Quiz 01

CMPE 58N, Monte Carlo Methods

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1. (β) (**Beta Distribution**) Given

$$x_i \sim \mathcal{G}(x_i; a_i, 1) = \exp((a_i - 1)\log x_i - x_i - \log\Gamma(a_i))$$

for $i = 1, 2$, find the distribution of

$$y_1 = \frac{x_1}{x_1 + x_2}$$

[Hint: $\mathcal{G}(x; a, b) = \exp((a-1)\log x - bx - \log\Gamma(a) + a\log b)$]

We define

$$\begin{aligned} y_1 &= \frac{x_1}{x_1 + x_2} \\ y_2 &= x_1 \end{aligned}$$

The inverse transformation is

$$\begin{aligned} x_1 &= y_2 \\ x_2 &= y_2/y_1 - y_2 = \frac{y_2(1 - y_1)}{y_1} \end{aligned}$$

The Jacobian determinant

$$\begin{aligned} J &= \begin{vmatrix} \partial x_1/\partial y_1 & \partial x_2/\partial y_1 \\ \partial x_1/\partial y_2 & \partial x_2/\partial y_2 \end{vmatrix} \\ &= \begin{vmatrix} 0 & -y_2/y_1^2 \\ 1 & 1/y_1 - 1 \end{vmatrix} = y_2/y_1^2 \end{aligned}$$

The density is

$$
\begin{aligned}
p(y_1, y_2) &= p(x_1(y_1, y_2), x_2(y_1, y_2))|J| \\
&= \exp((a_1 - 1)\log y_2 - y_2 - \log\Gamma(a_1)) \\
&\quad \exp((a_2 - 1)\log \frac{y_2(1-y_1)}{y_1} - \frac{y_2(1-y_1)}{y_1} - \log\Gamma(a_2))y_2/y_1^2 \\
&= \exp((a_1 - 1)\log y_2 - y_2 - \log\Gamma(a_1)) \\
&\quad \exp((a_2 - 1)\log y_2 + (a_2 - 1)\log(1-y_1) - (a_2 - 1)\log y_1 - \frac{y_2}{y_1} + y_2 - \log\Gamma(a_2)) \\
&\quad \exp(\log y_2 - 2\log y_1) \\
&= \exp((a_1 + a_2 - 1)\log y_2 - \frac{y_2}{y_1}) \\
&\quad \exp(-2\log y_1 - \log\Gamma(a_1) + (a_2 - 1)\log(1-y_1) - (a_2 - 1)\log y_1 - \log\Gamma(a_2)) \\
&= \exp((a_1 + a_2 - 1)\log y_2 - \frac{y_2}{y_1} - \log\Gamma(a_1 + a_2) - (a_1 + a_2)\log y_1) \\
&\quad \exp(+\log\Gamma(a_1 + a_2) + (a_1 + a_2)\log y_1) \\
&\quad \exp(-2\log y_1 - \log\Gamma(a_1) + (a_2 - 1)\log(1-y_1) - (a_2 - 1)\log y_1 - \log\Gamma(a_2)) \\
&= \mathcal{G}(y_2; a_1 + a_2, 1/y_1)\frac{\Gamma(a_1 + a_2)}{\Gamma(a_1)\Gamma(a_2)} y_1^{a_1 - 1}(1-y_1)^{a_2 - 1}
\end{aligned}
$$

Hence

$$
\begin{aligned}
p(y_1) &= \frac{\Gamma(a_1 + a_2)}{\Gamma(a_1)\Gamma(a_2)} y_1^{a_1 - 1}(1-y_1)^{a_2 - 1} \\
p(y_2|y_1) &= \mathcal{G}(y_2; a_1 + a_2, 1/y_1)
\end{aligned}
$$

2. (β) (**Bonus**) Given

$$x_i \sim \mathcal{G}(x_i; a_i, \lambda)$$

for $i = 1, 2$, find the distribution of

$$y_1 = \frac{x_1}{x_1 + x_2}$$