# Computer Vision
# Week 5

Segmentation: boundary based techniques

## Segmentation

Given a set of image pixels I and a uniformity predicate P(.); find a partition S of the image I into a set of n regions $R_i$ such that:

1. $\bigcup_i R_i = I \qquad R_i \bigcap R_j = \varnothing$
2. $\mathrm{P}(R_i) = \mathrm{True}$
3. $\mathrm{P}(R_i \bigcup R_j) = \mathrm{False}$ for $R_i$ adjacent to $R_j$

## Two Approaches for Segmentation

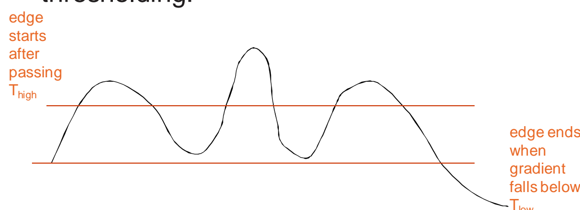
1. Detect discontinuities: Boundary based approach - edge detection followed by thinning, linking, etc.
2. Detect uniformity: Region growing

## Boundary based segmentation

- We have performed edge detection
- However, edges are:
  - broken,
  - too thin / too thick
  - Not closed
- We have to perform
  - Edge linking / edge following
  - Edge thinning
  - Hough transform
  - Functional approximation: Model-based approach
    - Curve fitting
    - Curve approximation

## Edge linking / edge following

- Canny does edge linking by: double thresholding:




## Edge linking as search problem

- Suppose you are given a gradient image that has undergone some thresholding; however, gradient magnitudes and angles of pixels that exceed the threshold are retained
- Then, edge linking becomes a search for a connected path from pixel A to pixel B
- Cost function: $f(x_i)=s(x_i,x_A)+h(x_i,x_B)$
  - $s(x_i,x_A)$: Cost from A to i: known
  - $h(x_i,x_B)$: Cost from i to B: unknown; to be estimated
- Possible cost functions: edge strength, curvature, proximity to goal

## Search alternatives

1. Heuristics: $h(x_i,x_B)$ =0 minimum cost search
2. Depth-first search
3. Least max cost: Keep the lowest cost branch
4. Pruning the tree when a threshold is reached
5. Branch & bound
6. Dynamic programming

## Dynamic Programming

- Dynamic programming is applicable when the cost function consists of linear terms:

$$\max_{x_i} h(x_1,x_2,x_3,x_4)$$

$$h(.) = h_1(x_1,x_2) + h(x_2,x_3) + h(x_3,x_4)$$

maximize over $x_1$ in $h_1$ and tabulate the best value

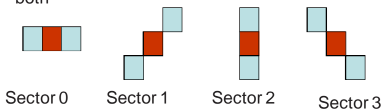
$$f_1(x_2) = \max_{x_1} h_1(x_1,x_2)$$

$$f_2(x_3) = \max_{x_2} [f_1(x_2) + h_2(x_2,x_3)]$$

$$f_3(x_4) = \max_{x_2} [f_2(x_3) + h_3(x_3,x_4)]$$

## Edge thinning

- Canny does edge thinning by nonmaxima suppression:
  - Classify gradient angle into one of 4 sectors:
    - 0: -22.5 to 22.5, 180-22.5 to 180+22.5
    - 1: 22.5 to 67.5, 180+22.5 to 180+67.5
    - 2: 67.5 to 112.5, 180+67.5 to 180+112.5
    - 3: 112.5 to 157.5, 180+112.5 to 180+157.5
  - Compare center with the 2 neighbors, set to 0 if not greater than both

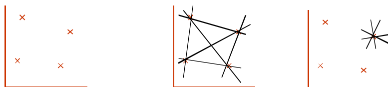



## Model-based Approaches:

- Hough Transform
- Regularization theory- inspired: GED
- Facet Model (Haralick and Shapiro)
- Curve fitting: Lines, circles, splines
- Curve approximation: Linear and nonlinear regression
- Deformable templates
- Snakes
- Active Contours

