

**CmpE 343**  
**Fall 2008**  
**Problem Session#3 Solution Key**

**Question1:** Two real-valued random variables,  $X$  and  $Y$ , have joint probability density function

$$f(x, y) = \frac{1}{2\pi\sqrt{(1-r^2)}} \exp\left(-\frac{x^2 - 2rxy + y^2}{2(1-r^2)}\right),$$

where  $-1 < r < 1$ .

- (a) Prove that each of  $X$  and  $Y$  is normally distributed with mean 0 and variance 1.  
 (b) Show that  $Cov[X, Y] = r$ .

**Solution:**

(a)

$$\begin{aligned} f(x) &= \int f(x, y) dy \\ &= \frac{1}{2\pi\sqrt{1-r^2}} \int \exp\left(-\frac{(y-rx)^2 + (1-r^2)x^2}{2(1-r^2)}\right) dy \\ &= \frac{1}{2\pi\sqrt{1-r^2}} \exp\left(-\frac{x^2}{2}\right) \int \exp\left(-\frac{(y-rx)^2}{2(1-r^2)}\right) dy \quad \left[\frac{y-rx}{\sqrt{1-r^2}} = t \Rightarrow dy = dt\sqrt{1-r^2}\right] \\ &= \frac{1}{2\pi} \exp\left(-\frac{x^2}{2}\right) \int \exp\left(-\frac{t^2}{2}\right) dt = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \sim \mathcal{N}(x; 0, 1) \end{aligned}$$

$$\begin{aligned} f(y) &= \int f(x, y) dx \\ &= \frac{1}{2\pi\sqrt{1-r^2}} \int \exp\left(-\frac{(x-ry)^2 + (1-r^2)y^2}{2(1-r^2)}\right) dx \\ &= \frac{1}{2\pi\sqrt{1-r^2}} \exp\left(-\frac{y^2}{2}\right) \int \exp\left(-\frac{(x-ry)^2}{2(1-r^2)}\right) dx \quad \left[\frac{x-ry}{\sqrt{1-r^2}} = t \Rightarrow dx = dt\sqrt{1-r^2}\right] \\ &= \frac{1}{2\pi} \exp\left(-\frac{y^2}{2}\right) \int \exp\left(-\frac{t^2}{2}\right) dt = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) \sim \mathcal{N}(y; 0, 1) \end{aligned}$$

(b)

$$\begin{aligned} E[XY] &= \int \int xyf(x, y) dy dx \\ &= \int x \int yf(x, y) dy dx \\ &= \int x \int y \frac{1}{2\pi\sqrt{1-r^2}} \exp\left(-\frac{(y-rx)^2 + (1-r^2)x^2}{2(1-r^2)}\right) dy dx \\ &= \int x \frac{1}{2\pi\sqrt{1-r^2}} \exp\left(-\frac{x^2}{2}\right) \int y \exp\left(-\frac{(y-rx)^2}{2(1-r^2)}\right) dy dx \end{aligned}$$

$$\begin{aligned}
&= \int x \frac{1}{2\pi\sqrt{1-r^2}} \exp\left(-\frac{x^2}{2}\right) \int (rx + t\sqrt{1-r^2}) \exp\left(-\frac{t^2}{2}\right) \sqrt{1-r^2} dt dx \\
&= \int x \frac{1}{2\pi\sqrt{1-r^2}} \exp\left(-\frac{x^2}{2}\right) rx\sqrt{2\pi}\sqrt{1-r^2} dx \\
&= \frac{r}{\sqrt{2\pi}} \int x^2 \exp\left(-\frac{x^2}{2}\right) dx \\
&= r
\end{aligned}$$

$$\text{Cov}[X, Y] = E[XY] - E[X]E[Y] = r$$

**Question2:** If  $X_1, X_2, X_3, X_4, X_5$  are independent and identically distributed exponential random variables with the parameter  $\beta$ .

- (a) Find  $\Pr(\min\{X_1, X_2, X_3, X_4, X_5\} \leq a)$ .  
(b) Find  $\Pr(\max\{X_1, X_2, X_3, X_4, X_5\} \leq a)$ .

**Solution:**

(a)

$$\begin{aligned}
\Pr(\min\{X_1, X_2, X_3, X_4, X_5\} \leq a) &= 1 - \Pr(\min\{X_1, X_2, X_3, X_4, X_5\} > a) \\
&= 1 - \prod_{i=1}^5 \Pr(X_i > a) \\
&= 1 - \exp\left(-\frac{5a}{\beta}\right)
\end{aligned}$$

(b)

$$\begin{aligned}
\Pr(\max\{X_1, X_2, X_3, X_4, X_5\} \leq a) &= \prod_{i=1}^5 \Pr(X_i < a) \\
&= \left\{1 - \exp\left(-\frac{a}{\beta}\right)\right\}^5
\end{aligned}$$

**Question3:** Let  $X_1, X_2, X_3$  be independent and exponential random variables with respective parameters  $\beta_1, \beta_2, \beta_3$ . Find  $\Pr(X_1 < X_2 < X_3)$ .

**Solution:**

$$\begin{aligned}
\Pr(\min\{X_2, X_3\} < x) &= 1 - \Pr(X_2 > x) \Pr(X_3 > x) \\
&= 1 - \exp\left(-\frac{1}{\beta_2}x\right) \exp\left(-\frac{1}{\beta_3}x\right) \\
&= 1 - \exp\left\{-\left(\frac{1}{\beta_2} + \frac{1}{\beta_3}\right)x\right\}
\end{aligned}$$

$$\begin{aligned}
\Pr(X_2 < X_3) &= \int_0^\infty \int_{x_2}^\infty \frac{1}{\beta_2} \exp\left(-\frac{1}{\beta_2}x_2\right) \frac{1}{\beta_3} \exp\left(-\frac{1}{\beta_3}x_3\right) dx_3 dx_2 \\
&= \frac{1}{\frac{1}{\beta_2} + \frac{1}{\beta_3}}
\end{aligned}$$

$$\begin{aligned} \Pr(X_1 < X_2 < X_3) &= \Pr(X_1 < \min\{X_2, X_3\}) \Pr(X_2 < X_3) \\ &= \frac{\frac{1}{\beta_1}}{\frac{1}{\beta_1} + \frac{1}{\beta_2} + \frac{1}{\beta_3}} \frac{\frac{1}{\beta_2}}{\frac{1}{\beta_2} + \frac{1}{\beta_3}} \end{aligned}$$

or we can calculate as follows:

$$\Pr(X_1 < X_2 < X_3) = \int_0^\infty \int_{x_1}^\infty \int_{x_2}^\infty \frac{1}{\beta_1} \exp\left(-\frac{1}{\beta_1}x_1\right) \frac{1}{\beta_2} \exp\left(-\frac{1}{\beta_2}x_2\right) \frac{1}{\beta_3} \exp\left(-\frac{1}{\beta_3}x_3\right) dx_3 dx_2 dx_1 .$$