CmpE 343
Fall 2008
Problem Session#1 Solution Key

Question1: You throw $6n$ dice at random. Find the probability that each number appears exactly n times.

Solution: There are 6^{6n} outcomes in total (six for each die), each has probability $\frac{1}{6^{6n}}$. We want n dice to show one dot, n two, and so forth. The number of such outcomes is counted by fixing first which dice show one: $\frac{(6n)!}{n!(5n)!}$. Given n dice showing one, we fix which remaining dice show two: $\frac{(5n)!}{n!(4n)!}$, etc. The total number of desired outcomes is the product that equals $\frac{(6n)!}{(n!)^6}$. This gives the answer $\frac{(6n)!}{(n!)^6}\frac{1}{6^{6n}}$.

Question2: Four mice are chosen (without replacement) from a litter containing two white mice. The probability that both white mice are chosen is twice the probability that neither is chosen. How many mice are there in the litter?

Solution: Let the number of mice in the litter be n. We use the notation $\Pr(2) = \Pr(\text{two white chosen})$ and $\Pr(0) = \Pr(\text{no white chosen})$. Then $\Pr(2) = \binom{n-2}{2}\Big/\binom{n}{4}$ and $\Pr(0) = \binom{n-2}{4}\Big/\binom{n}{4}$. Solving the equation $\frac{12}{n(n-1)} = 2\frac{(n-4)(n-5)}{n(n-1)}$, we get $n = (9 \pm 5)/2$; $n = 2$ is discarded as $n = 6$ (otherwise the second probability is 0). Hence, $n = 7$.

Question3: You have n urns, the rth of which contains $r-1$ red balls and $n-r$ blue balls, $r = 1, \ldots, n$.

(a) You pick an urn at random and remove two balls from it without replacement. Find the probability that the two balls are of different colors.

(b) Find the same probability when you put back a removed ball.

Solution:

(a) The totals of blue and red balls in all urns are equal. Hence, the first ball is equally likely to be any ball. So, $\Pr(\text{1st blue}) = \Pr(\text{1st red}) = \frac{1}{2}$.

$$\begin{aligned}
\Pr(\text{1st red, 2nd blue}) &= \sum_{k=1}^{n} \Pr(\text{1st red, 2nd blue} \mid \text{urn } k \text{ chosen})\frac{1}{n} \\
&= \frac{1}{n}\sum_{k=1}^{n}\frac{(k-1)(n-k)}{(n-1)(n-2)} \\
&= \frac{1}{n(n-1)(n-2)}\left[n\sum_{k=1}^{n}(k-1) - \sum_{k=1}^{n}k(k-1)\right] \\
&= \frac{1}{n(n-1)(n-2)}\left[\frac{n(n-1)n}{2} - \frac{(n+1)n(n-1)}{3}\right] \\
&= \frac{n(n-1)}{n(n-1)(n-2)}\left[\frac{n}{2} - \frac{(n+1)}{3}\right] = \frac{1}{6}
\end{aligned}$$

By symmetry, $\Pr(\text{different colors}) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

(b) If you return a removed ball, the probability that the two ball are of different colors becomes $\frac{1}{2}$.

Question4: Parliament contains a proportion p of Party A members, incapable of changing their opinions about anything, and $1-p$ of Party B members changing their minds at random, with probability r, between subsequent votes on the same issue. A randomly chosen parliamentarian is noticed to have voted twice in succession in the same way. Find the probability that (s)he will vote in the same way next time.

Solution:
$A = \{\text{Party } A \text{ chosen}\}$
$B = \{\text{Party } B \text{ chosen}\}$
$S = \{\text{The member chosen voted twice in the same way}\}$
We have $\Pr(A) = p$, $\Pr(B) = 1-p$, $\Pr(S|A) = 1$, and $\Pr(S|B) = 1-r$. We want to calculate $\Pr(A|S) = \frac{\Pr(A \cap S)}{\Pr(S)} = \frac{\Pr(A)\Pr(S|A)}{\Pr(S)}$ and $\Pr(B|S) = 1 - \Pr(A|S)$. Write $\Pr(S) = \Pr(A)\Pr(S|A) + \Pr(B)\Pr(S|B) = p+(1-p)(1-r)$. Then, $\Pr(A|S) = \frac{p}{p+(1-p)(1-r)}$, $\Pr(B|S) = \frac{(1-p)(1-r)}{p+(1-p)(1-r)}$, and the answer is given by $\Pr(\text{The member will vote in the same way}|S) = \frac{p+(1-p)(1-r)^2}{p+(1-p)(1-r)}$.

Question5: Two darts players throw alternately at a board and the first to score a bull wins. On each of their throws player A has probability p_A and player B p_B of success; the results of different throws are independent. If A starts, calculate the probability that (s)he wins.

Solution: If $q = \Pr(A \text{ wins})$, then

$$\begin{aligned} q &= p_A + (1-p_A)(1-p_B)p_A + (1-p_A)^2(1-p_B)^2 p_A + \dots \\ &= \frac{p_A}{1-(1-p_A)(1-p_B)} = \frac{p_A}{p_A + p_B - p_A p_B}. \end{aligned}$$

Question6: The joint probability density function of X and Y is given by

$$f(x,y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) \quad 0 < x < 1,\ 0 < y < 2\,.$$

(a) Verify that this is indeed a joint density function.

(b) Compute the density function of X.

(c) Find $\Pr(X > Y)$.

(d) Find $\Pr\left(Y > \frac{1}{2} \middle| X < \frac{1}{2}\right)$.

Solution:

(a) $f(x,y) \geq 0 \quad \forall (x,y)$ is trivially true. We need to show that $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)dxdy = 1$.

$$\begin{aligned} \int_0^2\int_0^1 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dxdy &= \frac{6}{7}\int_0^2 \left(\frac{x^3}{3} + \frac{x^2 y}{4}\right)\Bigg|_{x=0}^{1} dy \\ &= \frac{6}{7}\int_0^2 \left(\frac{1}{3} + \frac{y}{4}\right) dy \end{aligned}$$

$$= \frac{6}{7}\left(\frac{y}{3} + \frac{y^2}{8}\right)\Bigg|_{y=0}^{2}$$

$$= \frac{6}{7}\frac{7}{6} = 1$$

(b)

$$g(x) = \int_0^2 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dy$$

$$= \frac{6}{7}\left(x^2 y + \frac{xy^2}{4}\right)\Bigg|_{y=0}^{2}$$

$$= \frac{6}{7}\left(2x^2 + x\right)$$

(c)

$$\Pr(X > Y) = \int_0^1 \int_0^x \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dydx$$

$$= \frac{6}{7}\int_0^1 \left(x^2 y + \frac{xy^2}{4}\right)\Bigg|_{y=0}^{x} dx$$

$$= \frac{6}{7}\int_0^1 \frac{5}{4}x^3 dx$$

$$= \frac{6}{7}\frac{5}{16}x^4\Big|_{x=0}^{1} = \frac{15}{56}$$

(d)

$$\Pr\left(Y > \frac{1}{2}\Big| X < \frac{1}{2}\right) = \frac{\Pr\left(Y > \frac{1}{2}, X < \frac{1}{2}\right)}{\Pr\left(X < \frac{1}{2}\right)}$$

$$= \frac{\int_{1/2}^{2}\int_0^{1/2} \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dxdy}{\int_0^{1/2} \frac{6}{7}\left(2x^2 + x\right) dx}$$

$$= \frac{69}{80}$$