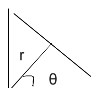
## Hough Transform

- Suppose you have some edge points
- Find out which lines pass through these points:



- The equation of a line that passes through (x,y):

$$x \cos\theta + y \sin\theta = r$$

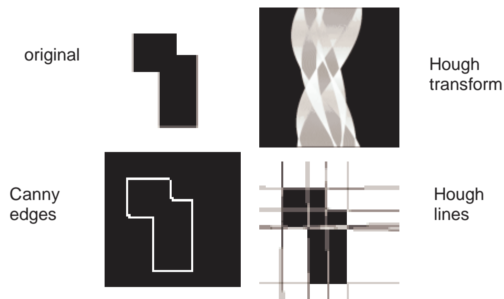



## Hough Transform (2)

Two-stage algorithm: 1. Fill an accumulator array A[r,theta]
2. Find the peaks in the accumulator array

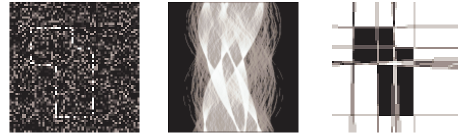
1. Apply an edge operator to image
   Apply thresholding to gradient image to find edge pixels
   For all edge pixels, {set theta to the edge direction
   quantize theta
   find corresponding value of r
   quantize r
   Increment A[r,theta] by 1}
2. Do until the desired number of lines is extracted:
   {Find the max of A[r,theta]
   Set peak value and neighbors to 0}

## Hough Transform (3)




## Hough Transform (4)

Noise performance:



## Generalized Hough transform

- Hough transform may be generalized to detect other curves such as circles, ellipses, etc.

## Curve fitting

- General approach: Segment arcs into simple segments
- Then, model these with simple curves such as:
  - Line segments
  - Circular arcs
  - Cubic polynomial curves
- Curve fitting vs. curve approximation

## Curve Representation

- 3 types of representation for curves:
  - Explicit
  - Implicit
  - Parametric
- Simplest curve: Line
  - Explicit: $y=mx+B$
  - Implicit: $ax+by+c=0$
  - Parametric: $x=x_1+(x_2-x_1)t$;
    $y=y_1+(y_2-y_1)t$

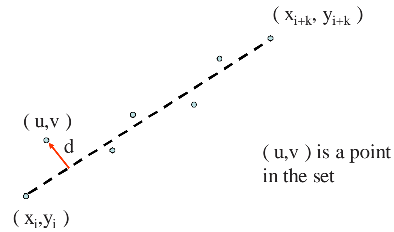
## Polyline Representation

- Sequence of line segments joined end to end
- Fit edge list with a sequence of line segments
- Polyline interpolates a selected subset of edge points, BUT ends of segments are members of original edge list
- Vertices: where end points join

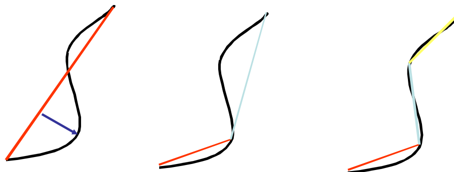
## Polyline Algorithm

- 1. Start with ordered list of edge points as input, { $(x_1,y_1)$, $(x_2,y_2)$,…, $(x_n,y_n)$ }
- 2. Join point $(x_i,y_i)$ and $(x_{i+k},y_{i+k})$, where k>1, with a straight line segment
- 3. Determine which point contained in the subset between $(x_i,y_i)$ and $(x_{i+k},y_{i+k})$ is maximum distance, d, from the constructed line segment
- 4. If d is less than the accepted maximum error, increase k to k+1 and repeat steps 2 and 3. If d is greater than the acceptable error, reduce k by 1. Recalculate line segment. This is the polyline segment for the subset of points.
- 5. Use point i+k as the starting point for the next polyline and continue the process until all points in the set have been joined by polylines

