CmpE 343
Fall 2008
Problem Session#2 Solution Key

Question1: To determine whether or not they have a certain disease, 100 people are to have their blood tested. However, rather than testing each individual separately, it has been decided first to group the people in groups of 10. The blood samples of the 10 people in each group will be pooled and analyzed together. If the test is negative, one test will suffice for the 10 people; whereas, if the test is positive each of the 10 people will also be individually tested and, in all, 11 tests will be made on this group. Assume the probability that a person has the disease is 0.1 for all people, independently of each other, and compute the expected number of tests necessary for each group.

Solution: Let T be the number of tests necessary for each group.

$$\begin{aligned}\Pr(\text{test is negative for the group}) &= \binom{10}{10}(0.9)^{10}(0.1)^{0}\\ &\approx 0.35\end{aligned}$$

$$\begin{aligned}E[T] &= 1(0.35) + 11(0.65)\\ &\approx 7.5\end{aligned}$$

Question2: Let X be a Poisson random variable with parameter λ, where $0 < \lambda < 1$. Find $E[X!]$.

Solution:

$$\begin{aligned}E[X!] &= \sum_{x=0}^{+\infty} x!\frac{e^{-\lambda}\lambda^x}{x!}\\ &= e^{-\lambda}\sum_{x=0}^{+\infty}\lambda^x\\ &= e^{-\lambda}\frac{1}{1-\lambda}\end{aligned}$$

Question3: A man aiming at a target receives 16 points if his shot is within 1 inch of the target, 6 points if it is between 1 and 3 inches from the target, 4 points if it is between 3 and 5 inches from the target, and no points if is further away. Find the expected number of points scored if the man's shot is uniformly distributed in a circle of radius 8 inches centered at the target.

Solution: Let S be the score received.

$$\begin{aligned}\Pr(16\text{ points}) &= \frac{\pi}{64\pi} = \frac{1}{64}\\ \Pr(6\text{ points}) &= \frac{9\pi-\pi}{64\pi} = \frac{8}{64}\\ \Pr(4\text{ points}) &= \frac{25\pi-9\pi}{64\pi} = \frac{16}{64}\\ \Pr(0\text{ points}) &= \frac{64\pi-25\pi}{64\pi} = \frac{39}{64}\end{aligned}$$

$$\begin{aligned}E[S] &= 16\frac{1}{64} + 6\frac{8}{64} + 4\frac{16}{64} + 0\frac{39}{64}\\ &= 2\end{aligned}$$

Question4: Alice has a matchbox in each pocket. Initially, each box contains n matches. At each time step, she reaches into the left pocket (with probability p) or into the right pocket (with probability $1-p$), and removes a match. Her choices at different times are independent. Let K be the first time that the match removed happens to be the last match of the selected box. Find the probability mass function of K.

Solution: Only two cases can happen the last matchis picked in the right pocket (event R) or in the left pocket (event L).

Observe that $n \leq K \leq 2n-1$. Let $k \in 0, \ldots, n-1$. If $K = n+k$, k picks in the nonemptied pocket have been chosen in the first $n+k-1$ picks, since the last pick was in the emptied pocket.

$$\Pr(K = n+k, L) = \binom{n+k-1}{k} p^n (1-p)^k$$
$$\Pr(K = n+k, R) = \binom{n+k-1}{k} p^k (1-p)^n$$
$$\Pr(K = n+k) = \binom{n+k-1}{k} \left[p^n (1-p)^k + p^k (1-p)^n\right]$$

Question5: An expedition is sent to the Himalayas with the objective of catching a pair of wild yaks for breeding. Assume yaks are loners and roam about the Himalayas at random. The probability $p \in (0,1)$ that a given trapped yak is male is independent of prior outcomes. Let N be the number of yaks that must be caught until a pair is obtained. Show that the expected value of N is $1+\frac{p}{q}+\frac{q}{p}$, where $q = 1-p$.

