CmpE 585 - 1KWQFA With Bounded Error

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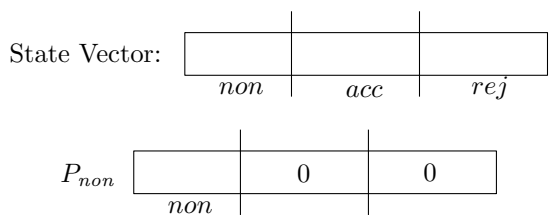
1KWQFA, $M = (Q, \Sigma, \delta.q_0, Q_{acc}, Q_{rej})$ $\Gamma = \Sigma \cup \{¢, \$\}$ The transition mapping δ:

- Unitary operator. $U_{\sigma \in \Gamma}$
- Measurement on the state space. $\mathbb{C}^{|Q|}$, Hilbert space. Inner product

$$l_2(Q) = E_{acc} \oplus E_{rej} \oplus E_{non}$$

$$E_{acc} = span\{|q\rangle | q \in Q_{acc}\}$$
$$E_{rej} = span\{|q\rangle | q \in Q_{rej}\}$$
$$E_{non} = span\{|q\rangle | q \in Q_{non}\}$$

Projection matrices: diagonal zero matrices, P_{acc}, P_{rej}, P_{non}.



Global state is a 3-tuple. $V = l_2(Q) \times R \times R$. at step i: (ψ, p_{acc}, p_{rej}) where ψ is the non-halting part of the global state vector, p_{acc} is the accept probability, and p_{rej} is the reject probability. The probability of continuation is $\|\psi\|_2^2$ (there will be another norm operator, so it is subscripted as 2).

$T_{\sigma \in \Gamma}$ on V is the evolution operator of M when reading σ.
$T_\sigma : (\psi, p_{acc}, p_{rej}) \mapsto (P_{non} U_\sigma \psi, p_{acc} + \|P_{acc} U_\sigma \psi\|_2^2, p_{rej} + \|p_{rej} U_\sigma \psi\|_2^2)$. For $w = w_1 \dots w_l \in \Gamma^*$, $(\psi, p_{acc}, p_{rej}) = T_w(|q_0\rangle, 0, 0)$ where $T_{w_l} \dots T_{w_2} T_{w_1}$. M accepts w with probabilty p_{acc}.

Define a norm on V as $\|(\psi, p_{acc}, p_{rej})\| = \frac{1}{2}(\|\psi\|_2 + |p_{acc}| + |p_{rej}|)$. Let $B = \{v \in V | \|v\| \leq 1\}$. Clearly, any representation of a 1KWQFA, v must be in B.

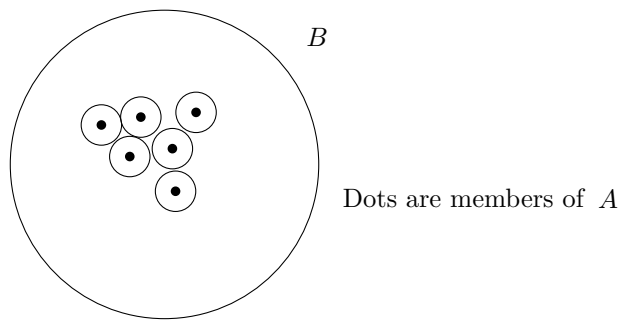
For every $v, v' \in B$ and $w \in \Gamma^*$, $\exists$ a fixed constant c,

$$\|T_w v - T_w v'\| \leq c\|v - v'\|$$

.

If a set $A \leq B$ satisfies the following property;

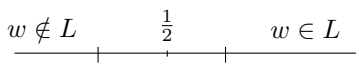
$$\exists \delta > 0 \text{ such that } \forall v, v' \in A,\ \|v - v'\| > \delta \text{ then, } A \text{ is a finite set}$$



Theorem 1. *Let L be any language recognizable by a 1KWQFA with bounded error, then L is regular.*

Proof. We will use the technique of Rabin (1963)

L, 1KWQFA, $p(M \text{ accepts } w) \notin [\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon]$



$w \equiv_L w'$ if $\forall y \in \Sigma^* \; wy \in L \leftrightarrow w'y \in L$. Where $\equiv_L$ is equivalence class and partitions Σ^* into finitely many equivalence classes if and only if L is regular.

$W \subseteq \Sigma^*$, the set of strings which are pairwise inequivalent with respect to $\equiv_L$.

$$\forall w \neq w' \in W, \; \exists y \in \Sigma^* \; wy \in L \leftrightarrow w'y \notin L$$

.

Let $v = T_{¢w}(|q_0\rangle, 0, 0)$ and $v' = T_{¢w'}(|q_0\rangle, 0, 0)$

$$2\epsilon < \|\underbrace{T_{y\$}v}_{p_{acc}} - \underbrace{T_{y\$}v'}_{p'_{acc}}\| \leq c\|v - v'\|$$

Remember: $|p_{acc} - p'_{acc}| \geq 2\epsilon$ so $\|v - v'\| \geq \frac{2\epsilon}{c}$.