## How do you determine error, d?



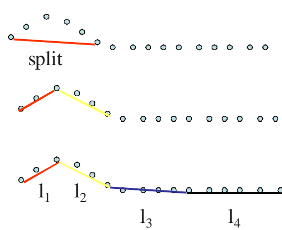

## Polyline Split Method Example



## Polyline: Hop Along Algorithm

- 1. Start with first k edges
- 2. Fit a line segment between first and last edges on sublist
- 3. If normalized maximum error too large, shorten sublist at maximum error, go to step 2.
- 4.If line fit succeeds, compare orientation with previous line segment, if similar, replace two line segments with one segment
- Continue to next k edges, go to step 2

## Hop Along Algorithm Example




Compare orientation of $l_3$ and $l_4$ and combine into one segment

## Curve approximation: Least Squares Fit

Consider fitting a straight line through a set of data points
- ease requirement of polyline approximation
- first and last point of the function approximation do not have to be edge points

# Least Squares

In function interpolation we match the original data exactly, in approximation we relax this requirement

The approach taken for 'best fit' is to minimize the square of the differences between the data value and the value of the approximation function

# Linear Least Squares

Approximate data by a straight line of the form $f(x) = ax + b$

We need to determine the coefficients a and b

Minimize sum of squares of the distance between given value $y_i$ ( original data ) and the computed function value:

$f_i = a*x_i + b$

Example: ( 1, 2.1 ), ( 2, 2.9 ), ( 5, 6.1 ), ( 7, 8.3 )
Problem is to find the coefficients a and b
The optimal coefficients of a straight line fit minimize the total squared error:

$$E = ( f(x_1) - y_1 )^2 + ( f(x_2) - y_2 )^2 + ( f(x_3) - y_3 )^2 + ( f(x_4) - y_4 )^2$$

$$E = [ ax_1 + b - y_1]^2 +[ ax_2 + b - y_2]^2 +[ ax_3 + b - y_3]^2 +[ ax_4 + b - y_4]^2$$

Minimize E with respect to two variables, a and b

Independent of the number of data points we want to fit a straight line that minimizes error in y-coordinate. We have two equations with two unknowns:

***For i = 1:n***

$a\sum xi2 + b\sum xi = \sum xi*yi$

$a\sum xi + b\sum 1 = \sum yi$ note: $b\sum 1 = b * n$

Need to calculate four different summations. In statistics literature:

$Sxx = \sum xi^2$ $Sx = \sum xi$

$Sxy = \sum xi*yi$ $Sy = \sum yi$

Therefore:

aSxx + bSx = Sxy

aSx + bn = Sy

We can also put the equations in matrix form. Starting with:

aSxx + bSx = Sxy

aSx + bn = Sy

$$\begin{pmatrix} \sum xi^2 & \sum xi \\ \sum xi & \sum 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sum xi*yi \\ \sum yi \end{pmatrix}$$

# Least squares vs. Total least squares

- Least squares does not minimize distance of point to line but minimizes error in the y coordinate
- Total least squares minimizes total error:

$d=ax+by+c$ $(a^2+b^2=1)$

Minimize $\Sigma d^2$

## Who came from which line?

- Assume we know how many lines there are - but which lines are they?
  - easy, if we know who came from which line
- Three strategies
  - Incremental line fitting
  - K-means
  - RANSAC

**Algorithm 15.1:** Incremental line fitting by walking along a curve, fitting a line to runs of pixels along the curve, and breaking the curve when the residual is too large

```
Put all points on curve list, in order along the curve
Empty the line point list
Empty the line list
Until there are too few points on the curve
  Transfer first few points on the curve to the line point list
  Fit line to line point list
  While fitted line is good enough
    Transfer the next point on the curve
      to the line point list and refit the line
  end
  Transfer last point(s) back to curve
  Refit line
  Attach line to line list
end
```

