to an edge of the tetrahedron when the sum of their volume is

a) a maximum b) a minimum

208. Find the slope of the line through the origin if the sum of its distances from the points $(2, 0)$, $(1, 3)$ and $(3, 3)$ is a

a) a minimum b) a maximum

209. Generalize the problem given Exercise 208 to n points $(a_1, b_1), \ldots, (a_n, b_n)$.

210. Given n points $(r_1 \cos\theta_1, r_1 \sin\theta_1), \ldots, (r_n \cos\theta_n, r_n \sin\theta_n)$ determine a circle, center at the origin, such that the sum of the squares of the distances of the points from the circumference be minimum.

Answers to even numbered exercises

126. a) $f'(0) = 0$, b) No derivative, c) No derivative, d) $-1/49$.

128. $-2\pi \cos 2x \cos(\pi \cos^2 x) - \pi^2 \sin^2 2x \sin(\pi \cos^2 x)$

130. a) $-\tan t$, $3 \sec^4 t \csc t$, b) $\sin t/(1-\cos t)$, $-1/(1-\cos t)^2$

132. a) 0, b) 0, c) 0, d) 0.

134. $2\sqrt{5}$ m/sec, $3/\sqrt{5}$ m/sec^2.

136. a) 1; 3-fold b) -2; 2-fold
c) -1; 4-fold 1; 2-fold 1; 2-fold

138. a) $2x - 9y = 4$, $9x + 2y = 18$,
b) $2\sqrt{2}\, x - y = 1$, $2x - 4\sqrt{2}\, y = -3\sqrt{2}$.

140. a) $y = 2x - 2, \quad y = 2x + 2$

b) $9x + 27y = -4\sqrt{2}, \quad 9x + 27y = 10\sqrt{2}$

142. $f'(x)$ is constant

144. $f'(x) = f(x)\, f'(0), \quad f''(x) = f(x)\, f'^2(0),$

$f^{(n)}(x) = f(x)\, f'^n(0).$

146. a) $-0,0\overline{3}$, b) $0,03$

148. a) $16\pi \, m^2$, b) $0,5 \, m^{-1}$

150. -9

152. $13\sqrt{3}/8 - \pi/9 \; m^2/sec$

154. $(19 - 10\sqrt{3})/122$ m/sec.

156. a) $\pm\sqrt{(4-\pi)/\pi}$, b) $3/2$, c) $-7/11$, d) $\pm 8/(3\sqrt{3})$.

158. a) 1, b) $\sqrt{2}$.

160. a) $2/3$, b) 0, c) No limit, d) 0

162. $a = 1, \quad b = 2, \quad c = 2$

164. a) 0, b) $6/\pi$ c) 0, d) $1/4$.

166. a) $-1/3$, b) -2.

168. a) 1,33 b) 0,68, c) 2,60 d) 1,09

170. a) -1, $1/5$ b) $0, \pm 1$, c) 0, 1, d) min 1

172. a) $a^2 \leq 3b$ b) $ab < 0$.

174. a) $f(b)$, $f(a)$, b) No rel. extr.

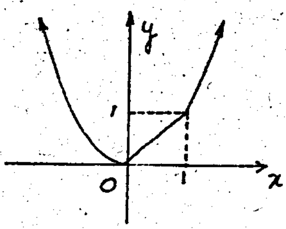
176. a) $t = 0$ (min), b) $t = \pi/2$ (max), $t = 3\pi/2$ (min)

180. $x = \sqrt{ab}, \quad f(\sqrt{ab}) = 2(b/a)^{k/2}$

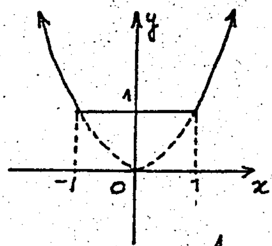
182. $m = 0, \quad M = 3/2$

186. $g(x) = x^2 - 7x + 10$

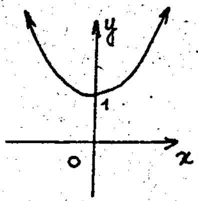
188. a)

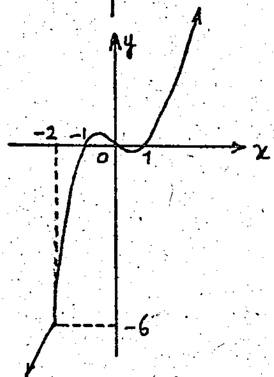


b)