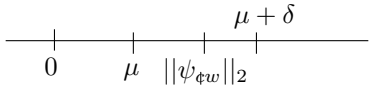
$T_{¢w}(|q_0\rangle, 0, 0)|w \in W)$ must be finite $\Rightarrow$ W must be finite. □

Theorem 2. *The language $L = \{a, b\}^* a$ cannot be recognized by a 1KWQFA with bounded error.*

Proof. $w \in \Gamma^* \; w = w_1 \ldots w_l \; \psi_w = (P_{non}U_{w_l}) \ldots (P_{non}U_{w_1})|q_0\rangle$.

Let $\mu = Inf\{\|\psi_{¢w}\|_2 | w \in \{a, b\}^*\}$ where Inf is the infimum (greatest lower bound of a set) then $\mu > 0$, otherwise M does not recognize L with bounded error.

Let $\delta > 0$, and chose w such that $\|\psi_{¢w}\|_2 < \mu + \delta$.



For every $y \in \{a,b\}^*$

$$\|\psi_{¢wy}\|_2 \in [\mu, \mu+\delta)$$

In particular, for any nonnegative integer j

$$\|(P_{non}U_b)^j \psi_{¢wa}\|_2 \in [\mu, \mu+\delta]$$

$\{(p_{non}U_b)^j \psi_{¢wa}\}_{j=0,1,\ldots}$ is a bounded sequence in $l_2(Q)$ therefore has a limit point u.

$\forall z > 0,\ \exists N,\ \forall j > N,\ \|a_j - a\| < z.$

$\exists j \geq 0$ and $k \geq 1$ $\|a_j - a_{j+k}\|_2 < \delta$

$$\|(P_{non}U_b)^j(\psi_{¢wa} - (P_{non}U_b)^k \psi_{¢wa})\|_2 < \delta$$

there is a fixed constant c' (independent of δ)

$$\|\psi_{¢wa} - (P_{non}U_b)^k \psi_{¢wa}\|_2 < c'\delta^{\frac{1}{4}}$$

and so $\|T_{¢wa\$}(|q_0\rangle, 0, 0) - T_{¢wab^k\$}(|q_0, 0, 0)\| < c''\delta^{\frac{1}{4}}$ for fixed c''. Since δ may be chosen arbitrarily small, and since M must accept wa and reject wab^k, M cannot have error probability bounded away from $\frac{1}{2}$. □

2KWQFA

6-tuple $(Q, \Sigma, \delta, q, Q_{acc}, Q_{rej})$ Details of δ: $U_{\sigma \in \Sigma \cup \{¢,\$\}}$. $\overrightarrow{q}$, $\overleftarrow{q}$, $\downarrow q$ (state specific head movement).

Configurations: $w \in \Sigma^*$

¢	w_1	w_2	$\ldots$	w_l	\$

$|w| = l$

$|¢w\$| = l + 2$

$$C^w = \{|c\rangle = |q, i\rangle | q \in Q, 1 \leq i \leq l+2\}$$

$$|\psi\rangle = \sum_{c_k \in C^w} \alpha_k |c_k\rangle$$

$$\sum_k |\alpha_k|^2 = 1$$

$$C^w_{acc} = Q_{acc} \times \{1, \ldots, l+2\}$$

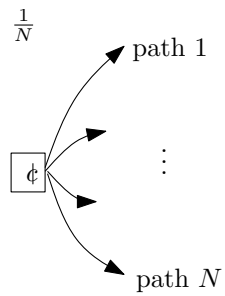
$$C^w_{rej} = Q_{rej} \times \{1, \ldots, l+2\}$$

$$C^w_{non} = Q_{non} \times \{1, \ldots, l+2\}$$

- 2PFA-polynomial=Regular
- 2PFA recognize $\{a^n, b^n\}$ in exponential time.
- 2KWQFA → 2RFA → 1DFA (exactly).

$L_{eq} = \{a^n b^n | n \geq 1\}$

1. Check $a^* b^*$, if not reject.
2. On path j: wait j step over a, wait $N - j + 1$ steps over b.



If input is $\{a^m b^n | m, n > 1\}$, path i, path j, $i \neq j$

For path i: $im + (N - i + 1)n$ steps. For path j: $jm + (N - j + 1)n$ steps.

$$\begin{aligned} (i-j)m &= (N - j + 1 - N + i - 1)n \\ (i = j)m &= (i - j)n \\ i = j &\leftrightarrow m = n \end{aligned}$$

If $m = n$, paths reach \$ at the same step.

3. 2-QFT:

$$\left.\begin{array}{l} \frac{1}{\sqrt{2}}\text{path-1} \rightarrow \frac{1}{\sqrt{2}}A + \frac{1}{\sqrt{2}}R = \frac{1}{2}A + \frac{1}{2}R \\ \frac{1}{\sqrt{2}}\text{path-2} \rightarrow \frac{1}{\sqrt{2}}A - \frac{1}{\sqrt{2}}R = \frac{1}{2}A - \frac{1}{2}R \end{array}\right\} 1A + 0R$$

N-QFT: If $m = n$, $1A + 0R$

If $m \neq n$, $\frac{1}{N}$ accept, $1 - \frac{1}{N}$ reject. $O(N|w|)$.