**Algorithm 15.2:** K-means line fitting by allocating points to the closest line and then refitting.

```
Hypothesize k lines (perhaps uniformly at random)
or
Hypothesize an assignment of lines to points
  and then fit lines using this assignment

Until convergence
  Allocate each point to the closest line
  Refit lines
end
```

## RANSAC

- Choose a small subset uniformly at random
- Fit to that
- Anything that is close to result is signal; all others are noise
- Refit
- Do this many times and choose the best

- Issues
  - How many times?
    - Often enough that we are likely to have a good line
  - How big a subset?
    - Smallest possible
  - What does close mean?
    - Depends on the problem
  - What is a good line?
    - One where the number of nearby points is so big it is unlikely to be all outliers

**Algorithm 15.4:** RANSAC: fitting lines using random sample consensus

```
Determine:
    n — the smallest number of points required
    k — the number of iterations required
    t — the threshold used to identify a point that fits well
    d — the number of nearby points required
      to assert a model fits well
Until k iterations have occurred
    Draw a sample of n points from the data
      uniformly and at random
    Fit to that set of n points
    For each data point outside the sample
      Test the distance from the point to the line
        against t; if the distance from the point to the line
        is less than t, the point is close
    end
    If there are d or more points close to the line
      then there is a good fit. Refit the line using all
      these points.
end
Use the best fit from this collection, using the
  fitting error as a criterion
```

## Other curves

- Implicit curves: $f(x,y)=0$
  - Example: parabola: $x^2-y=0$
    - $x^2+y^2-R^2=0$
- Parametric curves:
  - Polynomials; cubic polynomails are popular

## Cubic Splines

- Third degree polynomial curves that can be controlled via “control points”
  - B-splines,
  - Uniform B-splines,
  - non-uniform B-splines
  - NURBS

## Polynomials

- Polynomials are curved naturally and easy to work with
  - $x = c_{x0} + c_{x1}t + c_{x2}t^2 + \dots + c_{xn}t^n$
  - In general:

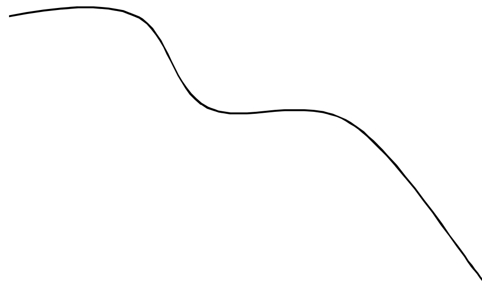
$$q(t) = \sum_{k=0}^{n} t^k c_k$$

  - where:

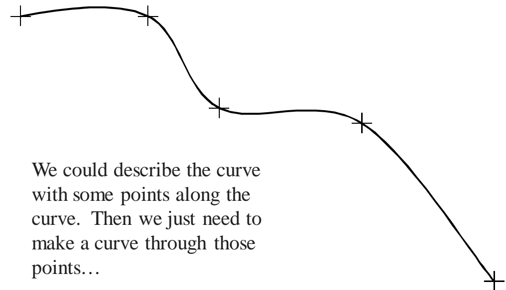
“n + 1 degrees of freedom” or “order n”

$$c_k = \begin{bmatrix} c_{xk} \\ c_{yk} \\ c_{zk} \end{bmatrix}$$

## Now, how to describe this curve?

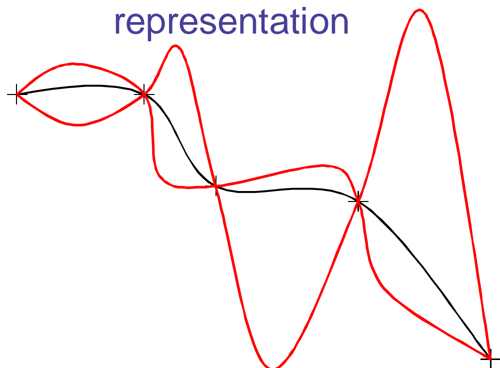


## Control Points

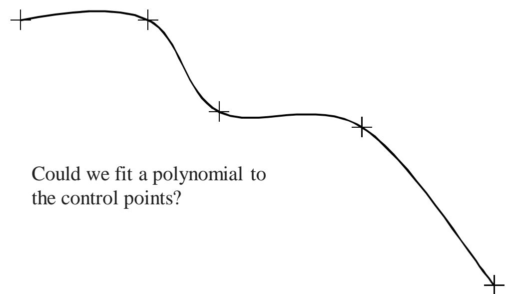


We could describe the curve with some points along the curve. Then we just need to make a curve through those points…

