CMPE 360: Numerical Methods
Midterm 1 Review Problems

Problem 1

State whether the following are True or False:

1. The floating point number `0.0123111` is in normalized form.
2. *You are given* LU factorization of a matrix A. The complexity of solving for x in $Ax = b$ is $O(n^2)$.
3. The complexity of Gaussian Elimination (without pivoting) for tridiagonal matrices (matrices in which you can only have non-zeros on diagonal, superdiagonal and subdiagonal) is $O(n^2)$.
4. If floating point numbers are not normalized, then we can have non-unique representation of a floating point number.
5. Gaussian elimination with partial pivoting takes $O(n^3)$.
6. Newton-Raphson method always has quadratic convergence.
7. The following matrix does not require any row or column interchanges in Gaussian Elimination.

$$\begin{bmatrix} 0.1 & 0.05 & .01 \\ 4 & 8 & 3 \\ 1 & 2 & 4 \end{bmatrix}$$

8. A singular matrix cannot have an LU decomposition.
9. The following expression defines a vector norm on R^n.

$$max\{|x_2|, |x_3|, \ldots, |x_n|\}$$

10. The following matrix has an **LU** decomposition.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

11. Given the **L** and **U** decomposition of an $n \times n$ matrix **A**, the determinant(**A**) cannot be calculated in $O(n)$ time.
12. A hidden bit enables us to get 24 bit accuracy from a 23 bit mantissa in a binary floating point system.
13. When Newton's method is applied to $f(x) = e^x - x - 1$, with proper choice of initial guess, the sequence generated will converge quadratically to the root.

Problem 2

Suppose you have n positive numbers, $x_1, \ldots, x_n$. Consider the following summation code which sums these numbers:

```
/* CODE1 */
sum = 0.0 ;
for(i=1 ; i <= n ; i = i+1) {
   sum = sum + x[i] ;
}
```

Now consider the following summation code:

```
/* CODE2 */
sort(x,n) ;
sum = 0.0 ;
for(i=1 ; i <= n ; i = i+1) {
   sum = sum + x[i] ;
}
```

The first code sorts the numbers and then sums them. How would you sort the numbers so that `CODE2` will have a lower *absolute error bound* than `CODE1`. Prove your answer by deriving absolute error bound in terms of machine epsilon, ϵ_M and x_i.

Problem 3

a) For $\mathbf{x}$ and $\mathbf{y}$ in R^n, show that

$$||\mathbf{x}+\mathbf{y}||_2^2 + ||\mathbf{x}-\mathbf{y}||_2^2 = 2||\mathbf{x}||_2^2 + 2||\mathbf{y}||_2^2$$

b) If $\mathbf{x}$ and $\mathbf{y} \in R^n$ are two orthogonal vectors. then show that:

$$||\mathbf{x}-\mathbf{y}||_2^2 = ||\mathbf{x}||_2^2 + ||\mathbf{y}||_2^2$$

Suppose that Gaussian elimination is performed on a 3 by 3 matrix **A** (without any pivoting). The matrix is stored in a two dimensinal C-language array called `matrixA[3][3]`. which has the following entries on input:

```
                      [ 1  4  7 ]
input  matrixA[3][3] = [ x  x  x ]
                      [ x  x  x ]
```

(You are supposed to find the missing entries given by `x`'s) Suppose that during the elimination process, we simply store the elimination multipliers in the 0 entries that are introduced, i.e. we overwrite on the original matrix entries (say, we do not care about keeping the original matrix). At the end of the forward elimination phase, the following is what is stored in the matrix:

```
                       [ x  x  x ]
output  matrixA[3][3] = [ 2 -3 -6 ]
                       [ 3  2  1 ]
```

Fill in the missing entries indicated by the `x`'s in BOTH the `input` and the `output matrixA`.

Problem 4

State two properties of the following matrix which enable Choleski factorization to be used. Prove that these properties indeed hold and perform Cholesky factorization on this matrix.

$$\begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$$

Problem 5

Suppose that B is an $n \times n$ symmetric positive definite matrix and x is an n-dimensional vector.

(a) Prove the following inequality:

$$|x^T By| \leq \sqrt{x^T Bx} \cdot \sqrt{y^T By}$$

(b) Prove that the function defined by

$$||x||_B = \sqrt{x^T Bx}$$

satisfies all the properties required for a vector norm.