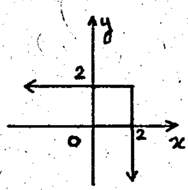


c) g(x):

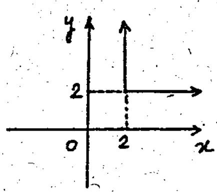




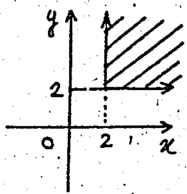
190. a)



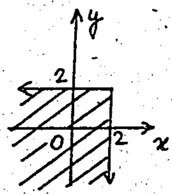
b)



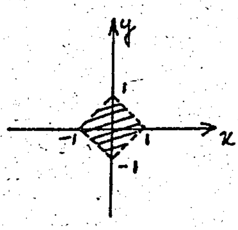
c)



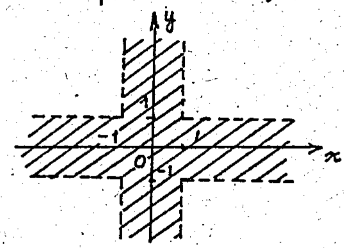
d)



192. a)

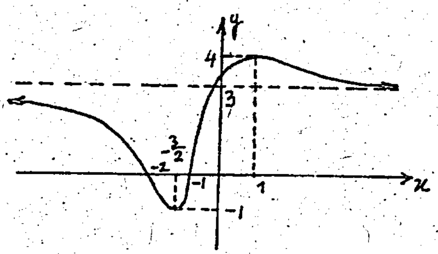


b)

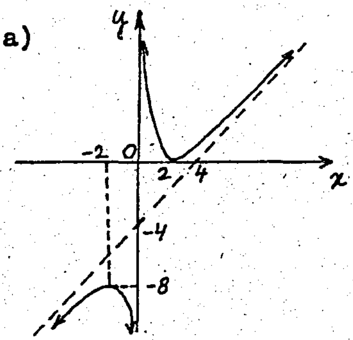


194. a) $y = x^2 - 6x + 7$ b) $x^2 + 2xy - x - y = 0$

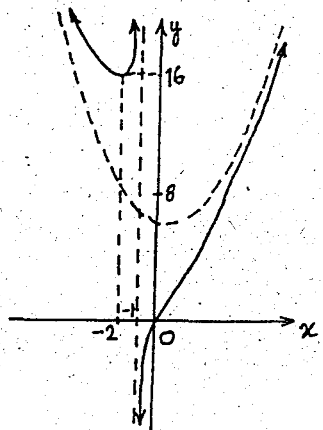
196.



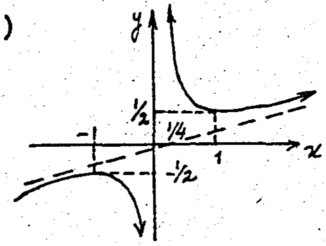
198. a)



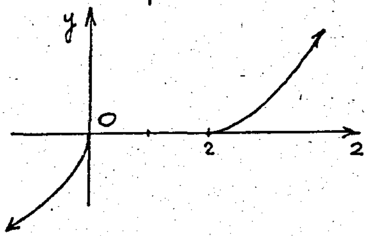
b)



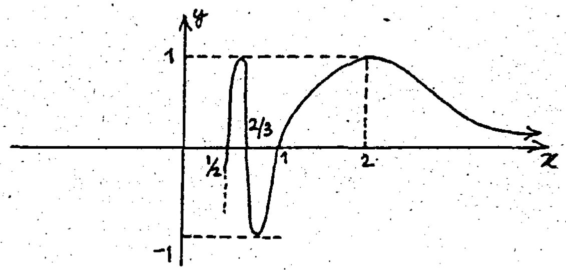
c)



d)



200. The curve is symmetric with respect to the origin:



$y = 0$ at $-x = 1/n$ for all $n \in N^{+}$

$y = 1$ at $x = \dfrac{2}{2n+1}$ (n even)

$y = -1$ at $x = \dfrac{2}{2n+1}$ (n odd)

202. a) PCO is an equilateral triangle, b) B

204. 4

206. $(\ell - a)\sqrt{(\ell^2 - a^2)/3}$.

208. a) 1, b) -1

210. $r = (r_1 + \dots + r_n)/n$.