## Control Points are not a unique representation



## Polynomial Fitting



Could we fit a polynomial to the control points?

## Problems with Polynomial Fitting

*Overfitting*: The tendency for curves to get very bumpy when forced to fit more and more points.

## Answer: Do the curve in segments

Use polynomials for each segment.

What are the criteria at the *join points*?

## Connections at join points…

- At the join points:
  - $G^0$ continuity – The ends touch (necessary)
  - $G^1$ continuity – The *slope* at the joint is the same (necessary)
  - $C^1$ continuity – The *first derivative* at the joint is the same (not necessary)

## Which degree polynomial?

- Linear, quadric, cubic..?
- Linear polynomials are line segments:

## Cubics are good because they have 4 free parameters

- We'll use cubic polynomials for curve segments
  - $x = c_{x0} + c_{x1}t + c_{x2}t^2 + c_{x3}t^3$
  - Can be expressed as:

$$q(t) = \sum_{k=0}^{3} t^k c_k \quad c_k = \begin{bmatrix} c_{kx} \\ c_{ky} \\ c_{kz} \end{bmatrix}$$

## A Bézier Curve

$p_1$ $p_2$ $p_0$ $p_3$

$P_1$ and $P_2$ are 1/3 and –1/3 along the tangent vectors as control points

## Blending function form for Bezier polynomials

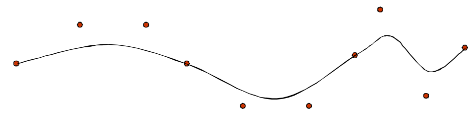
$$Q(t) = \sum_{i=0}^{3} c_i t^i = \sum_{i=0}^{3} P_i B_i(t)$$

$$Q(t) = \underbrace{(1-t)^3}_{B_0(t)} P_1 - \underbrace{3t(1-t)^2}_{B_1(t)} P_4 + \underbrace{3t^2(1-t)}_{B_2(t)} R_1 + \underbrace{t^3}_{B_3(t)} R_4$$

Bezier Blending polynomials: Bernstein polnomials

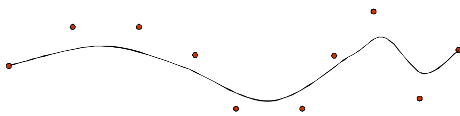
## Interpolating Curves

- An interpolating curve will hit certain points on the curve.
  - Bezier curves hit the end points
  - What if we have a lot of segments?



## What's bad about that?

- Specifying points on the curve (surface) is often difficult
  - We like to specify end points, but not other points
- What we would really like:



## What else do we want?

- We want local control
  - A control point effects a limited range of the curve
- We want more control points
  - Why only 4?

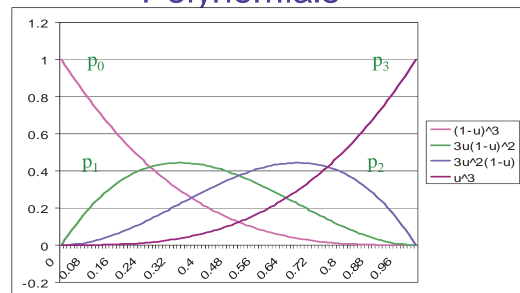
## Some ideas

- Remember that a Bézier curve can be defined as:

$$p(t) = \sum_{k=0}^{3} b_k(t) p_k$$

- This uses a function $b_k(t)$ that controls the influence of the points.