Solution: $\Pr(N = n) = p^{n-1}q + q^{n-1}p\,, \quad n \geq 2.$

$$E[N] = q\sum_{n=2}^{\infty} np^{n-1} + p\sum_{n=2}^{\infty} nq^{n-1} = \frac{q}{p}\sum_{n=2}^{\infty} np^n + \frac{p}{q}\sum_{n=2}^{\infty} nq^n$$
$$= \frac{q}{p}\left[\frac{p}{(1-p)^2} - p\right] + \frac{p}{q}\left[\frac{q}{(1-q)^2} - q\right]$$
$$= \frac{1}{p} + \frac{1}{q} - 1 = 1 + \frac{p}{q} + \frac{q}{p}$$

Question6: A fair die has two green faces, two red faces and two blue faces, and the die is thrown once. Let

$$X = \begin{cases} 1 & \text{if a green face is uppermost} \\ 0 & \text{otherwise} \end{cases}$$
$$Y = \begin{cases} 1 & \text{if a blue face is uppermost} \\ 0 & \text{otherwise.} \end{cases}$$

Find $Cov[X, Y]$.

Solution:

$$\Pr(\text{green}) = \Pr(\text{blue}) = \frac{1}{3}$$
$$E[X] = E[Y] = 1\frac{1}{3} + 0\frac{2}{3} = \frac{1}{3}$$
$$E[XY] = 0$$
$$Cov[X, Y] = 0 - \frac{1}{3}\frac{1}{3} = -\frac{1}{9}$$

Question7: Each of the random variables U and V takes the values ± 1. Their joint distribution is given by:

$$\begin{aligned}\Pr(U=+1) = Pr(U=-1) &= \frac{1}{2}\\ \Pr(V=+1|U=+1) = Pr(V=-1|U=-1) &= \frac{1}{3}\\ \Pr(V=-1|U=+1) = Pr(V=+1|U=-1) &= \frac{2}{3}\,.\end{aligned}$$

(a) Find the probability that $x^2 + Ux + V = 0$ has at least one real root.

(b) Find the expected value of the larger root of $x^2 + Ux + V = 0$ given that there is at least one real root.

(c) Find the probability that $x^2 + (U+V)x + (U+V) = 0$ has at least one real root.

Solution:

$$\Pr(U=+1, V=+1) = \frac{1}{6} \qquad \Pr(U=-1, V=+1) = \frac{1}{3}$$
$$\Pr(U=+1, V=-1) = \frac{1}{3} \qquad \Pr(U=-1, V=-1) = \frac{1}{6}$$

(a) $x^2 + Ux + V$ has a real root if and only if $U^2 - 4V \geq 0$ which means $V = -1$. Clearly, if $V = -1$, then $U^2 - 4V = 5$. So the probability of a real root is $\frac{1}{2}$.

(b) The expected value of the larger root is

$$\begin{aligned}&= \left(\frac{-1+\sqrt{5}}{2}\right)\Pr(U=+1|V=-1) + \left(\frac{1+\sqrt{5}}{2}\right)\Pr(U=-1|V=-1)\\ &= \frac{1}{\Pr(V=-1)}\left[\left(\frac{-1+\sqrt{5}}{2}\right)\frac{1}{3} + \left(\frac{-1+\sqrt{5}}{2}\right)\frac{1}{6}\right]\\ &= \frac{\sqrt{5}}{2} - \frac{1}{6}\end{aligned}$$

(c) $x^2 + Wx + W$ has a real root if and only if $W^2 - 4W \geq 0$. If $W = U + V$, W takes values +2, 0, -2 and the equation has a real root if $W = 0$ or $W = 2$. Then $\Pr(W=0) = \frac{2}{3}$ and $\Pr(W=-2) = \frac{1}{6}$. The asked probability is $\frac{2}{3} + \frac{1}{6} = \frac{5}{6}$.