Boğaziçi University, Dept. of Computer Engineering

CMPE 58N, MONTE CARLO METHODS

Spring 2009, Example Midterm

Name: ______________________

Student ID: ______________________

Signature: ______________________

- Please print your name and student ID number and write your signature to indicate that you accept the University honour code.
- During this examination, you may use any notes, books or laptops. You can even lookup resources on the internet; however communication with fellow students is not allowed.
- Read each question carefully and show all your work. Underline your final answer to each question.
- There are **about 8** questions. Point values are given in parentheses.
- You have **180 minutes** to do all the problems.

Q	1	2	3	4	5	6	Total
Score							
Max							

1. (**What is ...**) Use the space below each question. Give concise answers, long answers (> 2 sentences) don't get any points.

   (a) *(1 pts)* What is a Gibbs sampler?.

   (b) *(1 pts)* What is ergodicity?.

   (c) *(1 pts)* ...

2. (**could have been a quiz question**) Given

$$g_i \sim \mathcal{G}(g_i; a_i, \lambda)$$

for $i = 1, 2, 3$, find the distribution of

$$z = \frac{g_1}{g_1 + g_2 + g_3}$$

(12 points)

(blank page for further space)

3. **(Knuth and Yao Algorithm)** In preparation

(10 points)

4. **(Rejection Sampling)** In preparation

(10 points)

5. **(Importance sampling)** Consider the target distribution

$$p(x) = \mathcal{N}(x; 1, 1)$$

when using the proposal

$$q(x) = \mathcal{N}(x; \mu, \Sigma)$$

(a) Derive the weight function for the target distribution

(b) Derive the analytic expression for the variance of importance weights

(10 points)

6. **(Estimation of Gaussians)** Consider the following model

$$
\begin{aligned}
\beta &\sim \mathcal{G}(\beta; \nu, 1) \\
\mu &\sim \mathcal{N}(\mu; 0, 1000) \\
x_i &\sim \mathcal{N}(x_i; \mu, \beta^{-1})
\end{aligned}
$$

for $i = 1 \dots N$. $X = \{x_1, \dots, x_N\}$. Furthermore, use

$$
\begin{aligned}
\mathcal{G}(\beta; \nu, s) &\equiv \exp((\nu - 1)\log\beta - s\beta - \log\Gamma(\nu) + \nu\log s) \\
\mathcal{N}(x; m, s) &= \exp(-\frac{1}{2s}(x - m)^2 - \frac{1}{2}\log(2\pi s))
\end{aligned}
$$

(a) (4) Derive the full conditionals and give pseudocode of a Gibbs sampler to sample from

$$p(\mu, \beta | X, \nu)$$

(b) (4) Derive the expression for the acceptance probability of a Metropolis-Hastings algorithm to sample from

$$p(\nu | X, \mu, \beta)$$

Use the following proposal $q = \mathcal{G}(\nu; a, 1)$

(c) (2) Combine (a) and (b) and derive a MH algorithm to sample from

$$p(\nu, \mu, \beta | X)$$

(10 points)

(blank page for further space)

7. (**Metropolis and Gibbs**) x_1 and x_2 are two discrete random variables taking values in $\{-1, 1\}$. Suppose we have the joint distribution $p(x_1 = a, x_2 = b) = \pi_{a,b}$. We further have $g = \pi_{-1,1} = \pi_{1,-1} > \pi_{1,1} = \pi_{-1,-1}$.

   Suppose we implement a Metropolis algorithm to sample from this target distribution with the following proposal technique: Given the current configuration $x^{(n)} = (x_1^{(n)}, x_2^{(n)})$, for each n, we choose an index $i^{(n)} \in \{1, 2\}$ randomly with probability 0.5 and flip the sign of $x_{i^{(n)}}$.

   (a) *(1 pts)* Write down the state transition diagram of the proposal distribution and indicate the state transition probabilities,

   (b) *(1 pts)* Find an expression for the acceptance probability as a function of g,

   (c) *(1 pts)* Write the pseudocode for the Metropolis sampler,

   (d) *(2 pts)* Write down the state transition diagram of the transition Kernel T_M of this Metropolis algorithm and indicate the transition probabilities,

   (e) *(2 pts)* Verify if detailed balance condition is satisfied by this particular Metropolis algorithm (i.e., if $T_M(x|x')\pi(x') = T_M(x'|x)\pi(x)$) for all values of g.

   (f) *(1 pts)* Suppose we also implement a deterministic scan Gibbs sampler (that is we sample alternatingly from the full conditional distributions $p(x_1|x_2)$ and $p(x_2|x_1)$). Write down the pseudocode.

   (g) *(2 pts)* Write down an expression for the Gibbs transition Kernel T_G in terms of g.

   (h) *(2 pts)* Verify detailed balance is satisfied the Gibbs transition Kernel T_G for all values of g.

(12 points)