## The Bézier Blending Polynomials



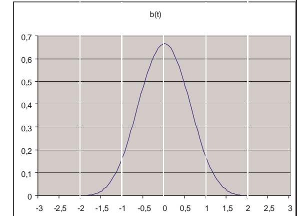

$$p(t) = \sum_{k=0}^{3} b_k(t) p_k$$

## Can we extend this?

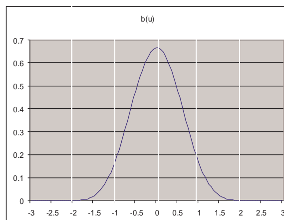
- Suppose we have 5 points?
  - Can we create a blending function of some kind?
- What do we want?
  - Smooth curve segments
  - Local control
  - Any number of control points
  - Lots of flexibility

## Cubic B-Spline Function

$$b(t)=\begin{cases}0 & if\ t\le -2\\ \frac{1}{6}(2+t)^3 & if\ -2<t\le -1\\ \frac{1}{6}(2+t)^3-\frac{2}{3}(1+t)^3 & if\ -1<t\le 0\\ -\frac{2}{3}(1-t)^3+\frac{1}{6}(2-t)^3 & if\ 0<t\le 1\\ \frac{1}{6}(2-t)^3 & if\ 1<t\le 2\\ 0 & if\ 2<t\end{cases}$$




## Some Observations




The points where we tie one function to the next are called "knots"

The knots are at integral values.

t values no longer 0 to 1

Clear "transition points" in the equation (but continuous at these points)

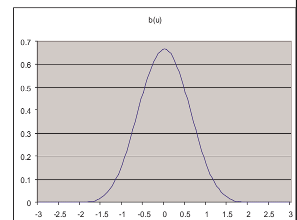
## Making this the appropriate function for a control point

- We'll have n+1 control points ($p_0$, …, $p_n$) corresponding to t values of 0 to n.

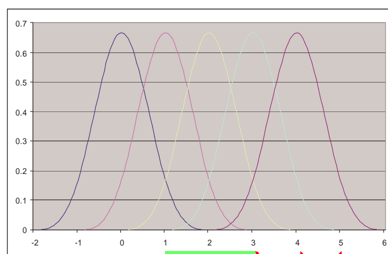
$$b_i(t)=b(t-i)$$

$$p(t)=\sum_{i=0}^{n} b_i(t)\,p_i$$

u is valid in the range 1 to n-1



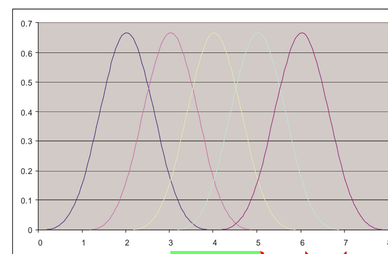

## Cubic B-Spline for 5 Point Curve



Knots

## Modifying the t value

t is valid in the range 3 to n+1

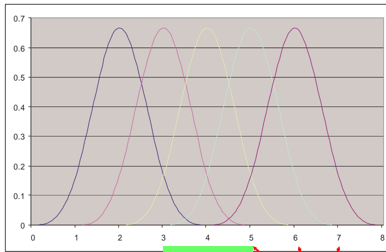


It's common to make t start at 0

For n+1 control points, there will be n+5 knots (here we have 5 control points, 9 knots)

Knots

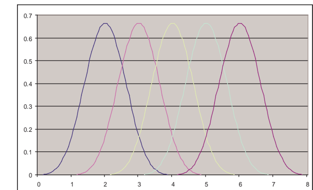
## Uniform B-Splines




Knots are evenly spaced

## Formal definition

- Parameters
  - $(p_0, \ldots, p_n)$ → n + 1 control points
  - $(t_0, \ldots, t_{n+4})$ → n + 5 knot values
  - d → Degree (we'll always use 3)



## Cox-deBoor recursion

- This function defines a spline curve
  - Not limited to degree 3, either…
  - With degree 3, gives the curves we used before

$$B_{k,0} = \begin{cases} 1, & \text{if } u_k \le u < u_{k+1} \\ 0, & \text{otherwise} \end{cases}$$

$$B_{k,d} = \frac{u - u_k}{u_{k+d} - u_k} B_{k,d-1}(u) + \frac{u_{k+d+1} - u}{u_{k+d+1} - u_{k+1}} B_{k+1,d-1}(u)$$

$$p(u) = \sum_{i=0}^{n} B_{i,d}(u)\, p_i$$

## Nonuniform B-Splines

- When we move to Cox-deBoor, it is no longer mandatory that the knots be evenly spaced!
  - If we move knots closer together, we increase the influence of some control points
  - Forcing duplicate knots can cause points to interpolate
  - Most common example:
    - Knots = {0, 0, 0, 0, 1, 1, 1, 1}
    - This is equivalent to a Bezier curve

## Non-uniform B-Spline common usages

- n=3, knots = (0, 0, 0, 0, 1, 1, 1, 1)
  - Bezier curve
- Arbitrary n,
  - knots = (0, 0, 0, 0, 1, 2, …, n-3, n-2, n-2, n-2, n-2)
  - Example: (0, 0, 0, 0, 1, 2, 3, 3, 3, 3)
    - n = 5, 6 control points….
  - These are called "open splines"
    - They interpolate the end points

## Non-uniform B-Spline advantages

- Very powerful
  - Can make not only polynomial curves, but also circles, ellipses, etc.
- Easily extended to surfaces

$$p(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{m} B_{i,d}(u) B_{j,d}(v)\, p_{i,j}$$

## NURBS

- <u>N</u>on-<u>U</u>niform <u>R</u>ational <u>B</u>-<u>S</u>plines
- Basically, Non-Uniform B-Splines, with one additional twist
  - If we consider $p_i$ as [x, y, z, 1], the result is invariant under perspective projection

$$p(u)=\frac{\sum_{i=0}^{n} B_{i,d}(u)\,p_i}{\sum_{i=0}^{n} B_{i,d}(u)}$$

← After projection

Note the division. This is the “rational” part.

## Another idea: Model surface, not edge

- View image intensity as a surface in 3D
- Model it with a planar surface
- Decide that there is an edge when slope of planar surface is large.

## Regularization

- Approximate intensity function by a smooth function
- Smooth function is nice because it is differentiable

## GED: Generalized Edge Detector Gokmen and Jain

- Approximate an image by a smooth function f by minimizing:

$$E_m(f)=\iint (f(x,y)-d(x,y))^2\,dxdy+\lambda\iint \underbrace{(f_x^{\,2}+f_y^{\,2})}_{\text{smoothness term}}dxdy$$

Membrane solution

$$E_p(f)=\iint (f(x,y)-d(x,y))^2\,dxdy+\lambda\iint (f_{xx}^{\,2}+2f_{xy}^{\,2}+f_{yy}^{\,2})\,dxdy$$

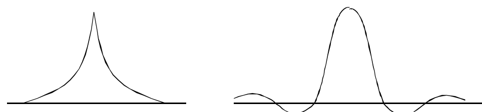
plate solution

## GED

- It is shown that this is equivalent to filtering with:

$$R_1(x,\lambda)=\frac{1}{2\lambda}e^{-|x|/\lambda} \quad \text{for the membrane}$$

$$R_2(x,\lambda)=\frac{1}{2\lambda^{1/4}}e^{-|x|/\sqrt{2}\lambda^{1/4}}\cos\left(\frac{|x|}{\sqrt{2}\lambda^{1/4}}-\frac{\pi}{4}\right) \quad \text{for the plate}$$



This looks like a Gaussian except for the tails

## GED

- Approximate d(x,y) by a smooth function f(x,y)
- Then, find the directional derivative in the direction of the tangent contour
- Use two thresholds to eliminate broken edge segments